mathematics

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A New Angular Measurement in Minkowski 3-Space

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Received: 20 November 2019; Accepted: 16 December 2019; Published: 2 January 2020

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Abstract: In Lorentz–Minkowski space, the angles between any two non-null vectors have been defined in the sense of the angles in Euclidean space. In this work, the angles relating to lightlike vectors are characterized by the Frenet frame of a pseudo null curve and the angles between any two non-null vectors in Minkowski 3-space. Meanwhile, the explicit measuring methods are demonstrated through several examples.

Keywords: angle; pseudo null curve; lightlike vector; Minkowski space

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1. Introduction

In Einstein's theory of relativity, time, together with the three dimension space, constitutes the four-dimension space–time. As one of the important space–time models in theory of relativity, Lorentz–Minkowski space–time is attracting the attentions of many physicians and mathematicians. One of the remarkable things is that some classical research topics with Riemannian metric are generalized into Lorentz–Minkowski space with a pseudo-Riemannian metric [1,2].

The Lorentz–Minkowski metric divides the vectors into timelike, lightlike (null) or spacelike vectors [1]. Due to the causal character of vectors in this space, some simple problems become a little complicated and strange, especially the ones relating to null vectors, such as null curves, pseudo null curves, B-scrolls, marginally trapped surfaces and so on [3–5]. One of the reasons is the angles relating to lightlike vectors that cannot be defined properly, which restrict some research, depending on angular measurement to some extent. As far as the authors know, this problem is still in the air at present.

The angles between any two nonzero vectors in Euclidean space can be defined through their scalar product. Naturally, the idea can be moved to Lorentz–Minkowski space. In Lorentz–Minkowski space, the angles between any two non-null vectors have been defined in the light of the angles in Euclidean space [6]. However, the method cannot be taken into the angles relating to lightlike vectors because of the character of lightlike vectors, i.e., the norm of lightlike vectors vanishes everywhere.

Considering the relationship between any two independent lightlike vectors and the existing definitions of angles between any two non-null vectors in Minkowski space, an appropriate method is proposed to define the angles relating to lightlike vectors by the Frenet frame of a pseudo null curve and the angles between any two non-null vectors. In Section 2, some fundamental facts about the pseudo null curves and the definitions of angles between any two non-null vectors are recalled. In Section 3, the angles between a lightlike vector and a spacelike vector, a timelike vector or another lightlike vector which is independent to it are defined, respectively. Last but not least, several examples are given explicitly.

Using the new angular measurement proposed in this paper, a lot of research works can be completed systematically. For example, the helix, k-type slant helix, the curves with constant precession, and the constant angle surfaces, which play important roles in the science of biology and physics, such as analyzing the structure of DNA and characterizing the motion of particles in a magnetic field [7,8]. It is of great significance to study the theory of relativity.

The curves in this paper are regular and smooth unless otherwise stated.

2. Preliminaries

A Minkowski 3-space $\mathbb{E}_1^3$ is provided with the standard flat metric given by

$$\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

in terms of the natural coordinate system (x_1, x_2, x_3). Recall that a vector v is said to be spacelike, timelike and lightlike (null), if $\langle v, v \rangle > 0$ or $v = 0$, $\langle v, v \rangle < 0$ and $\langle v, v \rangle = 0$, $(v \neq 0)$, respectively. The norm (modulus) of v is defined by $\|v\| = \sqrt{|\langle v, v \rangle|}$. Comparing to the vectors in Euclidean space, the existence of timelike and lightlike vectors gives some particular properties, as follows:

- Two lightlike vectors x and y are linearly dependent if and only if $\langle x, y \rangle = 0$;
- If x and y are two timelike or lightlike vectors with $\langle x, y \rangle = 0$, then they are lightlike vectors.

