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# New Inequalities of Weaving $K$-Frames in Subspaces

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**Abstract:** In the present paper, we obtain some new inequalities for weaving $K$-frames in subspaces based on the operator methods. The inequalities are associated with a sequence of bounded complex numbers and a parameter $\lambda \in \mathbb{R}$. We also give a double inequality for weaving $K$-frames with the help of two bounded linear operators induced by $K$-dual. Facts prove that our results cover those recently obtained on weaving frames due to Li and Leng, and Xiang.

**Keywords:** weaving frame; weaving $K$-frame; $K$-dual; pseudo-inverse

**MSC:** 42C15; 47B40

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## 1. Introduction

This paper adopts the following notations: $\mathbb{J}$ is a countable index set, $\mathbb{H}$ and $\mathbb{K}$ are complex Hilbert spaces, and $\mathrm{Id}_{\mathbb{H}}$ and $\mathbb{R}$ are used to denote respectively the identical operator on $\mathbb{H}$ and the set of real numbers. As usual, we denote by $B(\mathbb{H},\mathbb{K})$ the set of all bounded linear operators on $\mathbb{H}$ and, if $\mathbb{H}=\mathbb{K}$, then $B(\mathbb{H},\mathbb{K})$ is abbreviated to $B(\mathbb{H})$.

Frames were introduced by Duffin and Schaeffer [1] in their study of nonharmonic Fourier series, which have now been used widely not only in theoretical work [2,3], but also in many application areas such as quantum mechanics [4], sampling theory [5–7], acoustics [8], and signal processing [9]. As a generalization of frames, the notion of $K$-frames (also known as frames for operators) was proposed by L. Găvruţa [10] when dealing with atomic decompositions for a bounded linear operator $K$. Please check the papers [11–17] for further information of $K$-frames.

Recall that a family $\{\psi_j\}_{j\in\mathbb{J}} \subset \mathbb{H}$ is called a $K$-frame for $\mathbb{H}$, if there exist two positive numbers $A$ and $B$ satisfying

$$A\|K^* f\|^2 \le \sum_{j\in\mathbb{J}} |\langle f,\psi_j\rangle|^2 \le B\|f\|^2, \quad \forall f \in \mathbb{H}.$$

The constants $A$ and $B$ are called $K$-frame bounds. If $K=\mathrm{Id}_{\mathbb{H}}$, then a $K$-frame turns to be a frame. In addition, if only the right-hand inequality holds, then we call $\{\psi_j\}_{j\in\mathbb{J}}$ a Bessel sequence.

Inspired by a question arising in distributed signal processing, Bemrose et al. [18] introduced the concept of weaving frames, which have interested many scholars because of their potential applications such as in wireless sensor networks and pre-processing of signals; see [19–24]. Later on, Deepshikha and Vashisht [25] applied the idea of L. Găvruţa to the case of weaving frames and thus providing us the notion of weaving $K$-frames.

Balan et al. [26] obtained an interesting inequality when they further examined the remarkable identity for Parseval frames deriving from their work on signal reconstruction [27]. The inequality was then extended to alternate dual frames and general frames by P. Găvruţa [28], the results in which have already been applied in quantum information theory [29]. Recently, those inequalities have been extended to some generalized versions of frames such as continuous g-frames [30], fusion frames and continuous fusion frames [31,32], Hilbert–Schmidt frames [33], and weaving frames [34,35].

Motivated by the above-mentioned works, in this paper, we establish several new inequalities for weaving $K$-frames in subspaces from the operator-theoretic point of view, and we show that our results can naturally lead to some corresponding results in [34,35].

One says that two frames $\Psi_1 = \{\psi_{1j}\}_{j\in\mathbb{J}}$ and $\Psi_2 = \{\psi_{2j}\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are woven, if there are universal constants $C_\Psi$ and $D_\Psi$ such that, for any $\sigma \subset \mathbb{J}$, $\{\psi_{1j}\}_{j\in\sigma} \cup \{\psi_{2j}\}_{j\in\sigma^c}$ is a frame for $\mathbb{H}$ with bounds $C_\Psi$ and $D_\Psi$. If $C_\Psi = D_\Psi = 1$, then we call $\Psi_1$ and $\Psi_2$ 1-woven. Each family $\{\psi_{1j}\}_{j\in\sigma} \cup \{\psi_{2j}\}_{j\in\sigma^c}$ is said to be a waving frame, related to which there is an invertible operator $S_{\Psi_1\Psi_2} : \mathbb{H} \to \mathbb{H}$, called the frame operator, given by

$$S_{\Psi_1\Psi_2} f = \sum_{j\in\sigma} \langle f, \psi_{1j}\rangle \psi_{1j} + \sum_{j\in\sigma^c} \langle f, \psi_{2j}\rangle \psi_{2j}.$$

Recall also that a frame $\Psi_3 = \{\psi_{3j}\}_{j\in\mathbb{J}}$ is called an alternate dual frame of $\{\psi_{1j}\}_{j\in\sigma} \cup \{\psi_{2j}\}_{j\in\sigma^c}$, if for each $f \in \mathbb{H}$ we have

$$f = \sum_{j\in\sigma} \langle f, \psi_{1j}\rangle \psi_{3j} + \sum_{j\in\sigma^c} \langle f, \psi_{2j}\rangle \psi_{3j}, \quad \forall f \in \mathbb{H}.$$

**Lemma 1.** *Suppose that $P$, $Q$, and $K$ are bounded linear operators on $\mathbb{H}$ and $P + Q = K$. Then, for each $f \in \mathbb{H}$,*

$$\|Pf\|^2 + \mathrm{Re}\langle Qf, Kf\rangle \geq \frac{3}{4}\|Kf\|^2.$$

**Proof.** We have

$$\begin{aligned}
\|Pf\|^2 + \mathrm{Re}\langle Qf, Kf\rangle &= \langle (K-Q)f, (K-Q)f\rangle + \frac{1}{2}(\langle Qf, Kf\rangle + \langle Kf, Qf\rangle) \\
&= \langle (Q^*Q - (K^*Q + Q^*K) + \frac{1}{2}(K^*Q + Q^*K))f, f\rangle + \langle K^*Kf, f\rangle \\
&= \langle (Q - \frac{1}{2}K)^*(Q - \frac{1}{2}K)f, f\rangle + \frac{3}{4}\langle K^*Kf, f\rangle \geq \frac{3}{4}\|Kf\|^2
\end{aligned}$$

for any $f \in \mathbb{H}$. □

The next two lemmas are collected from the papers [36] and [32], respectively.

**Lemma 2.** *If $\Phi \in B(\mathbb{H}, \mathbb{K})$ has a closed range, then there is the pseudo-inverse $\Phi^\dagger \in B(\mathbb{K}, \mathbb{H})$ of $\Phi$ such that*

$$\Phi\Phi^\dagger\Phi = \Phi, \ \Phi^\dagger\Phi\Phi^\dagger = \Phi^\dagger, \ (\Phi\Phi^\dagger)^* = \Phi\Phi^\dagger, \ (\Phi^\dagger\Phi)^* = \Phi^\dagger\Phi.$$

**Lemma 3.** *If $P$ and $Q$ in $B(\mathbb{H})$ satisfy $P + Q = \mathrm{Id}_{\mathbb{H}}$, then, for any $\lambda \in \mathbb{R}$, we have*

$$P^*P + \lambda(Q^* + Q) = Q^*Q + (1-\lambda)(P^* + P) + (2\lambda - 1)\mathrm{Id}_{\mathbb{H}} \geq (2\lambda - \lambda^2)\mathrm{Id}_{\mathbb{H}}.$$

## 2. Main Results

We start with the definition on weaving $K$-frames due to Deepshikha and Vashisht [25].

