mathematics

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The Simultaneous Strong Resolving Graph and the Simultaneous Strong Metric Dimension of Graph Families

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Received: 5 December 2019; Accepted: 10 January 2020; Published: 14 January 2020

Abstract: We consider in this work a new approach to study the simultaneous strong metric dimension of graphs families, while introducing the simultaneous version of the strong resolving graph. In concordance, we consider here connected graphs G whose vertex sets are represented as $V(G)$, and the following terminology. Two vertices $u, v \in V(G)$ are strongly resolved by a vertex $w \in V(G)$, if there is a shortest $w - v$ path containing u or a shortest $w - u$ containing v. A set A of vertices of the graph G is said to be a strong metric generator for G if every two vertices of G are strongly resolved by some vertex of A. The smallest possible cardinality of any strong metric generator (SSMG) for the graph G is taken as the strong metric dimension of the graph G. Given a family $\mathcal{F}$ of graphs defined over a common vertex set V, a set $S \subset V$ is an SSMG for $\mathcal{F}$, if such set S is a strong metric generator for every graph $G \in \mathcal{F}$. The simultaneous strong metric dimension of $\mathcal{F}$ is the minimum cardinality of any strong metric generator for $\mathcal{F}$, and is denoted by $\mathrm{Sd}_s(\mathcal{F})$. The notion of simultaneous strong resolving graph of a graph family $\mathcal{F}$ is introduced in this work, and its usefulness in the study of $\mathrm{Sd}_s(\mathcal{F})$ is described. That is, it is proved that computing $\mathrm{Sd}_s(\mathcal{F})$ is equivalent to computing the vertex cover number of the simultaneous strong resolving graph of $\mathcal{F}$. Several consequences (computational and combinatorial) of such relationship are then deduced. Among them, we remark for instance that we have proved the NP-hardness of computing the simultaneous strong metric dimension of families of paths, which is an improvement (with respect to the increasing difficulty of the problem) on the results known from the literature.

Keywords: simultaneous strong resolving set; simultaneous strong metric dimension; simultaneous strong resolving graph

MSC: 05C12

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1. Introduction

Topics concerning distances in graphs are widely studied in the literature, and a high number of applications to real life problems can be found in the literature. As a sporadic example of a work that gives some ideas on the vastness of this topic we cite, for instance [1]. Metric graph theory is a significant area in graph theory that deals with distances in graphs, and a large number of works on this topic is nowadays being developed. One of the lines belonging to metric graph theory is that of the metric dimension parameters. Such topic is indeed a huge area of research that is lastly intensively dealt with. It is then not our goal to enter into citing several articles which are not connected exactly with our exposition. To those readers interested in metric dimension things, we suggest for instance the Ph.D. dissertation [2] (and references cited therein), which contains a good background on the topic.

For any given simple and connected graph G whose vertex set is represented as $V(G)$ and its edge set by $E(G)$, while considering it as a metric space, several styles of metrics over the vertex set

V, provided with the standard vertex distance, are nowadays defined and studied in the literature. For instance, the metric $d_G : V(G) \times V(G) \rightarrow \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ represents the set of positive integers numbers, and $d_G(x, y)$ is taken as the length of a shortest $u - v$ path, is one of the most commonly studied. In this sense, the pair $(V(G), d_G)$ is clearly a metric space. Concerning such a metric space, it is said that a vertex $v \in V(G)$ distinguishes (recognizes or determines are also used terms) two vertices x and y if $d_G(v, x) \neq d_G(v, y)$. A set $S \subset V(G)$ is said to be a *metric generator* for the graph G if it is satisfied that any pair of vertices of G is uniquely determined by some element of S. Consider that $S = \{w_1, w_2, \ldots, w_k\}$ is an ordered subset of vertices of G. The *metric vector* (or metric representation) of a given vertex $v \in V(G)$, with respect to S, is the vector of distance $(d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. In this sense, the subset of vertices S is called a *metric generator* for the graph G, if any two distinct vertices produce distinct metric vectors relative to such set S. A metric generator of G having the minimum possible cardinality is called a *metric basis*, and its cardinality is precisely the *metric dimension* of G, which is usually denoted by $\dim(G)$. The definitions of these concepts (for general metric spaces) are coming from the earliest 1950s from the work [3], although its popularity was not developed until relatively recently (about 15 years before). On the other hand, for the specific case of graphs, and motivated by a problem of uniquely recognizing intruder's locations in networks, these concepts were presented and studied by Slater in [4]. In such work, metric generators were called *locating sets*. On the other hand, Harary and Melter (see [5]) also independently came out with the same concept. In such work, metric generators were called *resolving sets*. It is interesting to remark that some examples of applications of the metric dimension concern navigation of robots in networks as discussed in the work [6], or to chemistry as appearing in [7–9].

An interesting variant of metric dimension in graphs was described by Sebö and Tannier in [10], where they have asked the following question. "*For a given metric generator T of a graph H, whenever H is a subgraph of a graph G, and the metric vectors of the vertices of H relative to T agree in both H and G, is H an isometric subgraph of G?*" The situation is that, despite the fact that metric vectors of all vertices of a graph G (relative to a given metric generator) distinguish all pairs of vertices in such graph, it happens that they do not always uniquely recognize all distances in this graph, a fact that was already shown in [10]. Addressed to give a positive answer to their own question, the authors of [10] replaced the notion of "metric generator" by a stronger one. This is described next.