Definition 1 ([9]). *Two vectors x, y in $\mathbb{E}_1^3$ are Lorentz orthogonal if and only if $\langle x, y \rangle = 0$.*

For any two vectors $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{E}_1^3$, the exterior product is given by

$$x \times y = (x_3 y_2 - x_2 y_3, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

An arbitrary curve $r(t)$ is spacelike, timelike or lightlike if all of its velocity vectors $r'(t)$ are spacelike, timelike or lightlike. A surface is said to be timelike, spacelike or lightlike if all of its normal vectors are spacelike, timelike or lightlike, respectively. Furthermore, the spacelike curves in $\mathbb{E}_1^3$ can be classified into three kinds according to their principal normal vectors are spacelike, timelike and lightlike, which are called the first and the second kind of spacelike curve and the pseudo null curve, respectively [9]. Among of them, the pseudo null curve is defined as follows:

Definition 2 ([10]). *A spacelike curve $r(t)$ framed by Frenet frame $\{\alpha, \beta, \gamma\}$ in $\mathbb{E}_1^3$ is called a pseudo null curve, if its principal normal vector β and binormal vector γ are linearly independent lightlike (null) vectors.*

Proposition 1 ([10]). *Let $r(s) : \mathbf{I} \to \mathbb{E}_1^3$ be a pseudo null curve parameterized by arc-length s, i.e., $\|r'(s)\| = 1$. Then there exists a unique Frenet frame $\{r'(s) = \alpha, \beta, \gamma\}$ such that*

$$\begin{pmatrix} \alpha'(s) \\ \beta'(s) \\ \gamma'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \kappa(s) & 0 \\ -1 & 0 & -\kappa(s) \end{pmatrix} \begin{pmatrix} \alpha(s) \\ \beta(s) \\ \gamma(s) \end{pmatrix}, \tag{1}$$

where $\langle \alpha, \alpha \rangle = \langle \beta, \gamma \rangle = 1$, $\langle \beta, \beta \rangle = \langle \gamma, \gamma \rangle = \langle \alpha, \beta \rangle = \langle \alpha, \gamma \rangle = 0$ and $\alpha \times \beta = \beta$, $\beta \times \gamma = \alpha$, $\gamma \times \alpha = \gamma$. In sequence, α, β, γ is called the tangent, principal normal and binormal vector field of $r(s)$, respectively. The function $\kappa(s)$ is called the curvature function.

Remark 1. *In some research papers for pseudo null curves such as [10], the function $\kappa(s)$ is also called torsion function. Throughout the paper, the pseudo null curves are parameterized by arc-length s.*

Motivated by the angular measurement in Euclidean 3-space, the angles between any two non-null vectors u and v are defined according to the classification of vectors in $\mathbb{E}_1^3$, as follows [6]:

Definition 3. *Let u and v be spacelike vectors in $\mathbb{E}_1^3$.*

- *If u and v span a timelike vector subspace. Then we have $|\langle u, v\rangle| > \|u\|\|v\|$ and hence, there is a unique positive real number θ such that*
$$|\langle u, v\rangle| = \|u\|\|v\| \cosh\theta. \tag{2}$$
The real number θ is called the Lorentz timelike angle between u and v.
- *If u and v span a spacelike vector subspace. Then we have $|\langle u, v\rangle| \leq \|u\|\|v\|$ and hence, there is a unique real number $\theta \in [0, \frac{\pi}{2}]$ such that*
$$|\langle u, v\rangle| = \|u\|\|v\| \cos\theta. \tag{3}$$
The real number θ is called the Lorentz spacelike angle between u and v.

Definition 4. *Let u and v be future pointing (past pointing) timelike vectors in $\mathbb{E}^3_1$. Then there is a unique non-negative real number θ such that*

$$|\langle u, v\rangle| = \|u\|\|v\| \cosh\theta. \tag{4}$$

The real number θ is called the Lorentz timelike angle between u and v.

Definition 5. *Let u be a spacelike vector and v a future pointing timelike vector in $\mathbb{E}^3_1$. Then there is a unique non-negative real number θ such that*

$$|\langle u, v\rangle| = \|u\|\|v\| \sinh\theta. \tag{5}$$

The real number θ is called the Lorentz timelike angle between u and v.

