mathematics

Article

On a New Generalization of Banach Contraction Principle with Application

Hüseyin Işık [1,2,*], Babak Mohammadi [3,*], Mohammad Reza Haddadi [4] and Vahid Parvaneh [5]

1 Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City 700000, Vietnam
2 Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City 700000, Vietnam
3 Department of Mathematics, Marand Branch, Islamic Azad University, Marand, Iran
4 Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran; haddadi@abru.ac.ir
5 Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran; zam.dalahoo@gmail.com
* Correspondence: huseyin.isik@tdtu.edu.vn (H.I.); bmohammadi@marandiau.ac.ir (B.M.)

Received: 10 August 2019; Accepted: 29 August 2019; Published: 18 September 2019

Abstract: The main purpose of the current work is to present firstly a new generalization of Caristi's fixed point result and secondly the Banach contraction principle. An example and an application is given to show the usability of our results.

Keywords: Banach contraction principle; Caristi fixed point; lower semi-continuous function; integral equation

1. Introduction and Preliminaries

Metric fixed point theory plays a crucial role in the field of functional analysis. It was first introduced by the great Polish mathematician Banach [1]. Over the years, due to its significance and application in different fields of science, a lot of generalizations have been done in different directions by several authors see, for example, [2–17] and references therein. Assuredly, the Caristi's fixed point theorem [18] is the most valuable generalization of this principle.

For any nonempty set Λ, set:

$$\Xi = \{\varrho : \Lambda \to \mathbb{R} \colon \varrho \text{ is a lower semi-continuous and bounded below function}\}.$$

Theorem 1. *[18] Let (Λ, d) be a complete metric space and $\Gamma : \Lambda \to \Lambda$ be a self-map. If there exists $\varrho \in \Xi$ such that:*

$$d(\eta, \Gamma\eta) \leq \varrho(\eta) - \varrho(\Gamma\eta)$$

for all $\eta \in \Lambda$. Then Γ has a fixed point.

Recently, Du [19] established a direct proof of Caristi's fixed point theorem without using Zorn's lemma. In the next section we introduce a new generalization of Caristi's fixed point theorem and provide the proof without using Zorn's lemma.

2. A Generalization of Caristi's Fixed Point Theorem

Let Ω be the collection of functions $\vartheta : \mathbb{R} \to (0, \infty)$ satisfying the following conditions:

(Ω_1) ϑ is strictly increasing and continuous;
(Ω_2) For every sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} \vartheta(\alpha_n) = 1$;
(Ω_3) For every $\alpha, \beta \in \mathbb{R}$, $\vartheta(\alpha + \beta) \leq \vartheta(\alpha)\vartheta(\beta)$;

Obviously, for a function ϑ, satisfying (Ω_2), $\vartheta(\alpha) = 1$ iff $\alpha = 0$.

Example 1. $\vartheta_1(t) = 1 + \tanh t \in \Omega$, $\vartheta_2(t) = e^t$,

$$\vartheta_3(t) = \begin{cases} 1 + \ln(1+t), & \text{if } t \in [0, \infty), \\ e^t, & \text{if } t \in (-\infty, 0], \end{cases}$$

are some elements in Ω.

Theorem 2. *Let* (Λ, d) *be a complete metric space and* $\Gamma : \Lambda \to \Lambda$ *be a self-map. If there exist* $\varrho \in \Xi$ *and* $\vartheta \in \Omega$ *such that:*

$$\vartheta(d(\eta, \Gamma\eta)) \leq \frac{\vartheta(\varrho(\eta))}{\vartheta(\varrho(\Gamma\eta))}, \tag{1}$$

for all $\eta \in \Lambda$, *then* Γ *has a fixed point.*

Proof. For any $\eta \in \Lambda$, define:

$$\Upsilon\eta = \{\mu \in \Lambda : \vartheta(d(\eta, \mu)) \leq \frac{\vartheta(\varrho(\eta))}{\vartheta(\varrho(\mu))}\}.$$