Given a pair of vertices $u, v \in V(G)$, the *interval* $I_G[u, v]$ between such two vertices u and v is defined as the collection of all vertices that belong to some shortest $u - v$ path. In this sense, a vertex w *strongly resolves* two other different vertices u and v, if it is satisfied that $v \in I_G[u, w]$ or $u \in I_G[v, w]$, or equivalently, if $d_G(u, w) = d_G(u, v) + d_G(v, w)$ or $d_G(v, w) = d_G(v, u) + d_G(u, w)$. In connection with this, it is also said that u, v are *strongly resolved* by w. From now on, all graphs considered are connected. A set S of vertices of G is a *strong metric generator* for G if any two distinct vertices x, y of such graph are strongly resolved by some vertex $u \in S$ (it could happen that u equals x or y). Then, the smallest possible cardinality of any set being a strong metric generator for G is called the *strong metric dimension* of G, and this cardinality is denoted by $\dim_s(G)$. In addition, a strong metric generator for G whose cardinality is precisely equal to $\dim_s(G)$ is called a *strong metric basis* of G. It is now readily observed that any strong metric generator of G also satisfies the property of being a metric generator for G. The computational problem concerning finding the strong metric dimension of a given graph is now relatively well studied, and one can find a rich literature concerning it. For more information on this issue, we suggest, for instance, the articles [11,12], the Ph.D. Thesis [13], the survey [14], and references cited therein.

More recently, an extension of the notion of the strong metric dimension of graphs to families of graphs was presented in [15]. The following was stated: Consider that $\mathcal{G} = \{G_1, G_2, ..., G_k\}$ is a family of connected graphs $G_i = (V, E_i)$ having a common vertex set V. Note that the edge sets of the graphs belonging to the family are not necessarily edge-disjoint, and also that the union of their edge sets is not necessarily the complete graph. Concerning such family, it was said in [15] that a *simultaneous strong metric generator* (SSMG for short) for the family $\mathcal{G}$ is taken as a set $S \subset V$ with the property that S forms a strong metric generator for every graph G_i of the family. As usual, an SSMG having the minimum possible cardinality for $\mathcal{G}$ is called a *simultaneous strong metric basis* of $\mathcal{G}$. This smallest cardinality is then precisely called the *simultaneous strong metric dimension* of $\mathcal{G}$, and this is denoted by $\mathrm{Sd}_s(\mathcal{G})$, or by $\mathrm{Sd}_s(G_1, G_2, ..., G_t)$ when it is necessary to clarify the graphs of the family. It is worthwhile mentioning that such concepts arise from a related version of simultaneity for the standard metric dimension studied in [16,17].

The notion of the simultaneous metric dimension of graphs families (and its strong related version) was first studied in the Ph.D. thesis [18], based on the following problem, which arises in relation with a similar problem for the standard metric dimension. It is assumed that the topology of robots navigation network changes within some amount of possible simple networks, say a set (or family) of graphs $\mathcal{F}$. Nodes of the networks remain the same, but their links could appear or disappear. This setting could require the use of a dynamic network whose links change over the time. In this sense, the problem concerning uniquely identifying the robots (by using the smallest resources) navigating in such a "variable" network can be understood as the problem of determining the minimum cardinality of a set of vertices that is simultaneously a metric generator for each graph belonging to this set $\mathcal{F}$. That is, if a set of vertices S gives a solution to this problem, then the position of a robot can be uniquely determined by the distance to the elements of S, independently of the graph which is being used in each moment in this dynamic network.

We now present some basic terminology and notation to beused throughout our exposition. Given a vertex v of a graph G, $N_G(v)$ denotes the *open neighborhood* of v in G, while the *closed neighborhood* is represented by $N_G[v]$ and it equals $N_G(v) \cup \{v\}$. If there is no confusion, we then simply use $N(v)$ or $N[v]$. Two vertices $x, y \in V(G)$ are called *twins* if they satisfy $N_G[x] = N_G[y]$ or $N_G(x) = N_G(y)$. Specifically, when $N_G[x] = N_G[y]$, they are known as *true twins*, and similarly whether $N_G(x) = N_G(y)$, they are called *false twins*. Now, if the open neighborhood $N(v)$ of a vertex v induces a complete graph, then such v is known as an *extreme vertex*. The set of extreme vertices of G is denoted by $\sigma(G)$. The largest possible distance between any two vertices of G is denoted by $D(G)$, also called the *diameter* of G. In this sense, a graph G is called 2-antipodal if, for every vertex $x \in V(G)$, there is exactly one other vertex $y \in V(G)$ satisfying the fact that $d_G(x, y) = D(G)$. Examples of 2-antipodal graphs are, for instance, even cycles C_{2k}, and the hypercubes Q_r. Finally, for a given set $W \subset V(G)$, by $\langle W \rangle_G$, we represent the subgraph of G induced by W. Any other definition used shall be introduced whenever a concept is firstly needed.

Since all the definitions above require the connectedness of the graph in question, throughout the whole exposition, we will consider that our graphs are connected; even so, we will not explicitly mention this fact.