Remark 2. *Physically, this designation of the future pointing and past pointing timelike vectors corresponds to a choice of an arrow of time at the given point, therefore Equations (4) and (5) include all the definitions of angles between non-null vectors and timelike vectors.*

Obviously, the angles recalled in Definitions 3–5 do not include the angles relating to lightlike vectors. In what follows, we will seek a method to fill this gap.

3. Main Conclusions

In this section, we will focus on the angle between any two lightlike vectors and the angles between a lightlike vector and a spacelike vector or a timelike vector, respectively.

3.1. The Angle between Any Two Lightlike Vectors

Let $r(s)$ be a pseudo null curve framed by $\{\alpha, \beta, \gamma\}$. From the lightlike vector β, we let $u = \omega_1(s)\beta$ be an arbitrary lightlike vector which is independent to γ. Similarly, from the lightlike vector γ, we can assume another arbitrary lightlike vector $v = \omega_2(s)\gamma$ which is independent to β, where $\omega_1(s)$ and $\omega_2(s)$ are non-zero smooth functions of arc-length s.

From the Frenet frame of $r(s)$, we know that $\langle u, v\rangle = \omega_1(s)\omega_2(s)$ and $\langle cu, v\rangle = c\omega_1(s)\omega_2(s)$, $(c \in \mathbb{R} - \{0\})$ which satisfy the axiom of angles. Thus, we have the following definition.

Definition 6. *The angle $\omega(u, v)$ between any two lightlike vectors $u = \omega_1(s)\beta$ and $v = \omega_2(s)\gamma$ is defined by*

$$\omega(u, v) = \omega_1(s)\omega_2(s).$$

Remark 3. *Obviously, when $\omega_1(s) = 1$, $\omega(u, v) = \langle\beta, v\rangle = \omega_2(s)$, and when $\omega_2(s) = 1$, $\omega(u, v) = \langle u, \gamma\rangle = \omega_1(s)$. Therefore, the function $\omega_1(s)$, $\omega_2(s)$ represents the rotation angle between u and γ, between v and β, respectively. In particular, when $\omega_1(s) = \omega_2(s) = 1$, $\omega(u, v) = \langle\beta, \gamma\rangle = 1$ which coincides with the Frenet frame of the pseudo null curve $r(s)$.*

3.2. The Angle between a Lightlike Vector and a Spacelike Vector

Let $r(s)$ be a pseudo null curve framed by $\{\alpha, \beta, \gamma\}$, $u = \omega_1(s)\beta$ and $v = \omega_2(s)\gamma$ be any two lightlike vectors. Assume $a_1 = \lambda_1\alpha + \lambda_2\beta + \lambda_3\gamma$ be an arbitrary unit spacelike vector, where $\lambda_i = \lambda_i(s)$, $(i = 1, 2, 3)$ are smooth functions and $\lambda_1^2 + 2\lambda_2\lambda_3 = 1$.

From the Frenet frame of $r(s)$, we know that $\langle u, a_1\rangle = \lambda_3(s)\omega_1(s)$, $\langle v, a_1\rangle = \lambda_2(s)\omega_2(s)$ and $\langle cu, a_1\rangle = c\lambda_3(s)\omega_1(s)$, $\langle cv, a_1\rangle = c\lambda_2(s)\omega_2(s)$, $(c \in \mathbb{R} - \{0\})$ which satisfy the axiom of angles. Then the angles $\omega(u, a_1)$ and $\omega(v, a_1)$ can be defined by

$$\begin{cases} \omega(u, a_1) = \lambda_3(s)\omega_1(s), \\ \omega(v, a_1) = \lambda_2(s)\omega_2(s). \end{cases} \tag{6}$$

Considering the Frenet frame of $r(s)$, we can fix two vectors, a spacelike vector $b_1(s) = \frac{\sqrt{2}}{2}(\beta + \gamma)$ and a timelike vector $b_2(s) = \frac{\sqrt{2}}{2}(\beta - \gamma)$. Due to a_1 is spacelike, b_2 is timelike, assuming θ is the angle between a_1 and b_2, from Equation (5), we have

$$|\langle b_2, a_1\rangle| = \frac{\sqrt{2}}{2}|\lambda_2 - \lambda_3| = \sinh\theta, \ (\theta \geq 0). \tag{7}$$