Obviously, $\Upsilon\eta \neq \emptyset$ for any $\eta \in \Lambda$, since $\eta \in \Upsilon\eta$. Let us firstly show that for any $\mu \in \Upsilon\eta$, we have $\varrho(\mu) \leq \varrho(\eta)$ and $\Upsilon\mu \subseteq \Upsilon\eta$. Suppose $\mu \in \Upsilon\eta$. Then,

$$1 \leq \vartheta(d(\eta, \mu)) \leq \frac{\vartheta(\varrho(\eta))}{\vartheta(\varrho(\mu))}, \tag{2}$$

which implies $\vartheta(\varrho(\mu)) \leq \vartheta(\varrho(\eta))$ and since ϑ is strictly increasing, we get $\varrho(\mu) \leq \varrho(\eta)$. Now let $\zeta \in \Upsilon\mu$. Then:

$$1 \leq \vartheta(d(\mu, \zeta)) \leq \frac{\vartheta(\varrho(\mu))}{\vartheta(\varrho(\zeta))}. \tag{3}$$

From Equation (2) and Equation (3), we get:

$$\begin{aligned} \vartheta(d(\eta, \zeta)) &\leq \vartheta(d(\eta, \mu) + d(\mu, \zeta)) \\ &\leq \vartheta(d(\eta, \mu))\vartheta(d(\mu, \zeta)) \\ &\leq \frac{\vartheta(\varrho(\eta))}{\vartheta(\varrho(\mu))} \frac{\vartheta(\varrho(\mu))}{\vartheta(\varrho(\zeta))} \\ &= \frac{\vartheta(\varrho(\eta))}{\vartheta(\varrho(\zeta))}. \end{aligned}$$

Therefore, $\zeta \in \Upsilon\eta$. Thus, $\Upsilon\mu \subseteq \Upsilon\eta$. Choose a point $\eta_1 \in \Lambda$ and construct a sequence $\{\eta_n\}$ in Λ in the following way: For any η_n there exists $\eta_{n+1} \in \Upsilon\eta_n$ such that:

$$\varrho(\eta_{n+1}) \leq \inf_{\zeta \in \Upsilon\eta_n} \varrho(\zeta) + \frac{1}{n}.$$

Since $\eta_{n+1} \in \Upsilon\eta_n$, we get $\varrho(\eta_{n+1}) \leq \varrho(\eta_n)$, for all $n \in \mathbb{N}$. Thus, the sequence $\{\varrho(\eta_n)\}$ is non-increasing. Since ϱ is bounded below, there exists $L \in \mathbb{R}$ such that $\lim \varrho(\eta_n) = L$. For any $n, m \in \mathbb{N}$ with $n < m$,

$$\begin{aligned}\vartheta(d(\eta_n,\eta_m)) &\leq \vartheta(\sum_{i=n}^{m-1} d(\eta_i,\eta_{i+1}))\\ &\leq \prod_{i=n}^{m-1}\vartheta(d(\eta_i,\eta_{i+1}))\\ &\leq \prod_{i=n}^{m-1}\frac{\vartheta(\varrho(\eta_i))}{\vartheta(\varrho(\eta_{i+1}))}\\ &= \frac{\vartheta(\varrho(\eta_n))}{\vartheta(\varrho(\eta_m))}\\ &\leq \frac{\vartheta(\varrho(\eta_n))}{\vartheta(L)}. \end{aligned} \tag{4}$$

Therefore, from continuity of ϑ and by taking the limit in both sides of Equation (4), we obtain that $\lim_{n,m\to\infty}\vartheta(d(\eta_n,\eta_m)) = 1$. Therefore, $\vartheta(\lim_{n,m\to\infty} d(\eta_n,\eta_m)) = 1$, which gives us $\lim_{n,m\to\infty} d(\eta_n,\eta_m) = 0$. Thus, we proved that $\{\eta_n\}$ is a Cauchy sequence. Completeness of Λ ensures that there exists $v \in \Lambda$ such that $\eta_n \to v$ as $n \to \infty$. We claim that v is a fixed point of Γ. Taking the limit in both sides of Equation (4) as $m \to \infty$, we obtain:

$$\vartheta(d(\eta_n,v)) \leq \frac{\vartheta(\varrho(\eta_n))}{\vartheta(\varrho(v))}.$$

