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Existence of a Unique Fixed Point for Nonlinear Contractive Mappings

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Abstract: In a recent work, we established the existence of a unique fixed point for nonlinear contractive self-mappings of a bounded and closed set in a Banach space. In the present paper we extend this result to the case of unbounded sets.

Keywords: Banach space; complete metric space; contractive mapping; fixed point; iterate

MSC: 47H09; 47H10; 54E35; 54E50

1. Introduction

For almost six decades, there has been considerable research activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for instance, [1–15] and the references cited therein. This activity stems from Banach's classical result [16] concerning the existence of a unique fixed point for a strict contraction. In addition, it also concerns, inter alia, the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility, common fixed point problems, and variational inequalities, which find important applications in engineering, medicine, and the natural sciences [14,15,17–21].

In a recent work of ours [22], we established the existence of a unique fixed point for nonlinear contractive self-mappings of a bounded and closed subset of a Banach space. In the present paper, we extend this result to the case of *unbounded* sets.

More precisely, in [22,23] we considered the following class of nonlinear mappings.

Assume that $(X, \|\cdot\|)$ is a Banach space and that $K \subset X$ is a bounded, closed, and convex set. Assume further, that $f : X \to [0, \infty)$ is a continuous function for which $f(0) = 0$, the set $f(K - K)$ is bounded, and the following three properties are valid:

(i) For every positive number ϵ, there is a positive number δ such that for each pair of points $x, y \in K$ satisfying $f(x - y) \leq \delta$, we have $\|x - y\| \leq \epsilon$;

(ii) For every number $\lambda \in (0, 1)$, there exists a number $\phi(\lambda) \in (0, 1)$ for which

$$f(\lambda(x - y)) \leq \phi(\lambda) f(x - y) \text{ for all } x, y \in K;$$

(iii) The function $(x, y) \mapsto f(x - y)$, $x, y \in K$, is uniformly continuous on the set $K \times K$.

Denote by $\mathcal{A}$ the set of all continuous mappings $A : K \to K$ which satisfy

$$f(Ax - Ay) \leq f(x - y) \text{ for all } x, y \in K.$$

For every pair of mappings $A, B \in \mathcal{A}$, define

$$d(A, B) := \sup\{\|Ax - Bx\| : x \in K\}.$$

Evidently, $(\mathcal{A}, d)$ is a complete metric space.

In [23] we established the existence of an everywhere dense and G_δ subset of $\mathcal{A}$, such that each one of its elements possesses a unique fixed point and all the iterates of such an element converge uniformly to this fixed point.

We remark in passing that the main result of [24] is a special case of this result of [23] for the case where $f = \|\cdot\|$. Clearly, the mappings considered in our papers are generalized nonexpansive mappings with respect to the function f. This approach, where the norm is replaced with a general function, had already been used in [25,26], in the study of generalized best approximation problems.

In [22] we improved the results of [23]. Namely, we introduced a notion of a contractive mapping, we showed that most of the mappings in $\mathcal{A}$ (in the sense of Baire category) are contractive, that every contractive mapping possesses a unique fixed point, and that all its iterates converge to this point uniformly. Note that all these results were obtained for a bounded set K. In the present paper we extend one of the main results of [22] to *unbounded* sets. More precisely, we show that even if K is unbounded, every contractive self-mapping of K possesses a unique fixed point and that all its iterates converge to this point, uniformly on bounded subsets of K. Moreover, for this result we do not need property (ii).

2. Main Result

Assume that $(X, \|\cdot\|)$ is a Banach space and that K be a nonempty and closed subset of X. Assume further that $f : X \to [0, \infty)$ is a continuous function with $f(0) = 0$ and that the following two properties are valid:

(P1) For every positive number ϵ, there is a positive number δ such that for every pair of points $x, y \in K$ satisfying $f(x - y) \le \delta$, we have $\|x - y\| \le \epsilon$;

(P2) The function $(x, y) \mapsto f(x - y)$, $x, y \in K$, is uniformly continuous on the set $K \times K$ and for each point $\xi \in K$, the function $f(x - \xi)$, $x \in D$, is bounded on every bounded set $D \subset K$.