On the other hand, $b_1 \times a_1 = \frac{1}{\sqrt{2}}[(\lambda_3 - \lambda_2)\alpha - \lambda_1(\beta - \gamma)]$, then

$$\langle b_1 \times a_1, b_1 \times a_1\rangle = \frac{1}{2}[(\lambda_2 + \lambda_3)^2 - 2].$$

Notice that $\lambda_2\lambda_3 \leq \frac{1}{2}$ from $\lambda_1^2 + 2\lambda_2\lambda_3 = 1$, then we have

Case 1: when $|\lambda_2 + \lambda_3| > \sqrt{2}$, $b_1 \times a_1$ is a spacelike vector, i.e., b_1 and a_1 span a timelike subspace. Assume ξ_1 is the angle between a_1 and b_1. From Equation (2), we know

$$|\langle b_1, a_1\rangle| = \frac{\sqrt{2}}{2}|\lambda_2 + \lambda_3| = \cosh\xi_1, \ (\xi_1 > 0). \tag{8}$$

From Equations (7) and (8), we have

1. when $\frac{1}{2\lambda_3} \leq \lambda_2 < -\sqrt{2} - \lambda_3(\lambda_3 < -\frac{\sqrt{2}}{2})$, we have

$$\begin{cases} \lambda_2 = -\frac{\sqrt{2}}{2}(\cosh\xi_1 - \sinh\theta), \\ \lambda_3 = -\frac{\sqrt{2}}{2}(\cosh\xi_1 + \sinh\theta); \end{cases} \tag{9}$$

2. when $\sqrt{2} - \lambda_3 < \lambda_2 \leq \frac{1}{2\lambda_3}(\lambda_3 > \frac{\sqrt{2}}{2})$, we have

$$\begin{cases} \lambda_2 = \frac{\sqrt{2}}{2}(\cosh\xi_1 - \sinh\theta), \\ \lambda_3 = \frac{\sqrt{2}}{2}(\cosh\xi_1 + \sinh\theta); \end{cases} \tag{10}$$

3. when $\sqrt{2} - \lambda_3 < \lambda_2 \leq \frac{1}{2\lambda_3}(0 < \lambda_3 < \frac{\sqrt{2}}{2})$ or $\lambda_2 > \sqrt{2} - \lambda_3(\lambda_3 \leq 0)$, we have

$$\begin{cases} \lambda_2 = \frac{\sqrt{2}}{2}(\cosh\xi_1 + \sinh\theta), \\ \lambda_3 = \frac{\sqrt{2}}{2}(\cosh\xi_1 - \sinh\theta); \end{cases} \tag{11}$$

4. when $\frac{1}{2\lambda_3} \leq \lambda_2 < -\sqrt{2} - \lambda_3(-\frac{\sqrt{2}}{2} < \lambda_3 < 0)$ or $\lambda_2 < -\sqrt{2} - \lambda_3(\lambda_3 \geq 0)$, we have

$$\begin{cases} \lambda_2 = -\frac{\sqrt{2}}{2}(\cosh\xi_1 + \sinh\theta), \\ \lambda_3 = -\frac{\sqrt{2}}{2}(\cosh\xi_1 - \sinh\theta). \end{cases} \tag{12}$$

Case 2: when $|\lambda_2 + \lambda_3| < \sqrt{2}$, $b_1 \times a_1$ is a timelike vector, i.e., b_1 and a_1 span a spacelike subspace. Assume ξ_2 is the angle between a_1 and b_1. From Equation (3), we know

$$|\langle b_1, a_1 \rangle| = \frac{\sqrt{2}}{2}|\lambda_2 + \lambda_3| = \cos\xi_2, \ (\xi_2 \in [0, \frac{\pi}{2}]). \tag{13}$$