This gives us $v \in \Upsilon\eta_n$, for all $n \in \mathbb{N}$ and so $v \in \cap_{n=1}^{\infty}\Upsilon\eta_n$. Also, for any $w \in \cap_{n=1}^{\infty}\Upsilon\eta_n$, we have:

$$\vartheta(d(\eta_n,w)) \leq \frac{\vartheta(\varrho(\eta_n))}{\vartheta(\varrho(w))} \leq \frac{\vartheta(\varrho(\eta_n))}{\inf_{\zeta\in\Upsilon\eta_n}\vartheta(\varrho(\zeta))} \leq \frac{\vartheta(\varrho(\eta_n))\vartheta(\frac{1}{n})}{\vartheta(\varrho(\eta_{n+1}))}. \tag{5}$$

Taking the limit in both sides of Equation (5) as $n \to \infty$, we obtain $\vartheta(d(v,w)) \leq 1$ and so $d(v,w) = 0$. Thus, $w = v$. Therefore, $\cap_{n=1}^{\infty}\Upsilon\eta_n = \{v\}$. On the other hand, $v \in \cap_{n=1}^{\infty}\Upsilon\eta_n$ implies $\Upsilon v \subseteq \cap_{n=1}^{\infty}\Upsilon\eta_n = \{v\}$. Thus $\Upsilon v = \{v\}$. Furthermore, from Equation (1), we have $\Gamma v \in \Upsilon v = \{v\}$. Therefore, $\Gamma v = v$. The proof is completed. □

Note that taking $\vartheta(t) = e^t$, Theorem (2) reduces to Carisi's fixed point theorem. Thus, Theorem (2) is a generalization of Caristi's theorem.

Theorem 3. *Let (Λ, d) be a complete metric space and $\Gamma : \Lambda \to \Lambda$ be a self-map. If there exist $\varrho \in \Xi$ and $\vartheta \in \Omega$ such that:*

$$\vartheta(\xi(d(\eta,\Gamma\eta))) \leq \frac{\vartheta(\varrho(\eta))}{\vartheta(\varrho(\Gamma\eta))}, \tag{6}$$

for all $\eta \in \Lambda$, where $\xi : [0,\infty) \to [0,\infty)$ is a continuous, non-decreasing, and concave downward function such that $\xi^{-1}(\{0\}) = \{0\}$, then Γ has a fixed point.

Proof. Define a function:

$$d'(\eta,\mu) = \xi(d(\eta,\mu))$$

for all $\eta, \mu \in \Lambda$. Then it is easy to check that (Λ, d') is a complete metric space and the conditions of Theorem (2) holds for (Λ, d'). Thus, by Theorem (2), Γ has a fixed point. □

3. A Generalization of Banach's Fixed Point Theorem

In this section, we introduce a generalization of Banach contraction principle via a different approach from Caristi's result.

Theorem 4. *Let (Λ, d) be a complete metric space and $\Gamma : \Lambda \to \Lambda$ be a continuous self-map. If there exists a function $\varrho : [0, \infty) \to [0, \infty)$ such that $\lim_{t\to 0^+} \varrho(t) = 0$, $\varrho(0) = 0$ and:*

$$d(\Gamma\eta, \Gamma\mu) \le \varrho(d(\eta, \mu)) - \varrho(d(\Gamma\eta, \Gamma\mu)), \tag{7}$$

for all $\eta, \mu \in \Lambda$, then Γ has a unique fixed point.

Proof. Consider an arbitrary element $\eta_0 \in \Lambda$. Construct a sequence $\{\eta_n\}$ in Λ with $\eta_{n+1} = \Gamma(\eta_n)$, for all $n \in \mathbb{N} \cup \{0\}$. Using Equation (7) for $\eta = \eta_n$ and $\mu = \eta_{n+1}$, we have:

$$\begin{aligned} 0 \le d(\eta_n, \eta_{n+1}) = d(\Gamma\eta_{n-1}, \Gamma\eta_n) &\le \varrho(d(\eta_{n-1}, \eta_n)) - \varrho(d(\Gamma\eta_{n-1}, \Gamma\eta_n)) \\ &= \varrho(d(\eta_{n-1}, \eta_n)) - \varrho(d(\eta_n, \eta_{n+1})). \end{aligned} \tag{8}$$