**Definition 1.** *Two $K$-frames $\Psi_1 = \{\psi_{1j}\}_{j\in\mathbb{J}}$ and $\Psi_2 = \{\psi_{2j}\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are said to be $K$-woven, if there are universal constants $C_\Psi$ and $D_\Psi$ such that, for any $\sigma \subset \mathbb{J}$, the family $\{\psi_{1j}\}_{j\in\sigma} \cup \{\psi_{2j}\}_{j\in\sigma^c}$ is a $K$-frame for $\mathbb{H}$ with $K$-frame bounds $C_\Psi$ and $D_\Psi$. In this case, the family $\{\psi_{1j}\}_{j\in\sigma} \cup \{\psi_{2j}\}_{j\in\sigma^c}$ is called a weaving $K$-frame.*

Given a weaving $K$-frame $\{\psi_{1j}\}_{j\in\sigma} \cup \{\psi_{2j}\}_{j\in\sigma^c}$ for $\mathbb{H}$, recall that a Bessel sequence $\Phi = \{\phi_j\}_{j\in\mathbb{J}}$ for $\mathbb{H}$ is said to be a $K$-dual of $\{\psi_{1j}\}_{j\in\sigma} \cup \{\psi_{2j}\}_{j\in\sigma^c}$, if

$$Kf = \sum_{j\in\sigma} \langle f, \psi_{1j}\rangle \phi_j + \sum_{j\in\sigma^c} \langle f, \psi_{2j}\rangle \phi_j, \quad \forall f \in \mathbb{H}.$$

Let $\Psi_1 = \{\psi_{1j}\}_{j\in\mathbb{J}}$ be a given $K$-frame for $\mathbb{H}$. For any $\sigma \subset \mathbb{J}$, we can define a positive operator $S^{\sigma}_{\Psi_1}$ in the following way:

$$S^{\sigma}_{\Psi_1} : \mathbb{H} \to \mathbb{H}, \quad S^{\sigma}_{\Psi_1} f = \sum_{j\in\sigma} \langle f, \psi_{1j}\rangle \psi_{1j}.$$

In the following, we show that, for given two $K$-woven frames, we can get some inequalities under the condition that $K$ has a closed range, which are related to a sequence of bounded complex numbers, the corresponding $K$-dual and a parameter $\lambda \in \mathbb{R}$.

**Theorem 1.** *Suppose that $K \in B(\mathbb{H})$ has a closed range and $K$-frames $\Psi_1 = \{\psi_{1j}\}_{j\in\mathbb{J}}$ and $\Psi_2 = \{\psi_{2j}\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are $K$-woven. Then,*

*(i) for any $f \in Range(K)$, for all $\sigma \subset \mathbb{J}$, $\{a_j\}_{j\in\mathbb{J}} \in \ell^\infty(\mathbb{J})$, and $\lambda \in \mathbb{R}$,*

$$\begin{aligned}
&\Big\| \sum_{j\in\sigma} a_j \langle K^\dagger f, \psi_{1j}\rangle \phi_j + \sum_{j\in\sigma^c} a_j \langle K^\dagger f, \psi_{2j}\rangle \phi_j \Big\|^2 \\
&\qquad + \mathrm{Re}\Big( \sum_{j\in\sigma} (1-a_j) \langle K^\dagger f, \psi_{1j}\rangle \langle \phi_j, f\rangle + \sum_{j\in\sigma^c} (1-a_j) \langle K^\dagger f, \psi_{2j}\rangle \langle \phi_j, f\rangle \Big) \\
&= \Big\| \sum_{j\in\sigma} (1-a_j) \langle K^\dagger f, \psi_{1j}\rangle \phi_j + \sum_{j\in\sigma^c} (1-a_j) \langle K^\dagger f, \psi_{2j}\rangle \phi_j \Big\|^2 \\
&\qquad + \mathrm{Re}\Big( \sum_{j\in\sigma} a_j \langle K^\dagger f, \psi_{1j}\rangle \langle \phi_j, f\rangle + \sum_{j\in\sigma^c} a_j \langle K^\dagger f, \psi_{2j}\rangle \langle \phi_j, f\rangle \Big) \\
&\geq (\lambda - \frac{\lambda^2}{4}) \mathrm{Re}\Big( \sum_{j\in\sigma} a_j \langle K^\dagger f, \psi_{1j}\rangle \langle \phi_j, f\rangle + \sum_{j\in\sigma^c} a_j \langle K^\dagger f, \psi_{2j}\rangle \langle \phi_j, f\rangle \Big) \\
&\qquad + (1 - \frac{\lambda^2}{4}) \mathrm{Re}\Big( \sum_{j\in\sigma} (1-a_j) \langle K^\dagger f, \psi_{1j}\rangle \langle \phi_j, f\rangle + \sum_{j\in\sigma^c} (1-a_j) \langle K^\dagger f, \psi_{2j}\rangle \langle \phi_j, f\rangle \Big),
\end{aligned} \tag{1}$$

*where $\Phi = \{\phi_j\}_{j\in\mathbb{J}}$ is a $K$-dual of $\{\psi_{1j}\}_{j\in\sigma} \cup \{\psi_{2j}\}_{j\in\sigma^c}$.*

*(ii) for any $f \in Range(K^*)$, for all $\sigma \subset \mathbb{J}$, $\{a_j\}_{j\in\mathbb{J}} \in \ell^\infty(\mathbb{J})$, and $\lambda \in \mathbb{R}$,*

$$\begin{aligned}
&\Big\| \sum_{j\in\sigma} a_j \langle (K^*)^\dagger f, \phi_j\rangle \psi_{1j} + \sum_{j\in\sigma^c} a_j \langle (K^*)^\dagger f, \phi_j\rangle \psi_{2j} \Big\|^2 \\
&\qquad + \mathrm{Re}\Big( \sum_{j\in\sigma} (1-a_j) \langle (K^*)^\dagger f, \phi_j\rangle \langle \psi_{1j}, f\rangle + \sum_{j\in\sigma^c} (1-a_j) \langle (K^*)^\dagger f, \phi_j\rangle \langle \psi_{2j}, f\rangle \Big) \\
&= \Big\| \sum_{j\in\sigma} (1-a_j) \langle (K^*)^\dagger f, \phi_j\rangle \psi_{1j} + \sum_{j\in\sigma^c} (1-a_j) \langle (K^*)^\dagger f, \phi_j\rangle \psi_{2j} \Big\|^2 \\
&\qquad + \mathrm{Re}\Big( \sum_{j\in\sigma} a_j \langle (K^*)^\dagger f, \phi_j\rangle \langle \psi_{1j}, f\rangle + \sum_{j\in\sigma^c} a_j \langle (K^*)^\dagger f, \phi_j\rangle \langle \psi_{2j}, f\rangle \Big) \\
&\geq (2\lambda - \lambda^2) \mathrm{Re}\Big( \sum_{j\in\sigma} a_j \langle (K^*)^\dagger f, \phi_j\rangle \langle \psi_{1j}, f\rangle + \sum_{j\in\sigma^c} a_j \langle (K^*)^\dagger f, \phi_j\rangle \langle \psi_{2j}, f\rangle \Big) \\
&\qquad + (1 - \lambda^2) \mathrm{Re}\Big( \sum_{j\in\sigma} (1-a_j) \langle (K^*)^\dagger f, \phi_j\rangle \langle \psi_{1j}, f\rangle + \sum_{j\in\sigma^c} (1-a_j) \langle (K^*)^\dagger f, \phi_j\rangle \langle \psi_{2j}, f\rangle \Big),
\end{aligned}$$