From Equations (7) and (13), we have

1. when $\lambda_3 \leq \lambda_2 < -\lambda_3(-\frac{\sqrt{2}}{2} < \lambda_3 < 0)$ or $-\sqrt{2} - \lambda_3 < \lambda_2 < -\lambda_3(\lambda_3 \leq -\frac{\sqrt{2}}{2})$, we have

$$\begin{cases} \lambda_2 = -\frac{\sqrt{2}}{2}(\cos\xi_2 - \sinh\theta), \\ \lambda_3 = -\frac{\sqrt{2}}{2}(\cos\xi_2 + \sinh\theta); \end{cases} \tag{14}$$

2. when $-\lambda_3 \leq \lambda_2 < \lambda_3(0 < \lambda_3 < \frac{\sqrt{2}}{2})$ or $-\lambda_3 \leq \lambda_2 < \sqrt{2} - \lambda_3(\lambda_3 \geq \frac{\sqrt{2}}{2})$, we have

$$\begin{cases} \lambda_2 = \frac{\sqrt{2}}{2}(\cos\xi_2 - \sinh\theta), \\ \lambda_3 = \frac{\sqrt{2}}{2}(\cos\xi_2 + \sinh\theta); \end{cases} \tag{15}$$

3. when $\lambda_3 \leq \lambda_2 < \sqrt{2} - \lambda_3(0 \leq \lambda_3 < \frac{\sqrt{2}}{2})$ or $-\lambda_3 \leq \lambda_2 < \sqrt{2} - \lambda_3(\lambda_3 < 0)$, we have

$$\begin{cases} \lambda_2 = \frac{\sqrt{2}}{2}(\cos\xi_2 + \sinh\theta), \\ \lambda_3 = \frac{\sqrt{2}}{2}(\cos\xi_2 - \sinh\theta); \end{cases} \tag{16}$$

4. when $-\sqrt{2} - \lambda_3 < \lambda_2 < \lambda_3(-\frac{\sqrt{2}}{2} < \lambda_3 < 0)$ or $-\sqrt{2} - \lambda_3 < \lambda_2 < -\lambda_3(\lambda_3 \geq 0)$, we have

$$\begin{cases} \lambda_2 = -\frac{\sqrt{2}}{2}(\cos\xi_2 + \sinh\theta), \\ \lambda_3 = -\frac{\sqrt{2}}{2}(\cos\xi_2 - \sinh\theta). \end{cases} \tag{17}$$

Case 3: when $|\lambda_2 + \lambda_3| = \sqrt{2}$, b_1 and a_1 span a lightlike subspace. And

$$|\langle b_1, a_1 \rangle| = \frac{\sqrt{2}}{2}|\lambda_2 + \lambda_3| = 1. \tag{18}$$

From Equations (7) and (18), we have

1. when $\lambda_2 = -\sqrt{2} - \lambda_3(\lambda_3 \leq -\frac{\sqrt{2}}{2})$, we have

$$\begin{cases} \lambda_2 = \frac{\sqrt{2}}{2}(\sinh\theta - 1), \\ \lambda_3 = -\frac{\sqrt{2}}{2}(\sinh\theta + 1); \end{cases} \tag{19}$$

2. when $\lambda_2 = \sqrt{2} - \lambda_3(\lambda_3 > \frac{\sqrt{2}}{2})$, we have

$$\begin{cases} \lambda_2 = -\frac{\sqrt{2}}{2}(\sinh\theta - 1), \\ \lambda_3 = \frac{\sqrt{2}}{2}(\sinh\theta + 1); \end{cases} \tag{20}$$

3. when $\lambda_2 = \sqrt{2} - \lambda_3(\lambda_3 \leq \frac{\sqrt{2}}{2})$, we have

$$\begin{cases} \lambda_2 = \frac{\sqrt{2}}{2}(\sinh\theta + 1), \\ \lambda_3 = -\frac{\sqrt{2}}{2}(\sinh\theta - 1); \end{cases} \tag{21}$$

4. when $\lambda_2 = -\sqrt{2} - \lambda_3(\lambda_3 > -\frac{\sqrt{2}}{2})$, we have

$$\begin{cases} \lambda_2 = -\frac{\sqrt{2}}{2}(\sinh\theta + 1), \\ \lambda_3 = \frac{\sqrt{2}}{2}(\sinh\theta - 1). \end{cases} \tag{22}$$

Substituting Equations (9)–(12), (14)–(17) and (19)–(22) to Equation (6), we have the following definition.