2. The Simultaneous Strong Resolving Graph

In this section, we describe an approach which was first presented in [19], in order to transform the problem of finding the strong metric dimension of a graph to computing the vertex cover number of another related graph. To this end, we need some terminology and notation. A vertex u of G is said to be *maximally distant* from other v, if every vertex $w \in N_G(u)$ satisfies that $d_G(v, w) \leq d_G(u, v)$. For a pair of vertices, u, v, if it happens that u is maximally distant from v and v is also maximally distant from u, then these u and v are called a pair of *mutually maximally distant vertices* (MMD for short). The set of vertices of G that are MMD with at least one other vertex of G is denoted by $\partial(G)$. The *strong resolving graph* of G, which is denoted by G_{SR}, is another graph whose vertex set is $V(G_{SR}) = V(G)$. In addition, there is an edge between two vertices u, v in G_{SR} if such vertices u and v are mutually maximally distant in the original graph G. Clearly, those vertices which are not MMD with any other vertex of G are isolated vertices in G_{SR}. The recent work [20] (a kind of survey) contains a number of results concerning characterizations, realizability, and several other properties of the strong resolving graphs of graphs.

Now, by a *vertex cover* set of a graph G, we mean a set of vertices S of G satisfying that every edge of G has at least one end vertex in the set S. The *vertex cover number* of G, which is denoted by $\alpha(G)$, is taken as the smallest possible cardinality of a subset of vertices of G being a vertex cover set of G. By an $\alpha(G)$-set, we represent a vertex cover set of cardinality $\alpha(G)$. In connection with this concept, the authors Oellermann and Peters-Fransen (see [19]) have proved that finding the strong metric dimension of a connected graph G is equivalent to finding the vertex cover number of G_{SR}, which is the next result.

Theorem 1 ([19])**.** *For any connected graph G, $\dim_s(G) = \alpha(G_{SR})$.*

There are several different and non trivial families of connected graphs for which the strong resolving graphs can relatively easily be obtained. We next mention some of these cases, mainly based on the fact that we further on shall refer to them. Such following observations have already appeared (in an identical presentation) in other works like, for instance [20].

Observation 1.

(a) *If $\partial(G) = \sigma(G)$, then $G_{SR} \cong K_{|\partial(G)|} \cup \left(\bigcup_{i=1}^{n-|\partial(G)|} K_1\right)$. In particular, $(K_n)_{SR} \cong K_n$ and for any tree T of order n with $l(T)$ leaves, $(T)_{SR} \cong K_{l(T)} \cup \left(\bigcup_{i=1}^{n-l(T)} K_1\right)$.*

(b) *For any 2-antipodal graph G of order n, $G_{SR} \cong \bigcup_{i=1}^{\frac{n}{2}} K_2$. Even cycles are 2-antipodal. Thus, $(C_{2k})_{SR} \cong \bigcup_{i=1}^{k} K_2$.*

(c) *For odd cycles, $(C_{2k+1})_{SR} \cong C_{2k+1}$.*

We now turn our attention to the simultaneous strong metric dimension of graph families and look for an equivalent version of the strong resolving graph in a simultaneous version. That is, given a family of graphs $\mathcal{G} = \{G_1, G_2, ..., G_k\}$ defined over the set of vertices V (as described above), we say that the *simultaneous strong resolving graph* of G, denoted by $\mathcal{G}_{SSR}$, is a graph whose vertex set is $V(\mathcal{G}_{SSR}) = V$. In addition, two vertices u, v are adjacent in $\mathcal{G}_{SSR}$ if the vertices u and v are mutually maximally distant in some graph $G_i \in \mathcal{G}$. It is readily seen that $\mathcal{G}_{SSR}$ can be obtained from the overlapping of the strong resolving graphs of the graphs $G_1, G_2, \ldots, G_k$. An equivalent result to that of Theorem 1 can be then derived for the simultaneous case. To this end, the next remarks make an important role.

Remark 1. *Let G be any connected graph. For any two vertices $x, y \in V(G)$, there are two MMD vertices x', y' of G, such that a shortest $x' - y'$ path contains the vertices x, y.*

We must recall that at least one of the vertices x, y in the result above could precisely be at least one of the vertices x', y', respectively (this could happen in case x, y are MMD or whether one of them is maximally distant from the other).

Theorem 2. *For any family of graphs, $\mathcal{G} = \{G_1, G_2, ..., G_k\}$, $Sd_s(\mathcal{G}) = \alpha(\mathcal{G}_{SSR})$.*

Proof. We shall prove that any set is an SSMG for $\mathcal{G}$ if and only if it is a vertex cover of $\mathcal{G}_{SSR}$. Assume each graph of $\mathcal{G}$ is defined over the set of vertices V. Let $W \subset V$ be an SSMG of $\mathcal{G}$ and let uv be an edge of $\mathcal{G}_{SSR}$. By the definition of $\mathcal{G}_{SSR}$, there is a graph $G_i \in \mathcal{G}$ such that u, v are MMD in G_i. Thus, $W \cap \{u, v\} \neq \emptyset$, which means that the edge uv is covered by W in $\mathcal{G}_{SSR}$. Thus, W is a vertex cover of $\mathcal{G}_{SSR}$.