Assume that $A : K \to K$ is a continuous mapping, $\psi : [0, \infty) \to [0, 1]$ is a decreasing function satisfying

$$\psi(t) < 1 \text{ for every positive number } t$$

and that

$$f(Ax - Ay) \le \psi(f(x - y))f(x - y) \text{ for each pair of points } x, y \in K. \tag{1}$$

In other words, the mapping A is contractive [13]. We denote the identity operator by A^0.

In Section 3 we establish the following result.

Theorem 1. *The mapping A has a unique fixed point $x_A \in K$ and $A^i x \to x_A$ as $i \to \infty$ for all $x \in K$, uniformly on bounded subsets of K.*

Note that in [27] a particular case of this theorem was obtained for $f(x) = \|x\|$.

3. Proof of Theorem 1

Let $x \in K$. In view of (1), for every integer $n \ge 0$, we have

$$\begin{aligned} f(A^{n+1}x - A^{n+2}x) &\le \psi(f(A^n x - A^{n+1}x))f(A^n x - A^{n+1}x) \\ &\le f(A^n x - A^{n+1}x). \end{aligned} \tag{2}$$

We claim that

$$\lim_{n\to\infty} f(A^n x - A^{n+1}x) = 0.$$

Suppose to the contrary that this does not hold. Then by (2), there exists $\epsilon > 0$ such that

$$f(A^n x - A^{n+1}x) \geq \epsilon \text{ for all integers } n \geq 0. \tag{3}$$

Since the function ψ is decreasing, it follows from (2) and (3) that for every integer $n \geq 0$,

$$f(A^{n+1}x - A^{n+2}x) \leq \psi(\epsilon) f(A^n x - A^{n+1}x).$$

This implies, in its turn, that $\lim_{n\to\infty} f(A^n x - A^{n+1}x) = 0$. This equality contradicts relation (3). Therefore,

$$\lim_{n\to\infty} f(A^n x - A^{n+1}x) = 0 \text{ for each } x \in K, \tag{4}$$

as claimed.

Next, we show that the following property holds:

(P3) for every positive number ϵ, there is a positive number δ such that for every pair of points $x, y \in K$ which satisfy

$$f(x - Ax) \leq \delta \text{ and } f(y - Ay) \leq \delta,$$

we have

$$\|x - y\| \leq \epsilon.$$

Let $\epsilon > 0$. By property (P1), there is

$$\delta_1 \in (0, \epsilon)$$

for which

$$\|x - y\| \leq \epsilon \text{ for all } x, y \in K \text{ satisfying } f(x - y) \leq \delta_1. \tag{5}$$

Property (P2) implies that there exists a number

$$\delta_2 \in (0, \delta_1)$$

such that

$$|f(z_1 - z_2) - f(\xi_1 - \xi_2)| \leq (1 - \psi(\delta_1))\delta_1/2 \tag{6}$$

for all $z_1, z_2, \xi_1, \xi_2 \in K$ satisfying

$$\|z_i - \xi_i\| \leq \delta_2,\ i = 1, 2.$$

By property (P1), there is

$$\delta \in (0, \delta_2)$$

such that the following property holds:

(P4) for each $z_1, z_2 \in K$, if $f(z_1 - z_2) \leq \delta$, then $\|z_1 - z_2\| \leq \delta_2$.

Let $x, y \in K$ satisfy

$$f(x - Ax) \leq \delta \text{ and } f(y - Ay) \leq \delta. \tag{7}$$

We will show that

$$\|x - y\| \leq \epsilon.$$

In view of (5), it is sufficient to prove that

$$f(x - y) \leq \delta_1.$$

Suppose, to the contrary, that this inequality does not hold. Then

$$f(x-y) > \delta_1. \tag{8}$$

Since the function ψ is decreasing, relations (2) and (8) imply that

$$f(Ax - Ay) \le \psi(f(x-y))f(x-y) \le \psi(\delta_1)f(x-y). \tag{9}$$

By (8) and (9), we have

$$\begin{aligned} &f(x-y) - f(Ax - Ay) \\ &\ge f(x-y) - \psi(\delta_1)f(x-y) \ge (1-\psi(\delta_1))\delta_1. \end{aligned} \tag{10}$$

Property (P4) and (7) imply that

$$\|x - Ax\| \le \delta_2 \text{ and } \|y - Ay\| \le \delta_2. \tag{11}$$

In view of (6) and (11), we have

$$|f(x-y) - f(Ax - Ay)| \le (1-\psi(\delta_1))\delta_1/2.$$

This inequality, however, contradicts (10). The contradiction we have reached proves that

$$\|x - y\| \le \epsilon$$

and that property (P3) holds.