Definition 7. *Let $r(s)$ be a pseudo null curve framed by $\{\alpha, \beta, \gamma\}$ in $\mathbb{E}^3_1$, $u = \omega_1(s)\beta$ and $v = \omega_2(s)\gamma$ any two lightlike vectors; $b_1 = \frac{\sqrt{2}}{2}(\beta + \gamma)$ a unit spacelike vector and $b_2 = \frac{\sqrt{2}}{2}(\beta - \gamma)$ a unit timelike vector; $a_1 = \lambda_1\alpha + \lambda_2\beta + \lambda_3\gamma$, $\lambda_i = \lambda_i(s), (i = 1, 2, 3)$ an arbitrary unit spacelike vector. Then the angles $\omega(u, a_1)$ and $\omega(v, a_1)$ can be defined explicitly as*

- *if a_1 and b_1 span a timelike subspace, then*

  1. *when $\frac{1}{2\lambda_3} \le \lambda_2 < -\sqrt{2} - \lambda_3(\lambda_3 < -\frac{\sqrt{2}}{2})$, we get*

$$\begin{cases} \omega(u, a_1) = -\frac{\sqrt{2}}{2}(\cosh\xi_1 + \sinh\theta)\omega_1(s), \\ \omega(v, a_1) = -\frac{\sqrt{2}}{2}(\cosh\xi_1 - \sinh\theta)\omega_2(s); \end{cases}$$

  2. *when $\sqrt{2} - \lambda_3 < \lambda_2 \le \frac{1}{2\lambda_3}(\lambda_3 > \frac{\sqrt{2}}{2})$, we get*

$$\begin{cases} \omega(u, a_1) = \frac{\sqrt{2}}{2}(\cosh\xi_1 + \sinh\theta)\omega_1(s), \\ \omega(v, a_1) = \frac{\sqrt{2}}{2}(\cosh\xi_1 - \sinh\theta)\omega_2(s); \end{cases}$$

  3. *when $\sqrt{2} - \lambda_3 < \lambda_2 \le \frac{1}{2\lambda_3}(0 < \lambda_3 < \frac{\sqrt{2}}{2})$ or $\lambda_2 > \sqrt{2} - \lambda_3(\lambda_3 \le 0)$, we get*

$$\begin{cases} \omega(u, a_1) = \frac{\sqrt{2}}{2}(\cosh\xi_1 - \sinh\theta)\omega_1(s), \\ \omega(v, a_1) = \frac{\sqrt{2}}{2}(\cosh\xi_1 + \sinh\theta)\omega_2(s); \end{cases}$$

  4. *when $\frac{1}{2\lambda_3} \le \lambda_2 < -\sqrt{2} - \lambda_3(-\frac{\sqrt{2}}{2} < \lambda_3 < 0)$ or $\lambda_2 < -\sqrt{2} - \lambda_3(\lambda_3 \ge 0)$, we get*

$$\begin{cases} \omega(u, a_1) = -\frac{\sqrt{2}}{2}(\cosh\xi_1 - \sinh\theta)\omega_1(s), \\ \omega(v, a_1) = -\frac{\sqrt{2}}{2}(\cosh\xi_1 + \sinh\theta)\omega_2(s); \end{cases}$$