*where $\Phi = \{\phi_j\}_{j\in\mathbb{J}}$ is a $K$-dual of $\{\psi_{1j}\}_{j\in\sigma} \cup \{\psi_{2j}\}_{j\in\sigma^c}$.*

**Proof.** We define two bounded linear operators $P_1$ and $P_2$ on $\mathbb{H}$ as follows:

$$\begin{aligned} P_1 f &= \sum_{j\in\sigma} a_j \langle f, \psi_{1j}\rangle \phi_j + \sum_{j\in\sigma^c} a_j \langle f, \psi_{2j}\rangle \phi_j, \\ P_2 f &= \sum_{j\in\sigma} (1-a_j) \langle f, \psi_{1j}\rangle \phi_j + \sum_{j\in\sigma^c} (1-a_j) \langle f, \psi_{2j}\rangle \phi_j. \end{aligned} \tag{2}$$

Then, clearly, $P_1 f + P_2 f = Kf$ for each $f \in \mathbb{H}$ and thus $P_1 + P_2 = K$. Since $K$ has a closed range, by Lemma 2, we have

$$P_1 K^\dagger + P_2 K^\dagger = KK^\dagger = P_{Range(K)},$$

where $P_{Range(K)}$ is the orthogonal projection onto $Range(K)$. Thus,

$$P_1 K^\dagger \mid_{Range(K)} + P_2 K^\dagger \mid_{Range(K)} = \mathrm{Id}_{Range(K)}.$$

By Lemma 3 (taking $\frac{\lambda}{2}$ instead of $\lambda$), we get

$$\|P_1 K^\dagger f\|^2 + \lambda \mathrm{Re}\langle P_2 K^\dagger f, f\rangle = \|P_2 K^\dagger f\|^2 + (2-\lambda)\mathrm{Re}\langle P_1 K^\dagger f, f\rangle + (\lambda - 1)\|f\|^2,$$

for any $f \in Range(K)$. Hence,

$$\begin{aligned} \|P_1 K^\dagger f\|^2 &= \|P_2 K^\dagger f\|^2 + 2\mathrm{Re}\langle P_1 K^\dagger f, f\rangle - \lambda(\mathrm{Re}\langle P_1 K^\dagger f, f\rangle + \mathrm{Re}\langle P_2 K^\dagger f, f\rangle) + (\lambda-1)\|f\|^2 \\ &= \|P_2 K^\dagger f\|^2 + 2\mathrm{Re}\langle P_1 K^\dagger f, f\rangle - \lambda\|f\|^2 + (\lambda-1)\|f\|^2 \\ &= \|P_2 K^\dagger f\|^2 + 2\mathrm{Re}\langle P_1 K^\dagger f, f\rangle - \mathrm{Re}\langle P_1 K^\dagger f, f\rangle - \mathrm{Re}\langle P_2 K^\dagger f, f\rangle. \end{aligned}$$

It follows that

$$\|P_1 K^\dagger f\|^2 + \mathrm{Re}\langle P_2 K^\dagger f, f\rangle = \|P_2 K^\dagger f\|^2 + \mathrm{Re}\langle P_1 K^\dagger f, f\rangle, \tag{3}$$

from which we arrive at

$$\begin{aligned} &\Big\|\sum_{j\in\sigma} a_j \langle K^\dagger f, \psi_{1j}\rangle \phi_j + \sum_{j\in\sigma^c} a_j \langle K^\dagger f, \psi_{2j}\rangle \phi_j\Big\|^2 \\ &\qquad + \mathrm{Re}\Big(\sum_{j\in\sigma}(1-a_j)\langle K^\dagger f, \psi_{1j}\rangle\langle \phi_j, f\rangle + \sum_{j\in\sigma^c}(1-a_j)\langle K^\dagger f, \psi_{2j}\rangle\langle \phi_j, f\rangle\Big) \\ &= \Big\|\sum_{j\in\sigma}(1-a_j)\langle K^\dagger f, \psi_{1j}\rangle \phi_j + \sum_{j\in\sigma^c}(1-a_j)\langle K^\dagger f, \psi_{2j}\rangle \phi_j\Big\|^2 \\ &\qquad + \mathrm{Re}\Big(\sum_{j\in\sigma} a_j\langle K^\dagger f, \psi_{1j}\rangle\langle \phi_j, f\rangle + \sum_{j\in\sigma^c} a_j\langle K^\dagger f, \psi_{2j}\rangle\langle \phi_j, f\rangle\Big). \end{aligned}$$

For the inequality in Equation (1), we apply Lemma 3 again,

$$\begin{aligned} \|P_1 K^\dagger f\|^2 &\geq (\lambda - \frac{\lambda^2}{4})\|f\|^2 - \lambda\mathrm{Re}\langle P_2 K^\dagger f, f\rangle \\ &= (\lambda - \frac{\lambda^2}{4})\mathrm{Re}\langle P_1 K^\dagger f + P_2 K^\dagger f, f\rangle - \lambda\mathrm{Re}\langle P_2 K^\dagger f, f\rangle \\ &= (\lambda - \frac{\lambda^2}{4})\mathrm{Re}\langle P_1 K^\dagger f, f\rangle - \frac{\lambda^2}{4}\mathrm{Re}\langle P_2 K^\dagger f, f\rangle. \end{aligned} \tag{4}$$

Thus, for any $f \in Range(K)$,

$$\begin{aligned}
&\left\|\sum_{j\in\sigma} a_j\langle K^\dagger f,\psi_{1j}\rangle\phi_j+\sum_{j\in\sigma^c} a_j\langle K^\dagger f,\psi_{2j}\rangle\phi_j\right\|^2\\
&\qquad+\mathrm{Re}\Big(\sum_{j\in\sigma}(1-a_j)\langle K^\dagger f,\psi_{1j}\rangle\langle\phi_j,f\rangle+\sum_{j\in\sigma^c}(1-a_j)\langle K^\dagger f,\psi_{2j}\rangle\langle\phi_j,f\rangle\Big)\\
&\ge(\lambda-\frac{\lambda^2}{4})\mathrm{Re}\langle P_1K^\dagger f,f\rangle+(1-\frac{\lambda^2}{4})\mathrm{Re}\langle P_2K^\dagger f,f\rangle\\
&=(\lambda-\frac{\lambda^2}{4})\mathrm{Re}\Big(\sum_{j\in\sigma} a_j\langle K^\dagger f,\psi_{1j}\rangle\langle\phi_j,f\rangle+\sum_{j\in\sigma^c} a_j\langle K^\dagger f,\psi_{2j}\rangle\langle\phi_j,f\rangle\Big)\\
&\qquad+(1-\frac{\lambda^2}{4})\mathrm{Re}\Big(\sum_{j\in\sigma}(1-a_j)\langle K^\dagger f,\psi_{1j}\rangle\langle\phi_j,f\rangle+\sum_{j\in\sigma^c}(1-a_j)\langle K^\dagger f,\psi_{2j}\rangle\langle\phi_j,f\rangle\Big).
\end{aligned}$$