Thus, the sequence $\{\varrho(d(\eta_n, \eta_{n+1}))\}$ is nonincreasing. Since ϱ is bounded below, there exists $L \in \mathbb{R}^+$ such that $\lim_{n\to\infty} \varrho(d(\eta_n, \eta_{n+1})) = L$. For any $n, m \in \mathbb{N}$ with $n < m$,

$$\begin{aligned} d(\eta_n, \eta_m) &\le \sum_{i=n}^{m-1} d(\eta_i, \eta_{i+1}) \\ &\le \sum_{i=n}^{m-1} \left(\varrho(d(\eta_{i-1}, \eta_i)) - \varrho(d(\eta_i, \eta_{i+1}))\right) \\ &= \varrho(d(\eta_{n-1}, \eta_n)) - \varrho(d(\eta_{m-1}, \eta_m)) \\ &\le \varrho(d(\eta_n, \eta_{n+1})) - L. \end{aligned} \tag{9}$$

Taking the limit in both sides of Equation (9), we obtain $\lim_{n,m\to\infty} d(\eta_n, \eta_m) = 0$. Thus, we proved that $\{\eta_n\}$ is a Cauchy sequence. Completeness of Λ ensures that there exists $\zeta \in \Lambda$ such that $\eta_n \to \zeta$ as $n \to \infty$. We claim that ζ is a fixed point of Γ. We have:

$$\begin{aligned} d(\zeta, \Gamma\zeta) = \lim_{n\to\infty} d(\eta_{n+1}, \Gamma\zeta) = \lim_{n\to\infty} d(\Gamma\eta_n, \Gamma\zeta) \\ &\le \lim_{n\to\infty} \varrho(d(\eta_n, \zeta)) - \varrho(d(\Gamma\eta_n, \Gamma\zeta)) \\ &\le \lim_{n\to\infty} \varrho(d(\eta_n, \zeta)) = 0. \end{aligned}$$

The proof is completed. □

Remark 1. *Note that Theorem 4 is a generalization of the Banach contraction principle. If $\Gamma : \Lambda \to \Lambda$ is a Banach contraction, there exists $k \in [0, 1)$ such that $d(\Gamma\eta, \Gamma\mu) \le kd(\eta, \mu)$, for all $\eta, \mu \in \Lambda$. Hence:*

$$d(\Gamma\eta, \Gamma\mu) \le kd(\eta, \mu) \le \frac{k}{1 + k - \sqrt{k}} d(\eta, \mu),$$

for all $\eta \in \Lambda$. Consequently,

$$kd(\Gamma\eta, \Gamma\mu) + (1 - \sqrt{k})d(\Gamma\eta, \Gamma\mu) \le kd(\eta, \mu)$$

and so,

$$(1 - \sqrt{k})d(\Gamma\eta, \Gamma\mu) \le kd(\eta, \mu) - kd(\Gamma\eta, \Gamma\mu).$$

Therefore,

$$d(\Gamma\eta, \Gamma\mu) \le \frac{k}{1 - \sqrt{k}} d(\eta, \mu) - \frac{k}{1 - \sqrt{k}} d(\Gamma\eta, \Gamma\mu).$$

Taking $\varrho(t) = \frac{k}{1-\sqrt{k}} t$, we have $d(\Gamma\eta, \Gamma\mu) \le \varrho(d(\eta, \mu)) - \varrho(d(\Gamma\eta, \Gamma\mu))$, for all $\eta, \mu \in \Lambda$.

Choosing $\varrho(t) = te^t$, for all $t \geq 0$, we deduce the following corollary.

Corollary 1. *Let (Λ, d) be a complete metric space and $\Gamma : \Lambda \to \Lambda$ be a continuous self-map. Let:*

$$\frac{d(\Gamma\eta, \Gamma\mu)(1 + e^{d(\Gamma\eta,\Gamma\mu)})}{d(\eta, \mu)e^{d(\eta,\mu)}} \leq 1, \tag{10}$$

for all $\eta, \mu \in \Lambda$ with $\eta \neq \mu$. Then Γ has a unique fixed point.