On the other hand, let $W' \subset V$ be a vertex cover of $\mathcal{G}_{SSR}$ and let x, y be any two different vertices of V. If x, y are MMD in some $G_j \in \mathcal{G}$, then xy is an edge of $\mathcal{G}_{SSR}$, which means that $W' \cap \{x, y\} \neq \emptyset$, since such edge must be covered by W'. Assume $x \in W'$. Thus, the pair of vertices x, y is strongly resolved by x in every $G_i \in \mathcal{G}$. On the contrary, if x, y are not MMD in every $G_i \in \mathcal{G}$, then the edge xy

does not exist in $\mathcal{G}_{SSR}$. Moreover, by Remark 1, in every graph $G_l \in \mathcal{G}$, there are two MMD vertices x_l, y_l such that a shortest $x_l - y_l$ path of G_l contains x and y. Clearly, for every $G_l \in \mathcal{G}$, the edge $x_l y_l$ belongs to G_{SSR} and so, $W' \cap \{x_l, y_l\} \neq \emptyset$. Hence, for any $G_l \in \mathcal{G}$, we observe that x, y are strongly resolved by x_l or by y_l. As a consequence, W' is a strong resolving set for any $G_i \in \mathcal{G}$ and therefore W' is a simultaneous strong resolving set of $\mathcal{G}$, which completes the proof of the equality $Sd_s(\mathcal{G}) = \alpha(\mathcal{G}_{SSR})$. $\square$

3. Realization of the Simultaneous Strong Resolving Graphs with Some Consequences

In the recent work [20], several results concerning the realization of the strong resolving graphs of graphs were presented. For instance, there was proved that there is not any graph G whose strong resolving graph is isomorphic to a complete bipartite graph $K_{2,r}$ for every $r \geq 2$. In contrast with these facts, we shall prove that every graph G can represent the simultaneous strong resolving graph of some family of graphs.

Proposition 1. *For any graph G of order n and size m with vertex set V, there exist a family of m paths $\mathcal{P} = \{P_n^1, P_n^2, \ldots, P_n^m\}$ defined over the set of vertices V such that $\mathcal{P}_{SSR}$ is isomorphic to G.*

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. For any edge $e_{ij} = v_i v_j$ of G, consider a path P_n^{ij} whose leaves are v_i and v_j and the remaining vertices are $V - \{v_i, v_j\}$. Since the strong resolving graph of P_n^{ij} is formed by a graph K_2 on the vertices $\{v_i, v_j\}$ and the $n-2$ isolated vertices $V - \{v_i, v_j\}$, it is readily seen that the union (overlapping) of the m paths P_2 constructed in this way, corresponding to the edges of G, together with the other $n-2$ isolated vertices, is precisely the graph G. $\square$

Since the realization family given above is formed only by paths, one may now wonder if a given graph can be realized as the simultaneous strong resolving graph of a family of other graphs different from paths. For instance, the following two results show two other different realizations. To this end, a *multisubdivided star* $S_{r,t}$ of order $r+t+1$ is obtained from a star $S_{1,t}$ by subdividing some edges with some vertices until we have a graph of order $r+t+1$ (clearly r vertices were used in this multisubdivision process). In addition, a *comet graph* $C_{r,t}$ (where $r \geq 4$ is an even integer and $t \geq 0$) is a unicyclic graph of order $r+t$ whose unique cycle has order r, and there is at most one vertex of degree three and at most one leaf. Note that this comet graph can be a cycle graph when $t = 0$ (in such case an even cycle indeed). In other words, a comet graph is obtained from a cycle C_r by attaching a path of order $t \geq 0$ to one of its vertices.

Proposition 2. *For any graph G of order n with vertex set V, there exists a family of multisubdivided star graphs $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ defined over the set of vertices V such that $\mathcal{F}_{SSR}$ is isomorphic to G.*

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. Now, for every clique Q_j of G of cardinality j, consider a multisubdivided star graph $S_{j,t}$ such that $j+t+1 = n$ whose leaves are the vertices of Q_j, and the remaining vertices are $V - Q_j$ (taken in any order). Since the strong resolving graph of every multisubdivided star graph $S_{j,t}$ is formed by a complete graph K_j on the vertices of Q_j and the $n - |Q_j|$ isolated vertices in $V - Q_j$ (by using Observation 1), it is readily seen that the overlapping of the strong resolving graphs of the graphs belonging to this set of multisubdivided star graphs constructed in this way, corresponding to the cliques of G, together with the other corresponding isolated vertices, gives precisely the graph G. $\square$

In order to present our next construction, we remark (which can be easily observed) that the strong resolving graph of a comet graph is given by the disjoint union of r graphs K_2 and t isolated vertices. We also need the following terminology. A *matching* in a graph G is a set of pairwise disjoint edges in the graph, and a *maximum matching* is a matching M such that the inclusion of any other edge of G to M leads to at least two not disjoint edges.