(ii) The proof is similar to (i), so we omit the details. □

**Corollary 1.** *Suppose that two frames $\Psi_1=\{\psi_{1j}\}_{j\in\mathbb{J}}$ and $\Psi_2=\{\psi_{2j}\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are woven. Then, for any $f\in\mathbb{H}$, for all $\sigma\subset\mathbb{J}$ and all $\lambda\in\mathbb{R}$, we have*

$$\begin{aligned}
&\sum_{j\in\sigma^c}|\langle f,\psi_{2j}\rangle|^2+\sum_{j\in\sigma}|\langle S^{\sigma}_{\Psi_1}f,S^{-1}_{\Psi_1\Psi_2}\psi_{1j}\rangle|^2+\sum_{j\in\sigma^c}|\langle S^{\sigma}_{\Psi_1}f,S^{-1}_{\Psi_1\Psi_2}\psi_{2j}\rangle|^2\\
&\quad=\sum_{j\in\sigma}|\langle f,\psi_{1j}\rangle|^2+\sum_{j\in\sigma}|\langle S^{\sigma^c}_{\Psi_2}f,S^{-1}_{\Psi_1\Psi_2}\psi_{1j}\rangle|^2+\sum_{j\in\sigma^c}|\langle S^{\sigma^c}_{\Psi_2}f,S^{-1}_{\Psi_1\Psi_2}\psi_{2j}\rangle|^2\\
&\quad\ge(\lambda-\frac{\lambda^2}{4})\sum_{j\in\sigma}|\langle f,\psi_{1j}\rangle|^2+(1-\frac{\lambda^2}{4})\sum_{j\in\sigma^c}|\langle f,\psi_{2j}\rangle|^2.
\end{aligned}$$

**Proof.** Letting $K^\dagger=\mathrm{Id}_{\mathbb{H}}$ and

$$\phi_j=\begin{cases} S^{-1/2}_{\Psi_1\Psi_2}\psi_{1j}, & j\in\sigma,\\ S^{-1/2}_{\Psi_1\Psi_2}\psi_{2j}, & j\in\sigma^c.\end{cases}$$

In addition, taking $S^{-1/2}_{\Psi_1\Psi_2}\psi_{1j}$, $S^{-1/2}_{\Psi_1\Psi_2}\psi_{2j}$ and $S^{1/2}_{\Psi_1\Psi_2}f$ instead of $\psi_{1j}$, $\psi_{2j}$ and $f$ respectively in (i) of Theorem 1 leads to

$$\begin{aligned}
&\left\|\sum_{j\in\sigma} a_j\langle f,\psi_{1j}\rangle S^{-1/2}_{\Psi_1\Psi_2}\psi_{1j}+\sum_{j\in\sigma^c} a_j\langle f,\psi_{2j}\rangle S^{-1/2}_{\Psi_1\Psi_2}\psi_{2j}\right\|^2\\
&\qquad+\mathrm{Re}\Big(\sum_{j\in\sigma}(1-a_j)\langle f,\psi_{1j}\rangle\langle\psi_{1j},f\rangle+\sum_{j\in\sigma^c}(1-a_j)\langle f,\psi_{2j}\rangle\langle\psi_{2j},f\rangle\Big)\\
&=\left\|\sum_{j\in\sigma}(1-a_j)\langle f,\psi_{1j}\rangle S^{-1/2}_{\Psi_1\Psi_2}\psi_{1j}+\sum_{j\in\sigma^c}(1-a_j)\langle f,\psi_{2j}\rangle S^{-1/2}_{\Psi_1\Psi_2}\psi_{2j}\right\|^2\\
&\qquad+\mathrm{Re}\Big(\sum_{j\in\sigma} a_j\langle f,\psi_{1j}\rangle\langle\psi_{1j},f\rangle+\sum_{j\in\sigma^c} a_j\langle f,\psi_{2j}\rangle\langle\psi_{2j},f\rangle\Big)\\
&\ge(\lambda-\frac{\lambda^2}{4})\mathrm{Re}\Big(\sum_{j\in\sigma} a_j\langle f,\psi_{1j}\rangle\langle\psi_{1j},f\rangle+\sum_{j\in\sigma^c} a_j\langle f,\psi_{2j}\rangle\langle\psi_{2j},f\rangle\Big)\\
&\qquad+(1-\frac{\lambda^2}{4})\mathrm{Re}\Big(\sum_{j\in\sigma}(1-a_j)\langle f,\psi_{1j}\rangle\langle\psi_{1j},f\rangle+\sum_{j\in\sigma^c}(1-a_j)\langle f,\psi_{2j}\rangle\langle\psi_{2j},f\rangle\Big). \qquad (5)
\end{aligned}$$

A direction calculation shows that

$$
\begin{aligned}
\Big\|\sum_{j\in\sigma}\langle f,\psi_{1j}\rangle S_{\Psi_1\Psi_2}^{-1/2}\psi_{1j}\Big\|^2 &= \Big\|S_{\Psi_1\Psi_2}^{-1/2}\sum_{j\in\sigma}\langle f,\psi_{1j}\rangle\psi_{1j}\Big\|^2 = \|S_{\Psi_1\Psi_2}^{-1/2}S_{\Psi_1}^{\sigma}f\|^2 \\
&= \langle S_{\Psi_1\Psi_2}^{-1/2}S_{\Psi_1}^{\sigma}f, S_{\Psi_1\Psi_2}^{-1/2}S_{\Psi_1}^{\sigma}f\rangle = \langle S_{\Psi_1\Psi_2}S_{\Psi_1\Psi_2}^{-1}S_{\Psi_1}^{\sigma}f, S_{\Psi_1\Psi_2}^{-1}S_{\Psi_1}^{\sigma}f\rangle \\
&= \sum_{j\in\sigma}\langle S_{\Psi_1\Psi_2}^{-1}S_{\Psi_1}^{\sigma}f,\psi_{1j}\rangle\langle\psi_{1j}, S_{\Psi_1\Psi_2}^{-1}S_{\Psi_1}^{\sigma}f\rangle + \sum_{j\in\sigma^c}\langle S_{\Psi_1\Psi_2}^{-1}S_{\Psi_1}^{\sigma}f,\psi_{2j}\rangle\langle\psi_{2j}, S_{\Psi_1\Psi_2}^{-1}S_{\Psi_1}^{\sigma}f\rangle \\
&= \sum_{j\in\sigma}|\langle S_{\Psi_1}^{\sigma}f, S_{\Psi_1\Psi_2}^{-1}\psi_{1j}\rangle|^2 + \sum_{j\in\sigma^c}|\langle S_{\Psi_1}^{\sigma}f, S_{\Psi_1\Psi_2}^{-1}\psi_{2j}\rangle|^2, \qquad (6)
\end{aligned}
$$

and, similarly,

$$
\Big\|\sum_{j\in\sigma^c}\langle f,\psi_{2j}\rangle S_{\Psi_1\Psi_2}^{-1/2}\psi_{2j}\Big\|^2 = \sum_{j\in\sigma}|\langle S_{\Psi_2}^{\sigma^c}f, S_{\Psi_1\Psi_2}^{-1}\psi_{1j}\rangle|^2 + \sum_{j\in\sigma^c}|\langle S_{\Psi_2}^{\sigma^c}f, S_{\Psi_1\Psi_2}^{-1}\psi_{2j}\rangle|^2. \qquad (7)
$$