Example 2. *Let $\Lambda = \{\kappa_j = \frac{j(j+1)}{2} : j = 1, 2, \cdots\}$, $d(\eta, \mu) = |\eta - \mu|$ and:*

$$\Gamma\eta = \{ \begin{array}{ll} \kappa_1, & \eta = \kappa_1, \\ \kappa_{j-1}, & \eta = \kappa_j, j \geq 2. \end{array}$$

We need only check the following two cases:

Case 1: *$\eta = \kappa_j, j \geq 2$ and $\mu = \kappa_1$.*

$$d(\Gamma\eta, \Gamma\mu) = |\kappa_{j-1} - 1|$$

and $d(\eta, \mu) = |\kappa_j - 1|$. Then,

$$\begin{aligned} \frac{d(\Gamma\eta, \Gamma\mu)(1 + e^{d(\Gamma\eta,\Gamma\mu)})}{d(\eta, \mu)e^{d(\eta,\mu)}} &= \frac{(\kappa_{j-1} - 1)(1 + e^{\kappa_{j-1}-1})}{(\kappa_j - 1)e^{\kappa_j - 1}} \\ &= \frac{(\frac{j(j-1)}{2} - 1)(1 + e^{\frac{j(j-1)}{2}-1})}{(\frac{j(j+1)}{2} - 1)e^{\frac{j(j+1)}{2}-1}} \\ &\leq \frac{2(e^{\frac{j(j-1)}{2}-1})}{e^{\frac{j(j+1)}{2}-1}} \\ &< 2e^{-j} \leq 1. \end{aligned}$$

Case 2: *$\eta = \kappa_j, \mu = \kappa_l, j > l$. So,*

$$d(\Gamma\eta, \Gamma\mu) = |\kappa_{j-1} - \kappa_{l-1}|$$

and $d(\eta, \mu) = |\kappa_j - \kappa_l|$. Then,

$$\begin{aligned} \frac{d(\Gamma\eta, \Gamma\mu)(1 + e^{d(\Gamma\eta,\Gamma\mu)})}{d(\eta, \mu)e^{d(\eta,\mu)}} &= \frac{(\kappa_{j-1} - \kappa_{m-1})(1 + e^{\kappa_{j-1}-\kappa_{l-1}})}{(\kappa_j - \kappa_l)e^{\kappa_j - \kappa_l}} \\ &= \frac{(\frac{j(j-1)}{2} - \frac{l(l-1)}{2})(1 + e^{\frac{j(j-1)}{2} - \frac{l(l-1)}{2}})}{(\frac{j(j+1)}{2} - \frac{l(l+1)}{2})e^{\frac{j(j+1)}{2} - \frac{l(l+1)}{2}}} \\ &\leq \frac{j + l - 1}{j + l + 1} \frac{2(e^{\frac{j(j-1)}{2} - \frac{l(l-1)}{2}})}{e^{\frac{j(j+1)}{2} - \frac{l(l+1)}{2}}} \\ &< 2e^{-(j-l)} \leq 2e^{-1} \leq 1. \end{aligned}$$

So, by Corollary 1, Γ has a unique fixed point. Here $\Gamma\kappa_1 = \kappa_1$.

Note that Γ *is not a Banach contraction. Since,*

$$\sup \frac{d(\Gamma\kappa_j, \Gamma\kappa_1)}{d(\kappa_j, \kappa_1)} = \sup \frac{\kappa_{j-1} - 1}{\kappa_j - 1}$$
$$= \sup \frac{\frac{j(j-1)}{2} - 1}{\frac{j(j+1)}{2} - 1} = 1.$$

4. Application to Integral Equations

Take $\mathcal{I} = [0, \mathcal{T}]$. Let $\Lambda = \mathcal{C}(\mathcal{I}, \mathbb{R})$ be the set of all real valued continuous functions with domain $\mathcal{I}$. Define:

$$d(\eta, \mu) = \sup_{t \in \mathcal{I}} (|\eta(t) - \mu(t)|) = ||\eta - \mu||.$$