Proposition 3. *For any graph G of order n with vertex set V, there exist a family $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ containing comet graphs and paths, defined over the set of vertices V such that $\mathcal{F}_{SSR}$ is isomorphic to G.*

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. We consider all possible maximum matchings of G. If there is a maximum matching which has only one edge $e = uv$, then we consider a path graph P_n, for the family $\mathcal{F}$, whose leaves are u and v. Clearly, the strong resolving graph of this path is the graph K_2 on the vertex set $\{u, v\}$ and $n-2$ isolated vertices. Next, we take every other maximum matching M_i of G having $i \geq 2$ edges. Now, consider a comet graph $C_{2i,t}$ with $2i + t = n$, such that any two vertices being an edge of M_i are diametral in the cycle C_{2i} of $C_{2i,t}$, and the remaining vertices are $V - M_i$ (taken in any order) forming the path of $C_{2i,t}$ of order t that are attached to one vertex of C_{2i}. Note that the strong resolving graph of any comet graph is given by the disjoint union of i graphs K_2 and t isolated vertices. Thus, by using all the graphs constructed as mentioned above for all the maximum matchings of G, it is readily seen that the overlapping of the strong resolving graphs of the graphs belonging to this set constructed in this way, corresponding to the maximum matchings of G, together with the other corresponding isolated vertices, gives precisely the graph G. □

Based on the constructions above, it looks like several different families of graphs can produce the same simultaneous strong resolving graph. In this sense, it is natural to raise the following question, which roughly speaking, seems to be very challenging.

Open question: Given a graph G, is it possible to characterize all the possible families of graphs $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ such that $\mathcal{F}_{SSR}$ is isomorphic to G?

It was proved in [15] that computing the simultaneous strong metric dimension of graph families is NP-hard, even when restricted to families of trees. An interesting consequence of Proposition 1 shows that such problem, which next appears, remains NP-hard, even when restricted to a couple of simpler families.

Simultaneous Strong Metric Dimension Problem (SSD Problem for Short)
INSTANCE: A graph family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ defined on a common vertex set V and an integer k, $1 \leq k \leq \lvert V \rvert - 1$. PROBLEM: Deciding whether $\mathrm{Sd}_s(\mathcal{G})$ is less than k.

By using Proposition 1, we next present a reduction of the problem of computing the vertex cover number of graph to the problem of computing the simultaneous strong metric dimension of families of paths.

Theorem 3. *The SSD problem is NP-complete for families of paths or multisubdivided star graphs.*

Proof. The problem is clearly in NP since verifying that a given set of vertices is indeed an SSMG for a graph family can be done in polynomial time. Now, let G be any graph with vertex set V of order n and size m. From Proposition 1 (resp. Proposition 2), we know there is a family of m paths $\mathcal{P} = \{P_n^1, P_n^2, \ldots, P_n^m\}$ (resp. of multisubdivided star graphs) defined over the set of vertices V such that $\mathcal{P}_{SSR}$ is isomorphic to G. Therefore, from Theorem 2, we have that $Sd_s(\mathcal{P}) = \alpha(\mathcal{P}_{SSR}) = \alpha(G)$, which completes the NP-completeness reduction based on the fact that the decision problem concerning the vertex cover number of graphs is an NP-complete problem (see [21]). □

Another interesting consequence of Theorem 2 concerns the approximation of computing the simultaneous strong metric dimension of graphs families. We first note that finding the simultaneous strong resolving graph of a graph family can be polynomially done. This fact, together with the fact that computing the vertex cover number of graphs admits a polynomial-time 2-approximation, allows for claiming that computing the simultaneous strong metric dimension of graphs families also admits a polynomial-time 2-approximation.

4. Applications of the Simultaneous Strong Resolving Graph

Since computing the simultaneous strong metric dimension of graph families is NP-hard even when restricted to very specific families, it is then desirable to describe as many families as possible for which its simultaneous strong metric dimension can be computed. In this sense, from now on in this section we are devoted to make this so, and a fundamental tool for it shall precisely be the simultaneous strong resolving graph and, connected with it, Theorem 2.

Proposition 4. *If $\mathcal{F}$ is a family of bipartite graphs, each of them is defined over the common bipartition sets U, V, then* $\mathrm{Sd}_s(\mathcal{F}) \leq |U| + |V| - 2$.

Proof. If any two vertices are MMD in some graph $G_i \in \mathcal{F}$, then they belong to the same bipartition set of G_i. Thus, it must happen that $\mathcal{F}_{SSR}$ is a subgraph of a graph with two connected components isomorphic to $K_{|U|}$ and $K_{|V|}$. By using Theorem 2, we obtain that $\mathrm{Sd}_s(\mathcal{F}) = \alpha(\mathcal{F}_{SSR}) \leq \alpha(K_{|U|} \cup K_{|V|}) = |U| + |V| - 2$. □

Next, we particularize the result above and show that such bound is achieved in several situations.

Proposition 5. *If $\mathcal{F}$ is a family of bipartite graphs, each of them is defined over the common bipartition sets U, V, and such that it contains the complete bipartite graph $K_{|U|,|V|}$, then* $\mathrm{Sd}_s(\mathcal{F}) = |U| + |V| - 2$.