Thus, the result follows if, in Equation (5), we take $a_j = \begin{cases} 1, & j\in\sigma, \\ 0, & j\in\sigma^c. \end{cases}$ □

**Corollary 2.** *Suppose that two frames $\Psi_1 = \{\psi_{1j}\}_{j\in\mathbb{J}}$ and $\Psi_2 = \{\psi_{2j}\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are woven. Then, for any $\sigma\subset\mathbb{J}$, for all $\lambda\in\mathbb{R}$ and all $f\in\mathbb{H}$, we have*

$$
\begin{aligned}
&\Big\|\sum_{j\in\sigma}\langle f,\phi_j\rangle\psi_{1j}\Big\|^2 + \mathrm{Re}\sum_{j\in\sigma^c}\langle f,\phi_j\rangle\langle\psi_{2j},f\rangle \\
&\quad= \Big\|\sum_{j\in\sigma^c}\langle f,\phi_j\rangle\psi_{2j}\Big\|^2 + \mathrm{Re}\sum_{j\in\sigma}\langle f,\phi_j\rangle\langle\psi_{1j},f\rangle \\
&\quad\geq (2\lambda-\lambda^2)\mathrm{Re}\sum_{j\in\sigma}\langle f,\phi_j\rangle\langle\psi_{1j},f\rangle + (1-\lambda^2)\mathrm{Re}\sum_{j\in\sigma^c}\langle f,\phi_j\rangle\langle\psi_{2j},f\rangle,
\end{aligned}
$$

*where $\Phi = \{\phi_j\}_{j\in\mathbb{J}}$ is an alternate dual of $\{\psi_{1j}\}_{j\in\sigma}\cup\{\psi_{2j}\}_{j\in\sigma^c}$.*

**Proof.** The result follows immediately from (ii) in Theorem 1 when taking $K^{\dagger} = \mathrm{Id}_{\mathbb{H}}$ and

$$
a_j = \begin{cases} 1, & j\in\sigma, \\ 0, & j\in\sigma^c. \end{cases}
$$

□

Suppose that two frames $\Psi_1 = \{\psi_{1j}\}_{j\in\mathbb{J}}$ and $\Psi_2 = \{\psi_{2j}\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are 1-woven. For any $\sigma\subset\mathbb{J}$ and any $j\in\mathbb{J}$, taking $\phi_j = \begin{cases} \psi_{1j}, & j\in\sigma, \\ \psi_{2j}, & j\in\sigma^c. \end{cases}$ Then, obviously, $\Phi = \{\phi_j\}_{j\in\mathbb{J}}$ is an alternate dual of the frame $\{\psi_{1j}\}_{j\in\sigma}\cup\{\psi_{2j}\}_{j\in\sigma^c}$. Thus, Corollary 2 provides us a direct consequence as follows.

**Corollary 3.** *Let the two frames $\Psi_1 = \{\psi_{1j}\}_{j\in\mathbb{J}}$ and $\Psi_2 = \{\psi_{2j}\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ be 1-woven. Then, for any $\sigma\subset\mathbb{J}$, for all $\lambda\in\mathbb{R}$ and all $f\in\mathbb{H}$, we have*

$$
\begin{aligned}
\Big\|\sum_{j\in\sigma}\langle f,\psi_{1j}\rangle\psi_{1j}\Big\|^2 + \sum_{j\in\sigma^c}|\langle f,\psi_{2j}\rangle|^2 &= \Big\|\sum_{j\in\sigma^c}\langle f,\psi_{2j}\rangle\psi_{2j}\Big\|^2 + \sum_{j\in\sigma}|\langle f,\psi_{1j}\rangle|^2 \\
&\geq (2\lambda-\lambda^2)\sum_{j\in\sigma}|\langle f,\psi_{1j}\rangle|^2 + (1-\lambda^2)\sum_{j\in\sigma^c}|\langle f,\psi_{2j}\rangle|^2.
\end{aligned}
$$

**Remark 1.** *Corollaries 1 and 2 are respectively Theorems 7 and 9 in [34], and Theorem 5 in [34] can be obtained if we put $\lambda = \frac{1}{2}$ in Corollary 3.*

**Theorem 2.** *Suppose that $K \in B(\mathbb{H})$ has a closed range and that K-frames $\Psi_1 = \{\psi_{1j}\}_{j\in\mathbb{J}}$ and $\Psi_2 = \{\psi_{2j}\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are K-woven. Then, for any $f \in Range(K)$, for all $\sigma \subset \mathbb{J}$, $\{a_j\}_{j\in\mathbb{J}} \in \ell^\infty(\mathbb{J})$, and $\lambda \in \mathbb{R}$,*

$$\begin{aligned}&\Big\|\sum_{j\in\sigma} a_j\langle K^\dagger f,\psi_{1j}\rangle\phi_j + \sum_{j\in\sigma^c} a_j\langle K^\dagger f,\psi_{2j}\rangle\phi_j\Big\|^2 + \Big\|\sum_{j\in\sigma}(1-a_j)\langle K^\dagger f,\psi_{1j}\rangle\phi_j + \sum_{j\in\sigma^c}(1-a_j)\langle K^\dagger f,\psi_{2j}\rangle\phi_j\Big\|^2\\ &\geq (2\lambda-\frac{\lambda^2}{2}-1)\mathrm{Re}\Big(\sum_{j\in\sigma} a_j\langle K^\dagger f,\psi_{1j}\rangle\langle\phi_j,f\rangle + \sum_{j\in\sigma^c} a_j\langle K^\dagger f,\psi_{2j}\rangle\langle\phi_j,f\rangle\Big)\\ &\quad + (1-\frac{\lambda^2}{2})\mathrm{Re}\Big(\sum_{j\in\sigma}(1-a_j)\langle K^\dagger f,\psi_{1j}\rangle\langle\phi_j,f\rangle + \sum_{j\in\sigma^c}(1-a_j)\langle K^\dagger f,\psi_{2j}\rangle\langle\phi_j,f\rangle\Big),\end{aligned}$$

*where $\Phi = \{\phi_j\}_{j\in\mathbb{J}}$ is a K-dual of $\{\psi_{1j}\}_{j\in\sigma} \cup \{\psi_{2j}\}_{j\in\sigma^c}$.*

*Moreover, if $(P_1K^\dagger)^*P_2K^\dagger$ is a positive operator, then*

$$\begin{aligned}&\Big\|\sum_{j\in\sigma} a_j\langle K^\dagger f,\psi_{1j}\rangle\phi_j + \sum_{j\in\sigma^c} a_j\langle K^\dagger f,\psi_{2j}\rangle\phi_j\Big\|^2\\ &\quad + \Big\|\sum_{j\in\sigma}(1-a_j)\langle K^\dagger f,\psi_{1j}\rangle\phi_j + \sum_{j\in\sigma^c}(1-a_j)\langle K^\dagger f,\psi_{2j}\rangle\phi_j\Big\|^2 \leq \|f\|^2\end{aligned}$$