Consider the integral equation:

$$\eta(t) = p(t) + \int_0^{\mathcal{T}} \mathcal{G}(t,s)\mathcal{K}(s, \eta(s))ds, \quad t \in [0, \mathcal{T}] \tag{11}$$

Assume that the following conditions hold:

(A) $p : \mathcal{I} \to \mathbb{R}$ and $\mathcal{K} : \mathcal{I} \times \mathbb{R} \to \mathbb{R}$ are continuous;
(B) $\mathcal{G} : \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ is continuous and measurable at $s \in \mathcal{I}$ for all $t \in \mathcal{I}$;
(C) $\mathcal{G}(t,s) \geq 0$ for all $t, s \in \mathcal{I}$ and $\int_0^{\mathcal{T}} \mathcal{G}(t,s)ds \leq 1$ for all $t \in \mathcal{I}$;
(D) For each $t \in \mathcal{I}$ and for all $\eta, \mu \in \Lambda$.

$$|\mathcal{K}(t, \eta(t)) - \mathcal{K}(t, \mu(t))| \leq \frac{-1 + \sqrt{1 + 4(\eta(t) - \mu(t))^2}}{2}$$

Theorem 5. *Under the assumptions (A)–(D), the integral Equation (7) has a solution in* Λ*.*

Proof. Define $\Upsilon : \Lambda \to \Lambda$ as:

$$\Upsilon\eta(t) = p(t) + \int_0^{\mathcal{T}} \mathcal{G}(t,s)\mathcal{K}(s, \eta(s))ds, \quad t \in [0, \mathcal{T}].$$

We have:

$$|\Upsilon\eta(t) - \Upsilon\mu(t)| = |\int_0^{\mathcal{T}} \mathcal{G}(t,s)(\mathcal{K}(s, \eta(s)) - \mathcal{K}(s, \mu(s))ds|$$
$$\leq \int_0^{\mathcal{T}} \mathcal{G}(t,s)|\mathcal{K}(s, \eta(s)) - \mathcal{K}(s, \mu(s))|ds$$
$$\leq \int_0^{\mathcal{T}} \mathcal{G}(t,s)(\frac{-1 + \sqrt{1 + 4(\eta(s) - \mu(s))^2}}{2})ds$$
$$\leq \int_0^{\mathcal{T}} \mathcal{G}(t,s)(\frac{-1 + \sqrt{1 + 4||\eta - \mu||^2}}{2})ds$$
$$= \frac{-1 + \sqrt{1 + 4||\eta - \mu||^2}}{2}$$
$$= \frac{-1 + \sqrt{1 + 4[d(\eta, \mu)]^2}}{2}$$

for every $t \in [0,1]$. Take sup to find that:

$$d(\Upsilon\eta, \Upsilon\mu) = ||\Upsilon\eta - \Upsilon\mu|| \leq \frac{-1 + \sqrt{1 + 4[d(\eta,\mu)]^2}}{2}.$$

From the above inequality, we obtain:

$$(1 + 2d(\Upsilon\eta, \Upsilon\mu))^2 \leq 1 + 4[d(\eta,\mu)]^2.$$

This is equivalent to:

$$d(\Upsilon\eta, \Upsilon\mu)) + [d(\Upsilon\eta, \Upsilon\mu)]^2 \leq [d(\eta,\mu)]^2.$$

Therefore,

$$d(\Upsilon\eta, \Upsilon\mu)) \leq [d(\eta,\mu)]^2 - [d(\Upsilon\eta, \Upsilon\mu)]^2.$$

Taking $\varrho(t) = t^2$, we get:

$$d(\Upsilon\eta, \Upsilon\mu) \leq \varrho(d(\eta,\mu)) - \varrho(d(\Upsilon\eta, \Upsilon\mu)),$$

for all $\eta, \mu \in \Lambda$, which is Equation (7). Therefore, by Theorem 4, Υ has a fixed point. Hence there is a solution for Equation (11). □

5. Conclusions

In this paper, we introduced a new generalization of the Banach contraction principle. The new contraction will be a powerful tool for the existence solution of integral equations, differential equations, and also the fractional integro-differential equations. We think that the multi-valued version of this new contraction can be considered by researchers. The new multi-valued contraction will be a powerful tool for the existence solution of Volterra-integral inclusions.