Proof. The result directly follows from the fact that, if $K_{|U|,|V|} \in \mathcal{F}$, then $(\mathcal{F})_{SR}$ is isomorphic to $K_{|U|} \cup K_{|V|}$. Thus, from Theorem 2, we get the desired result, since $\alpha(K_{|U|} \cup K_{|V|}) = |U| + |V| - 2$. □

Let $C_r = v_0 v_1 \dots v_{r-1}$, with $r \geq 4$ and even be a cycle. Then, let $\mathcal{F}_C$ be a family of cycles defined on a common vertex set with $C_r \in \mathcal{F}_C$, and every other cycle $C \in \mathcal{F}_C$ is obtained from C_r, by making a permutation of two vertices v_i, v_j of C_r such that either $i, j \in \{0, \dots, r/2 - 1\}$ or $i, j \in \{r/2, \dots, r - 1\}$.

Proposition 6. *If $\mathcal{F}_C$ is a family of cycles obtained from a cycle C_r as described above, then* $\mathrm{Sd}_s(\mathcal{F}_C) = \frac{r}{2}$.

Proof. We first note (by Observation 1) that the strong resolving graph of any cycle $C_r^{(j)} \in \mathcal{F}_C$ is isomorphic to $\cup_{i=1}^{r/2} K_2$. Now, since any cycle of $\mathcal{F}_C$ is obtained from C_r, by making a permutation of two vertices v_i, v_j of C_r such that either $i, j \in \{0, \dots, r/2 - 1\}$ or $i, j \in \{r/2, \dots, r - 1\}$, we deduce that there are no edges in $(\mathcal{F}_C)_{SSR}$ between any two vertices of the set $\{v_0, v_1, \dots, v_{r/2-1}\}$, in addition to no edges between any two vertices of the set $\{v_{r/2}, v_{(r/2+1}, \dots, v_{r-1}\}$. Moreover, for any vertex of the set $v_j \in \{v_0, v_1, \dots, v_{r/2-1}\}$, there is an edge joining v_j with a vertex $v_k \in \{v_{r/2}, v_{r/2+1}, \dots, v_{r-1}\}$ and vice versa. As a consequence of such facts, we obtain that $(\mathcal{F}_C)_{SSR}$ is a bipartite graph, which is a subgraph of the complete bipartite graph $K_{r/2,r/2}$. Since $\alpha(K_{r/2,r/2}) = r/2$, by using Theorem 2, we have $\mathrm{Sd}_s(\mathcal{F}_C) = \alpha((\mathcal{F}_C)_{SSR}) \leq \alpha(K_{r/2,r/2}) = \frac{r}{2}$. On the other hand, since $C_r \in \mathcal{F}_C$, the edges $v_0 v_{r/2}, v_1 v_{r/2+1}, \dots, v_{r/2-1} v_{r-1}$ belong to $(\mathcal{F}_C)_{SSR}$. Consequently, in order to cover such edges, it must happen $\alpha((\mathcal{F}_C)_{SSR}) \geq r/2$. Therefore, by using again Theorem 2, we complete the proof. □

Let T be a tree having all the vertices with a degree larger than two unless they are leaves. Hence, let $\mathcal{F}_T$ be the family of trees defined on a common vertex set with $T \in \mathcal{F}_T$, and every other tree $T' \in \mathcal{F}_T$ is obtained from T, by making a permutation of two vertices u, v of T such that u is a leaf of T and v is not a leaf.

Proposition 7. *If $\mathcal{F}_T$ is a family of trees obtained from a tree T with t leaves as described above, then $t - 1 \leq$ $\mathrm{Sd}_s(\mathcal{F}_T) \leq t$. Moreover, $\mathrm{Sd}_s(\mathcal{F}_T) = t$ if and only if every leaf of T has been used to make a permutation with other non leaf vertex of T in order to obtain another tree of $\mathcal{F}_T$.*

Proof. By Observation 1, the strong resolving graph of the tree $T \in \mathcal{F}_T$ is isomorphic to the complete graph K_t together with $n - t$ isolated vertices. Thus, K_t is a subgraph of the graph $(\mathcal{F}_T)_{SSR}$ and so, by using Theorem 2, we obtain $\mathrm{Sd}_s(\mathcal{F}_T) = \alpha((\mathcal{F}_T)_{SSR}) \geq \alpha(K_t) = t-1$, which is the lower bound. Now, observe that the strong resolving graph of any tree, other than T, is obtained from the strong resolving graph of T by removing all edges incident to one vertex, say y, corresponding to a leaf of T, choosing an isolated vertex x corresponding to a non leaf of T, and adding all the possible edges between x and the vertices corresponding to leaves of T other than y. Moreover, since there are no edges between any two of such chosen isolated vertices mentioned above, it is clear that the set of t vertices corresponding to the leaves of T represents a vertex cover set of $(\mathcal{F}_T)_{SSR}$. By using again Theorem 2, we obtain $\mathrm{Sd}_s(\mathcal{F}_T) = \alpha((\mathcal{F}_T)_{SSR}) \leq t$, which is the upper bound.