*for any $f \in Range(K)$, where $P_1$ and $P_2$ are given in Equation (2).*

**Proof.** For any $f \in Range(K)$, for all $\sigma \subset \mathbb{J}$, $\{a_j\}_{j\in\mathbb{J}} \in \ell^\infty(\mathbb{J})$, and $\lambda \in \mathbb{R}$, we know, by combining Equation (3) and Lemma 3, that

$$\begin{aligned}&\Big\|\sum_{j\in\sigma} a_j\langle K^\dagger f,\psi_{1j}\rangle\phi_j + \sum_{j\in\sigma^c} a_j\langle K^\dagger f,\psi_{2j}\rangle\phi_j\Big\|^2 + \Big\|\sum_{j\in\sigma}(1-a_j)\langle K^\dagger f,\psi_{1j}\rangle\phi_j + \sum_{j\in\sigma^c}(1-a_j)\langle K^\dagger f,\psi_{2j}\rangle\phi_j\Big\|^2\\ &= \|P_1K^\dagger f\|^2 + \|P_2K^\dagger f\|^2 = 2\|P_2K^\dagger f\|^2 + \mathrm{Re}\langle P_1K^\dagger f,f\rangle - \mathrm{Re}\langle P_2K^\dagger f,f\rangle\\ &\geq (2-\frac{\lambda^2}{2})\|f\|^2 - (4-2\lambda)\mathrm{Re}\langle P_1K^\dagger f,f\rangle + \mathrm{Re}\langle P_1K^\dagger f,f\rangle - \mathrm{Re}\langle P_2K^\dagger f,f\rangle\\ &= (2\lambda-\frac{\lambda^2}{2}-1)\mathrm{Re}\langle P_1K^\dagger f,f\rangle + (1-\frac{\lambda^2}{2})\mathrm{Re}\langle P_2K^\dagger f,f\rangle\\ &= (2\lambda-\frac{\lambda^2}{2}-1)\mathrm{Re}\Big(\sum_{j\in\sigma} a_j\langle K^\dagger f,\psi_{1j}\rangle\langle\phi_j,f\rangle + \sum_{j\in\sigma^c} a_j\langle K^\dagger f,\psi_{2j}\rangle\langle\phi_j,f\rangle\Big)\\ &\quad + (1-\frac{\lambda^2}{2})\mathrm{Re}\Big(\sum_{j\in\sigma}(1-a_j)\langle K^\dagger f,\psi_{1j}\rangle\langle\phi_j,f\rangle + \sum_{j\in\sigma^c}(1-a_j)\langle K^\dagger f,\psi_{2j}\rangle\langle\phi_j,f\rangle\Big).\end{aligned}$$

For the "Moreover" part, we have for any $f \in Range(K)$ that

$$\begin{aligned}\|P_1K^\dagger f\|^2 &= \|P_2K^\dagger f\|^2 - \mathrm{Re}\langle P_2K^\dagger f,f\rangle + \mathrm{Re}\langle P_1K^\dagger f,f\rangle\\ &= \mathrm{Re}\langle P_2K^\dagger f,P_2K^\dagger f\rangle - \mathrm{Re}\langle P_2K^\dagger f,f\rangle + \mathrm{Re}\langle P_1K^\dagger f,f\rangle\\ &= -(\mathrm{Re}\langle P_2K^\dagger f,P_1K^\dagger f + P_2K^\dagger f\rangle - \mathrm{Re}\langle P_2K^\dagger f,P_2K^\dagger f\rangle) + \mathrm{Re}\langle P_1K^\dagger f,f\rangle\\ &= -\mathrm{Re}\langle P_2K^\dagger f,P_1K^\dagger f\rangle + \mathrm{Re}\langle P_1K^\dagger f,f\rangle \leq \mathrm{Re}\langle P_1K^\dagger f,f\rangle.\end{aligned}$$

With a similar discussion, we can show that $\|P_2K^{\dagger}f\|^2 \leq \mathrm{Re}\langle P_2K^{\dagger}f, f\rangle$. Thus,

$$\begin{aligned}&\Big\|\sum_{j\in\sigma}a_j\langle K^{\dagger}f,\psi_{1j}\rangle\phi_j+\sum_{j\in\sigma^c}a_j\langle K^{\dagger}f,\psi_{2j}\rangle\phi_j\Big\|^2+\Big\|\sum_{j\in\sigma}(1-a_j)\langle K^{\dagger}f,\psi_{1j}\rangle\phi_j+\sum_{j\in\sigma^c}(1-a_j)\langle K^{\dagger}f,\psi_{2j}\rangle\phi_j\Big\|^2\\&\quad\leq \mathrm{Re}\langle P_1K^{\dagger}f,f\rangle+\mathrm{Re}\langle P_2K^{\dagger}f,f\rangle=\mathrm{Re}\langle P_1K^{\dagger}f+P_2K^{\dagger}f,f\rangle=\|f\|^2.\end{aligned}$$

□

**Corollary 4.** *Suppose that two frames $\Psi_1 = \{\psi_{1j}\}_{j\in\mathbb{J}}$ and $\Psi_2 = \{\psi_{2j}\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are woven. Then, for any $\sigma \subset \mathbb{J}$, for all $\lambda \in \mathbb{R}$ and all $f \in \mathbb{H}$, we have*

$$\begin{aligned}(2\lambda-\frac{\lambda^2}{2}-1)&\sum_{j\in\sigma}|\langle f,\psi_{1j}\rangle|^2+(1-\frac{\lambda^2}{2})\sum_{j\in\sigma^c}|\langle f,\psi_{2j}\rangle|^2\\&\leq\sum_{j\in\sigma}|\langle S^{\sigma}_{\Psi_1}f,S^{-1}_{\Psi_1\Psi_2}\psi_{1j}\rangle|^2+\sum_{j\in\sigma^c}|\langle S^{\sigma}_{\Psi_1}f,S^{-1}_{\Psi_1\Psi_2}\psi_{2j}\rangle|^2\\&\quad+\sum_{j\in\sigma}|\langle S^{\sigma^c}_{\Psi_2}f,S^{-1}_{\Psi_1\Psi_2}\psi_{1j}\rangle|^2+\sum_{j\in\sigma^c}|\langle S^{\sigma^c}_{\Psi_2}f,S^{-1}_{\Psi_1\Psi_2}\psi_{2j}\rangle|^2\\&\leq\sum_{j\in\sigma}|\langle f,\psi_{1j}\rangle|^2+\sum_{j\in\sigma^c}|\langle f,\psi_{2j}\rangle|^2.\end{aligned}\tag{8}$$

**Proof.** Letting $K^{\dagger} = \mathrm{Id}_{\mathbb{H}}$ and for any $\sigma \subset \mathbb{J}$, taking

$$a_j=\begin{cases}1, & j\in\sigma,\\ 0, & j\in\sigma^c,\end{cases}\qquad \phi_j=\begin{cases}S^{-1/2}_{\Psi_1\Psi_2}\psi_{1j}, & j\in\sigma,\\ S^{-1/2}_{\Psi_1\Psi_2}\psi_{2j}, & j\in\sigma^c.\end{cases}$$