Let $x \in K$. In view of (4),

$$\lim_{n\to\infty} f(A^n x - A^{n+1}x) = 0.$$

When combined with property (P3), this implies that $\{A^n x\}_{n=1}^{\infty}$ is a Cauchy sequence. Therefore, there exists the limit $\lim_{n\to\infty} A^n x$. Since the mapping A is continuous, we have

$$A(\lim_{n\to\infty} A^n x) = \lim_{n\to\infty} A^n x$$

and $\lim_{n\to\infty} A^n x$ is a fixed point of the mapping A. Property (P3) now implies the uniqueness of the fixed point of A. Therefore there exists a point $x_A \in K$ such that

$$Ax_A = x_A \tag{12}$$

and for each $x \in K$,

$$\lim_{n\to\infty} A^n x = x_A. \tag{13}$$

We claim that

$$A^n x \to x_A \text{ as } n \to \infty,$$

uniformly on all bounded subsets of K.

Let $M, \epsilon > 0$. By (1) and (12), for all $x \in K$,

$$f(Ax - x_A) \le \psi(f(x - x_A))f(x - x_A). \tag{14}$$

By property (P1), there is $\epsilon_1 \in (0, \epsilon)$ for which

$$\|x - y\| \le \epsilon \text{ for all } x, y \in K \text{ satisfying } f(x-y) \le \epsilon_1. \tag{15}$$

In view of (P2), there is $c_0 > 0$ for which

$$f(z - x_A) \le c_0 \text{ for all } x \in K \text{ satisfying } \|z - x_A\| \le M. \tag{16}$$

Choose an integer

$$n(M, \epsilon) > 1 + (c_0 + M)\epsilon_1^{-1}(1 - \psi(\epsilon_1))^{-1} \tag{17}$$

and let a point $x \in K$ satisfy

$$\|x - x_A\| \le M. \tag{18}$$

We claim that for all integers $n \ge n(M, \epsilon)$, we have

$$\|x_A - A^n x\| \le \epsilon.$$

In view of (15), it suffices to show that for all integers $n \ge n(M, \epsilon)$,

$$f(A^n x - x_A) \le \epsilon_1.$$

By (14), in order to establish this inequality, it is enough to prove that there is an integer

$$m \in [0, n(M, \epsilon)]$$

for which

$$f(A^m x - x_A) \le \epsilon_1.$$

Suppose, to the contrary, that this is not true. Then for each $i \in \{0, \dots, n(M, \epsilon)\}$, we have

$$f(A^i x - x_A) > \epsilon_1. \tag{19}$$

Since the function ψ is decreasing, it follows from (14) and (19) that for each $i \in \{0, \dots, n(M, \epsilon)\}$, we have

$$\begin{aligned} f(A^{i+1}x - x_A) &\le \psi(f(A^i x - x_A)) f(A^i x - x_A) \\ &\le \psi(\epsilon_1) f(A^i x - x_A) \end{aligned}$$

and

$$\begin{aligned} &f(A^i x - x_A) - f(A^{i+1}x - x_A) \\ &\ge (1 - \psi(\epsilon_1)) f(A^i x - x_A) \ge \epsilon_1(1 - \psi(\epsilon_1)). \end{aligned} \tag{20}$$

By (16), (18), and (20),

$$\begin{aligned} c_0 &\ge f(x - x_A) \ge f(x - x_A) - f(A^{n(M,\epsilon)}x - x_A) \\ &= \sum_{i=0}^{n(M,\epsilon)-1} (f(A^i x - x_A) - f(A^{i+1)} x - x_A)) \\ &\ge \epsilon_1(1 - \psi(\epsilon_1)) n(M, \epsilon) \end{aligned}$$

and

$$n(M, \epsilon) \le c_0 \epsilon_1^{-1}(1 - \psi(\epsilon_1))^{-1}.$$

This, however, contradicts (17). The contradiction we have reached completes the proof of Theorem 1.

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