On the other hand, in order for this set of t vertices to correspond to the leaves of T, which will represent a vertex cover set of minimum cardinality in $(\mathcal{F}_T)_{SSR}$, it is required that all such vertices will have a neighbor not in this set of leaves. This means that every leaf of T has been used to make a permutation with another non leaf vertex of T in order to obtain a tree of $\mathcal{F}_T$ other than T. The opposite direction is straightforward to observe. Therefore, the proof is complete. □

Again, let T be a tree having all the vertices with a degree larger than two unless they are leaves. Let $\mathcal{H}_T$ be the family of at least two unicyclic graphs H defined on a common vertex set such that every graph $H \in \mathcal{H}_T$ is obtained from T, by adding an edge between any two vertices u, v of T. Before studying the simultaneous strong metric dimension of $\mathcal{H}_T$, we introduce some terminology and basic properties of the strong resolving graph of unicyclic graphs. Given a unicyclic graph $G = (V, E)$ with the unique cycle C_r, we denote by $c_2(G)$ the set of vertices of the cycle C_r having degree two. By $T(G)$, we represent the set of vertices of degree one in G.

Remark 2. *Let G be a unicyclic graph. For every vertex $x \in c_2(G)$, there exists at least one vertex $y \in c_2(G) \cup T(G)$ such that x, y are mutually maximally distant in G.*

Remark 3. *Let G be a unicyclic graph. Then, two vertices x, y are mutually maximally distant in G if and only if $x, y \in c_2(G) \cup T(G)$.*

We note that any graph $H \in \mathcal{H}_T$ satisfies that $c_2(H)$ is empty (whether the unicyclic graph H has been obtained from T by adding an edge between two non leaf vertices of T), or has cardinality 1 (whether H has been obtained from T by adding an edge between a leaf a non leaf vertex of T), or cardinality 2 (whether H has been obtained by an added edge between two leaves of T).

Proposition 8. *If $\mathcal{H}_T$ is a family of unicyclic graphs obtained from a tree T with t leaves as described above, then $\mathrm{Sd}_s(\mathcal{H}_T) = t-1$.*

Proof. By Remarks 2 and 3, and the fact that $0 \leq |c_2(H)| \leq 2$, we deduce that, for any graph $H \in \mathcal{H}_T$, it follows that H_{SR} contains $|V(G)| - t$ isolated vertices together with either

(i) a subgraph isomorphic to K_t or,
(ii) a subgraph isomorphic to K_{t-1} and one extra vertex adjacent to a subset of vertices of the subgraph K_{t-1} or,
(iii) a subgraph isomorphic to K_{t-2} and two extra vertices (in which case such two vertices are not adjacent), which are adjacent to two different subsets of vertices of the subgraph K_{t-2}.

We note that these, one or two, extra vertices are precisely the leaves of T, for which an incident edge has been added to the tree T in order to obtain H. According to these facts, the simultaneous strong resolving graph $(\mathcal{H}_T)_{SSR}$ has a connected component which is a subgraph of a complete graph K_t whose vertex set is precisely the set of leaves of T. In this sense, by using Theorem 2, we obtain $\mathrm{Sd}_s(\mathcal{H}_T) = \alpha((\mathcal{H}_T)_{SSR}) \leq t-1$.

On the other hand, consider Q is the subgraph induced by the set of vertices corresponding to the leaves of T. If item (i) above occurs for every $H \in \mathcal{H}_T$, then clearly $(\mathcal{H}_T)_{SSR} \cong K_t$ and so

$\mathrm{Sd}_s(\mathcal{H}_T) = \alpha((\mathcal{H}_T)_{SSR}) = t-1$. Suppose now that $\mathrm{Sd}_s(\mathcal{H}_T) < t-1$. This means that there are two vertices x, y corresponding to two leaves of T which are not adjacent in $(\mathcal{H}_T)_{SSR}$. In consequence, there is a unicyclic graph $G_{x,y} \in \mathcal{H}_T$ which was obtained from T by adding the edge xy. However, since $\mathcal{H}_T$ has at least two graphs, there is at least a graph $G' \in \mathcal{H}_T$, other than $G_{x,y}$ in which the vertices x, y are not adjacent. Thus, x, y are MMD in G' and so the edge xy exists in $(\mathcal{H}_T)_{SSR}$, a contradiction with our supposition. Therefore, $\mathrm{Sd}_s(\mathcal{H}_T) \geq t-1$ and we have the desired equality. $\square$

A similar process as described above can be developed in order to get families of graphs for which the simultaneous strong metric dimension of graphs can be computed. However, one may need to use several assumptions while constructing such families. This is based on the fact that computing the simultaneous strong metric dimension is NP-complete for several "very simple" families of graphs (like families of paths, for instance). In connection with this, it would be desirable to find some "properties" satisfied by a graph family in order to decide if it is "easy" to compute its simultaneous strong metric dimension or not.

5. The Particular Case of Cartesian Product Graphs Families

Given two graph families $\mathcal{F}_1 = \{G_1, \ldots, G_r\}$ and $\mathcal{F}_2 = \{H_1, \ldots, H_t\}$ defined over the common sets of vertices V_1 and V_2, respectively, the *Cartesian product graph family* $\mathcal{F}_1 \square \mathcal{F}_2$ is given by the family $\{G_i \square H_j \,:\, G_i \in \mathcal{F}_1,\ H_j \in \mathcal{F}_2\}$. In order to study the simultaneous strong metric dimension of Cartesian product graphs families, we also need the following definition. The *direct product graph family* $\mathcal{F}_1 \times \mathcal{F}_2$ is given by the family $\{G_i \times H_j \,:\, G_i \in \mathcal{F}_1,\ H_j \in \mathcal{F}_2\}$.