If, now, we replace $\psi_{1j}$, $\psi_{2j}$ and $f$ in the left-hand inequality of Theorem 2 respectively by $S^{-1/2}_{\Psi_1\Psi_2}\psi_{1j}$, $S^{-1/2}_{\Psi_1\Psi_2}\psi_{2j}$ and $S^{1/2}_{\Psi_1\Psi_2}f$, then

$$\begin{aligned}&\Big\|\sum_{j\in\sigma}\langle f,\psi_{1j}\rangle S^{-1/2}_{\Psi_1\Psi_2}\psi_{1j}\Big\|^2+\Big\|\sum_{j\in\sigma^c}\langle f,\psi_{2j}\rangle S^{-1/2}_{\Psi_1\Psi_2}\psi_{2j}\Big\|^2\\&\quad\geq(2\lambda-\frac{\lambda^2}{2}-1)\mathrm{Re}\sum_{j\in\sigma}\langle f,\psi_{1j}\rangle\langle\psi_{1j},f\rangle+(1-\frac{\lambda^2}{2})\mathrm{Re}\sum_{j\in\sigma^c}\langle f,\psi_{2j}\rangle\langle\psi_{2j},f\rangle\\&\quad=(2\lambda-\frac{\lambda^2}{2}-1)\sum_{j\in\sigma}|\langle f,\psi_{1j}\rangle|^2+(1-\frac{\lambda^2}{2})\sum_{j\in\sigma^c}|\langle f,\psi_{2j}\rangle|^2.\end{aligned}$$

This along with Equations (6) and (7) gives the left-hand inequality in Equation (8), and the proof of the right-hand inequality is similar and we omit the details. □

**Theorem 3.** *Suppose that $K \in B(\mathbb{H})$ has a closed range and that $K$-frames $\Psi_1 = \{\psi_{1j}\}_{j\in\mathbb{J}}$ and $\Psi_2 = \{\psi_{2j}\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are $K$-woven. Then, for all $\sigma \subset \mathbb{J}$, for any $\{a_j\}_{j\in\mathbb{J}} \in \ell^{\infty}(\mathbb{J})$, $\lambda \in \mathbb{R}$ and $f \in Range(K)$,*

$$\begin{aligned}&\mathrm{Re}\Big(\sum_{j\in\sigma}a_j\langle K^{\dagger}f,\psi_{1j}\rangle\langle\phi_j,f\rangle+\sum_{j\in\sigma^c}a_j\langle K^{\dagger}f,\psi_{2j}\rangle\langle\phi_j,f\rangle\Big)-\Big\|\sum_{j\in\sigma}a_j\langle K^{\dagger}f,\psi_{1j}\rangle\phi_j+\sum_{j\in\sigma^c}a_j\langle K^{\dagger}f,\psi_{2j}\rangle\phi_j\Big\|^2\\&\quad\leq(1-\frac{\lambda}{2})^2\mathrm{Re}\Big(\sum_{j\in\sigma}a_j\langle K^{\dagger}f,\psi_{1j}\rangle\langle\phi_j,f\rangle+\sum_{j\in\sigma^c}a_j\langle K^{\dagger}f,\psi_{2j}\rangle\langle\phi_j,f\rangle\Big)\\&\qquad+\frac{\lambda^2}{4}\mathrm{Re}\Big(\sum_{j\in\sigma}(1-a_j)\langle K^{\dagger}f,\psi_{1j}\rangle\langle\phi_j,f\rangle+\sum_{j\in\sigma^c}(1-a_j)\langle K^{\dagger}f,\psi_{2j}\rangle\langle\phi_j,f\rangle\Big),\end{aligned}$$

*where* $\Phi = \{\phi_j\}_{j\in\mathbb{J}}$ *is a K-dual of* $\{\psi_{1j}\}_{j\in\sigma} \cup \{\psi_{2j}\}_{j\in\sigma^c}$.

*Moreover, if* $(P_1K^\dagger)^*P_2K^\dagger \geq 0$, *then*

$$\mathrm{Re}\Big(\sum_{j\in\sigma} a_j\langle K^\dagger f, \psi_{1j}\rangle\langle\phi_j, f\rangle + \sum_{j\in\sigma^c} a_j\langle K^\dagger f, \psi_{2j}\rangle\langle\phi_j, f\rangle\Big) - \Big\|\sum_{j\in\sigma} a_j\langle K^\dagger f, \psi_{1j}\rangle\phi_j + \sum_{j\in\sigma^c} a_j\langle K^\dagger f, \psi_{2j}\rangle\phi_j\Big\|^2 \geq 0$$

*for any* $f \in Range(K)$, *where* $P_1$ *and* $P_2$ *are given in Equation (2).*

**Proof.** For all $\sigma \subset \mathbb{J}$, for any $\{a_j\}_{j\in\mathbb{J}} \in \ell^\infty(\mathbb{J})$, $\lambda \in \mathbb{R}$ and $f \in Range(K)$, we see from Equation (4) that

$$\begin{aligned}
&\mathrm{Re}\Big(\sum_{j\in\sigma} a_j\langle K^\dagger f, \psi_{1j}\rangle\langle\phi_j, f\rangle + \sum_{j\in\sigma^c} a_j\langle K^\dagger f, \psi_{2j}\rangle\langle\phi_j, f\rangle\Big) - \Big\|\sum_{j\in\sigma} a_j\langle K^\dagger f, \psi_{1j}\rangle\phi_j + \sum_{j\in\sigma^c} a_j\langle K^\dagger f, \psi_{2j}\rangle\phi_j\Big\|^2 \\
&= \mathrm{Re}\langle P_1K^\dagger f, f\rangle - \|P_1K^\dagger f\|^2 \\
&\leq \mathrm{Re}\langle P_1K^\dagger f, f\rangle - (\lambda - \frac{\lambda^2}{4})\mathrm{Re}\langle P_1K^\dagger f, f\rangle + \frac{\lambda^2}{4}\mathrm{Re}\langle P_2K^\dagger f, f\rangle \\
&= (1-\frac{\lambda}{2})^2\mathrm{Re}\Big(\sum_{j\in\sigma} a_j\langle K^\dagger f, \psi_{1j}\rangle\langle\phi_j, f\rangle + \sum_{j\in\sigma^c} a_j\langle K^\dagger f, \psi_{2j}\rangle\langle\phi_j, f\rangle\Big) \\
&\quad + \frac{\lambda^2}{4}\mathrm{Re}\Big(\sum_{j\in\sigma}(1-a_j)\langle K^\dagger f, \psi_{1j}\rangle\langle\phi_j, f\rangle + \sum_{j\in\sigma^c}(1-a_j)\langle K^\dagger f, \psi_{2j}\rangle\langle\phi_j, f\rangle\Big).
\end{aligned}$$