Author Contributions: H.I. analyzed and prepared/edited the manuscript, B.M. analyzed and prepared/edited the manuscript, V.P. analyzed and prepared the manuscript, M.R.H. analyzed and prepared the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare that they have no competing interests regarding the publication of this paper.

References

1. Banach, S. Sur les operations dans les ensembles abstraits et leur applications aux equations integrales. *Fundam. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Berinde, V.; Păcurar, M. An iterative method for approximating fixed points of Prešić nonexpansive mappings. *Rev. Anal. Numer. Theor. Approx.* **2009**, *38*, 144–153.
3. Asl, J.H.; Rezapour, S.; Shahzad, N. On fixed points of $\alpha - \psi$ contractive multifunctions. *Fixed Point Theory Appl.* **2012**, *2012*, 212. [CrossRef]
4. Işık, H. Solvability to coupled systems of functional equations via fixed point theory. *TWMS J. Appl. Eng. Math.* **2018**, *8*, 230–237. [CrossRef]
5. Abbas, M.; Illic, D.; Nazir, T. Iterative approximation of fixed points of generalized weak Presic type k-step iterative method for a class of operators. *Filomat* **2015**, *29*, 713–724. [CrossRef]
6. Chen, Y.Z.; A Prešić type contractive condition and its applications. *Nonlinar Anal.* **2009**, *71*, 2012–2017. [CrossRef]
7. Berinde, V. General constructive fixed point theorem for Ćirić-type almost contractions in metric spaces. *Carpath. J. Math.* **2008**, *24*, 10–19.

8. Işık, H.; Turkoglu, D. Some fixed point theorems in ordered partial metric spaces. *J. Inequalities Spec. Funct.* **2013**, *4*, 13–18.
9. Ahmad, J.; Al-Rawashdeh, A.; Azam, A. Fixed point results for $\{\alpha, \xi\}$-expansive locally contractive mappings. *J. Inequalities Appl.* **2014**, *2014*, 364. [CrossRef]
10. Berinde, V.; Păcurar, M. Two elementary applications of some Prešić type fixed point theorems. *Creat. Math. Inform.* **2011**, *20*, 32–42.
11. Lu, N.; He, F.; Huang, H. Answers to questions on the generalized Banach contraction conjecture in b-metric spaces. *J. Fixed Point Theory Appl.* **2019**, *21*, 43. [CrossRef]
12. Işık, H.; Turkoglu, D. Generalized weakly α-contractive mappings and applications to ordinary differential equations. *Miskolc Math. Notes* **2016**, *17*, 365–379. [CrossRef]
13. Choudhury, B.S.; Metiya, N.; Bandyopadhyay, C. Fixed points of multivalued α-admissible mappings and stability of fixed point sets in metric spaces. *Rend. Circ. Mat. Palermo* **2015**, *64*, 43–55. [CrossRef]
14. Ahmad, J.; Al-Rawashdeh, A.; Azam, A. New Fixed Point Theorems for Generalized F-Contractions in Complete Metric Spaces. *Fixed Point Theory Appl.* **2015**, *2015*, 80. [CrossRef]
15. Boyd, D.W.; Wong, J.S.W. On nonlinear contractions. *Proc. Am. Math. Soc.* **1969**, *20*, 458–464. [CrossRef]
16. Işık, H.; Ionescu, C. New type of multivalued contractions with related results and applications. *UPB Sci. Bull. Ser. A* **2018**, *80*, 13–22.
17. Abbas, M.; Berzig, M.; Nazir, T.; Karapinar, E. Iterative approximation of fixed points for Prešić type F-Contraction Operators. *UPB Sci. Bull. Ser. A* **2016**, *78*, 147–160.
18. Caristi, J. Fixed point theorems for mappings satisfying inwardness conditions. *Trans. Amer. Math. Soc.* **1976**, *215*, 241–251. [CrossRef]
19. Du, W.-S. A direct proof of Caristi's fixed point theorem. *Appl. Math. Sci.* **2016**, *46*, 2289–2294. [CrossRef]