We shall next need the following definitions. Given two graphs G and H, the *Cartesian product graph* of G and H is a graph, denoted by $G \square H$, having vertex set $V(G \square H) = V(G) \times V(H)$. In addition, there is an edge between two vertices $(a,b),(c,d) \in V(G \square H)$ if it is satisfied that either ($a = c$ and $bd \in E(H)$) or ($b = d$ and $ac \in E(G)$). In a similar way, the direct product of graphs can be defined. That is, the *direct product graph* of G and H is a graph, denoted by $G \times H$, having vertex set $V(G \times H) = V(G) \times V(H)$. Now, two vertices $(a,b),(c,d)$ are adjacent in the direct product $G \times H$ whenever $ac \in E(G)$ and $bd \in E(H)$.

The first result concerning Cartesian product graphs families is a relationship between the simultaneous strong resolving graph of Cartesian product graphs families and that of its factors. The following equivalent result for the strong metric dimension of graph was given in [22].

Theorem 4 ([22])**.** *Let G and H be two connected graphs. Then, $(G \square H)_{SR} \cong G_{SR} \times H_{SR}$.*

By using the result above and the fact that the simultaneous strong resolving graph of a graph family equals the union (overlapping) of the strong resolving graph of each graph of the family, we deduce the next result.

Theorem 5. *Let $\mathcal{F}_1 = \{G_1, \ldots, G_r\}$ and $\mathcal{F}_2 = \{H_1, \ldots, H_t\}$ be two graph families defined over the common sets of vertices V_1 and V_2, respectively. Then, $(\mathcal{F}_1 \square \mathcal{F}_2)_{SSR} \cong (\mathcal{F}_1)_{SSR} \times (\mathcal{F}_2)_{SSR}$.*

Proof. Since $(\mathcal{F}_1 \square \mathcal{F}_2)_{SSR}$ is given by $\bigcup_{G_i \in \mathcal{F}_1, H_j \in \mathcal{F}_2} (G_i \square H_j)_{SR}$ and $(G_i \square H_j)_{SR} \cong (G_i)_{SR} \times (H_j)_{SR}$ (by Theorem 4), we get that

$$(\mathcal{F}_1 \square \mathcal{F}_2)_{SSR} \cong \bigcup_{G_i \in \mathcal{F}_1, H_j \in \mathcal{F}_2} (G_i \square H_j)_{SR} \cong \bigcup_{G_i \in \mathcal{F}_1, H_j \in \mathcal{F}_2} (G_i)_{SR} \times (H_j)_{SR} \cong (\mathcal{F}_1 \times \mathcal{F}_2)_{SSR},$$

which gives our claim. $\square$

We next give several results concerning the simultaneous strong metric dimension of Cartesian product graph families. In this sense, the result above plays an important role. Now, our next result, which is obtained by using Theorem 2 and Theorem 5, shall be used as an important tool to develop our exposition.

Corollary 1. *Let $\mathcal{F}_1 = \{G_1, \ldots, G_r\}$ and $\mathcal{F}_2 = \{H_1, \ldots, H_t\}$ be two graph families defined over the common sets of vertices V_1 and V_2, respectively. Then,* $\mathrm{Sd}_s(\mathcal{F}_1 \Box \mathcal{F}_2) = \alpha((\mathcal{F}_1)_{SSR} \times (\mathcal{F}_2)_{SSR})$.

Due to the similarity of the results above (in this section) with those obtained in [22] concerning the strong metric dimension of graphs, we note that some analogous reasonings as that ones in [22] shall lead to several results concerning the simultaneous strong metric dimension of graph families. In this sense, we now close our exposition with some problems that are of interest in our point of view.

6. Conclusions and Open Problems

A new approach to study the simultaneous strong metric dimension of graphs families has been presented in this work. That is, we have introduced the notion of simultaneous strong resolving graph of graphs families, and proved the computing the simultaneous strong metric dimension of a family of graphs is equivalent to compute the vertex cover number of this newly introduced simultaneous strong resolving graph. Based on this equivalence, several computational and combinatorial results have been deduced. For instance, we have proved that computing the simultaneous strong metric dimension of families of paths and families of multisubdivided star graphs is NP-hard. As a consequence of the study, a number of open questions have been raised. We next point out several of the most interesting ones:

- Since finding the simultaneous strong metric dimension of graph families is NP-hard, even for relatively simple families of graphs (like families of paths for instance), it would be desirable to describe several other graph families in which this problem could be solved in polynomial time.
- Based on the fact that computing the simultaneous strong metric dimension is NP-hard for several "very simple" families of graphs (like families of paths for instance), it would be desirable to find some structural properties satisfied by a graph family in order to claim that computing its simultaneous strong metric dimension can be efficiently done.
- One of the families studied in [15] was that one containing a graph G and its complement $\overline{G}$. In this sense, it would be interesting to consider the problem of describing the strong resolving graph of $\overline{G}$ and its possible relationship with the strong resolving graph of G, in order to construct the simultaneous strong resolving graph of $\{G, \overline{G}\}$ and thus study its simultaneous strong metric dimension. With this problem, we would also contribute to some open problem presented in [20] concerning describing the structure of the strong resolving graph of several graphs.

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

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