Suppose now that $(P_1K^\dagger)^*P_2K^\dagger$ is a positive operator. Then

$$\begin{aligned}
&\mathrm{Re}\Big(\sum_{j\in\sigma} a_j\langle K^\dagger f, \psi_{1j}\rangle\langle\phi_j, f\rangle + \sum_{j\in\sigma^c} a_j\langle K^\dagger f, \psi_{2j}\rangle\langle\phi_j, f\rangle\Big) - \Big\|\sum_{j\in\sigma} a_j\langle K^\dagger f, \psi_{1j}\rangle\phi_j + \sum_{j\in\sigma^c} a_j\langle K^\dagger f, \psi_{2j}\rangle\phi_j\Big\|^2 \\
&= \mathrm{Re}\langle P_1K^\dagger f, f\rangle - \|P_1K^\dagger f\|^2 = \mathrm{Re}\langle P_1K^\dagger f, P_1K^\dagger f + P_2K^\dagger f\rangle - \mathrm{Re}\langle P_1K^\dagger f, P_1K^\dagger f\rangle \\
&= \mathrm{Re}\langle P_1K^\dagger f, P_2K^\dagger f\rangle = \mathrm{Re}\langle f, (P_1K^\dagger)^*P_2K^\dagger f\rangle \geq 0.
\end{aligned}$$

□

**Corollary 5.** *Let the two frames* $\Psi_1 = \{\psi_{1j}\}_{j\in\mathbb{J}}$ *and* $\Psi_2 = \{\psi_{2j}\}_{j\in\mathbb{J}}$ *in* $\mathbb{H}$ *be woven. Then, for any* $\sigma \subset \mathbb{J}$, *for all* $\lambda \in \mathbb{R}$ *and all* $f \in \mathbb{H}$, *we have*

$$\begin{aligned}
0 &\leq \sum_{j\in\sigma}|\langle f, \psi_{1j}\rangle|^2 - \sum_{j\in\sigma}|\langle S^\sigma_{\Psi_1}f, S^{-1}_{\Psi_1\Psi_2}\psi_{1j}\rangle|^2 - \sum_{j\in\sigma^c}|\langle S^\sigma_{\Psi_1}f, S^{-1}_{\Psi_1\Psi_2}\psi_{2j}\rangle|^2 \\
&\leq (1-\frac{\lambda}{2})^2\sum_{j\in\sigma}|\langle f, \psi_{1j}\rangle|^2 + \frac{\lambda^2}{4}\sum_{j\in\sigma^c}|\langle f, \psi_{2j}\rangle|^2.
\end{aligned}$$

**Proof.** The proof is similar to Corollary 4 by using Theorem 3, so we omit it. □

**Remark 2.** *Corollaries 4 and 5 are respectively Theorems 15 and 14 in [34].*

We conclude the paper with a double inequality for $K$-weaving frames stated as follows.

**Theorem 4.** *Suppose that K-frames $\Psi_1 = \{\psi_{1j}\}_{j\in\mathbb{J}}$ and $\Psi_2 = \{\psi_{2j}\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are K-woven. Then, for any $\sigma \subset \mathbb{J}$, for all $\{a_j\}_{j\in\mathbb{J}} \in \ell^\infty(\mathbb{J})$ and all $f \in \mathbb{H}$, we have*

$$\begin{aligned}\frac{3}{4}\|Kf\|^2 \le & \left\|\sum_{j\in\sigma} a_j\langle f,\psi_{1j}\rangle\phi_j + \sum_{j\in\sigma^c} a_j\langle f,\psi_{2j}\rangle\phi_j\right\|^2 \\ & + \mathrm{Re}\left(\sum_{j\in\sigma}(1-a_j)\langle f,\psi_{1j}\rangle\langle\phi_j,Kf\rangle + \sum_{j\in\sigma^c}(1-a_j)\langle f,\psi_{2j}\rangle\langle\phi_j,Kf\rangle\right) \\ \le & \frac{3\|K\|^2+\|P_1-P_2\|^2}{4}\|f\|^2,\end{aligned}$$

*where $P_1$ and $P_2$ are given in Equation (2), and $\Phi = \{\phi_j\}_{j\in\mathbb{J}}$ is a K-dual of $\{\psi_{1j}\}_{j\in\sigma} \cup \{\psi_{2j}\}_{j\in\sigma^c}$.*

**Proof.** For any $\sigma \subset \mathbb{J}$, for all $\{a_j\}_{j\in\mathbb{J}} \in \ell^\infty(\mathbb{J})$ and all $f \in \mathbb{H}$, it is easy to check that $P_1 + P_2 = K$. By Lemma 1, we get

$$\begin{aligned}& \left\|\sum_{j\in\sigma} a_j\langle f,\psi_{1j}\rangle\phi_j + \sum_{j\in\sigma^c} a_j\langle f,\psi_{2j}\rangle\phi_j\right\|^2 + \mathrm{Re}\left(\sum_{j\in\sigma}(1-a_j)\langle f,\psi_{1j}\rangle\langle\phi_j,Kf\rangle + \sum_{j\in\sigma^c}(1-a_j)\langle f,\psi_{2j}\rangle\langle\phi_j,Kf\rangle\right) \\ & = \|P_1f\|^2 + \mathrm{Re}\langle P_2f,Kf\rangle \ge \frac{3}{4}\|Kf\|^2.\end{aligned}$$

We also have

$$\begin{aligned}& \left\|\sum_{j\in\sigma} a_j\langle f,\psi_{1j}\rangle\phi_j + \sum_{j\in\sigma^c} a_j\langle f,\psi_{2j}\rangle\phi_j\right\|^2 + \mathrm{Re}\left(\sum_{j\in\sigma}(1-a_j)\langle f,\psi_{1j}\rangle\langle\phi_j,Kf\rangle + \sum_{j\in\sigma^c}(1-a_j)\langle f,\psi_{2j}\rangle\langle\phi_j,Kf\rangle\right) \\ & = \langle P_1f,P_1f\rangle + \frac{1}{2}\langle P_2f,Kf\rangle + \frac{1}{2}\langle Kf,P_2f\rangle \\ & = \langle P_1f,P_1f\rangle + \frac{1}{2}\langle (K-P_1)f,Kf\rangle + \frac{1}{2}\langle Kf,(K-P_1)f\rangle \\ & = \langle Kf,Kf\rangle - \frac{1}{2}[\langle P_1f,Kf\rangle - \langle P_1f,P_1f\rangle] - \frac{1}{2}[\langle Kf,P_1f\rangle - \langle P_1f,P_1f\rangle] \\ & = \langle Kf,Kf\rangle - \frac{1}{2}\langle P_1f,P_2f\rangle - \frac{1}{2}\langle P_2f,P_1f\rangle \\ & = \frac{3}{4}\langle Kf,Kf\rangle + \frac{1}{4}\langle P_1f+P_2f,P_1f+P_2f\rangle - \frac{1}{2}\langle P_1f,P_2f\rangle - \frac{1}{2}\langle P_2f,P_1f\rangle \\ & = \frac{3}{4}\langle Kf,Kf\rangle + \frac{1}{4}\langle (P_1-P_2)f,(P_1-P_2)f\rangle \\ & \le \frac{3}{4}\|K\|^2\|f\|^2 + \frac{1}{4}\|P_1-P_2\|^2\|f\|^2 = \frac{3\|K\|^2+\|P_1-P_2\|^2}{4}\|f\|^2,\end{aligned}$$

and the proof is over. □

**Remark 3.** *Theorem 3 in [35] can be obtained when taking $K = \mathrm{Id}_{\mathbb{H}}$ in Theorem 4.*

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