- *if a_1 and b_1 span a spacelike subspace, then*

  1. *when $\lambda_3 \le \lambda_2 < -\lambda_3(-\frac{\sqrt{2}}{2} < \lambda_3 < 0)$ or $-\sqrt{2} - \lambda_3 < \lambda_2 < -\lambda_3(\lambda_3 \le -\frac{\sqrt{2}}{2})$, we get*

$$\begin{cases} \omega(u, a_1) = -\frac{\sqrt{2}}{2}(\cos\xi_2 + \sinh\theta)\omega_1(s), \\ \omega(v, a_1) = -\frac{\sqrt{2}}{2}(\cos\xi_2 - \sinh\theta)\omega_2(s); \end{cases}$$

  2. *when $-\lambda_3 \le \lambda_2 < \lambda_3(0 < \lambda_3 < \frac{\sqrt{2}}{2})$ or $-\lambda_3 \le \lambda_2 < \sqrt{2} - \lambda_3(\lambda_3 \ge \frac{\sqrt{2}}{2})$, we get*

$$\begin{cases} \omega(u, a_1) = \frac{\sqrt{2}}{2}(\cos\xi_2 + \sinh\theta)\omega_1(s), \\ \omega(v, a_1) = \frac{\sqrt{2}}{2}(\cos\xi_2 - \sinh\theta)\omega_2(s); \end{cases}$$

3. *when* $\lambda_3 \le \lambda_2 < \sqrt{2} - \lambda_3 (0 \le \lambda_3 < \frac{\sqrt{2}}{2})$ *or* $-\lambda_3 \le \lambda_2 < \sqrt{2} - \lambda_3 (\lambda_3 < 0)$, *we get*

$$\begin{cases} \omega(u, a_1) = \frac{\sqrt{2}}{2}(\cos \xi_2 - \sinh \theta)\omega_1(s), \\ \omega(v, a_1) = \frac{\sqrt{2}}{2}(\cos \xi_2 + \sinh \theta)\omega_2(s); \end{cases}$$

4. *when* $-\sqrt{2} - \lambda_3 < \lambda_2 < \lambda_3 (-\frac{\sqrt{2}}{2} < \lambda_3 < 0)$ *or* $-\sqrt{2} - \lambda_3 < \lambda_2 < -\lambda_3 (\lambda_3 \ge 0)$, *we get*

$$\begin{cases} \omega(u, a_1) = -\frac{\sqrt{2}}{2}(\cos \xi_2 - \sinh \theta)\omega_1(s), \\ \omega(v, a_1) = -\frac{\sqrt{2}}{2}(\cos \xi_2 + \sinh \theta)\omega_2(s); \end{cases}$$

- *if* a_1 *and* b_1 *span a lightlike subspace, then*

1. *when* $\lambda_2 = -\sqrt{2} - \lambda_3 (\lambda_3 \le -\frac{\sqrt{2}}{2})$, *we get*

$$\begin{cases} \omega(u, a_1) = -\frac{\sqrt{2}}{2}(\sinh \theta + 1)\omega_1(s), \\ \omega(v, a_1) = \frac{\sqrt{2}}{2}(\sinh \theta - 1)\omega_2(s); \end{cases}$$

2. *when* $\lambda_2 = \sqrt{2} - \lambda_3 (\lambda_3 > \frac{\sqrt{2}}{2})$, *we get*

$$\begin{cases} \omega(u, a_1) = \frac{\sqrt{2}}{2}(\sinh \theta + 1)\omega_1(s), \\ \omega(v, a_1) = -\frac{\sqrt{2}}{2}(\sinh \theta - 1)\omega_2(s); \end{cases}$$

3. *when* $\lambda_2 = \sqrt{2} - \lambda_3 (\lambda_3 \le \frac{\sqrt{2}}{2})$, *we get*

$$\begin{cases} \omega(u, a_1) = -\frac{\sqrt{2}}{2}(\sinh \theta - 1)\omega_1(s), \\ \omega(v, a_1) = \frac{\sqrt{2}}{2}(\sinh \theta + 1)\omega_2(s); \end{cases}$$

4. *when* $\lambda_2 = -\sqrt{2} - \lambda_3 (\lambda_3 > -\frac{\sqrt{2}}{2})$, *we get*

$$\begin{cases} \omega(u, a_1) = \frac{\sqrt{2}}{2}(\sinh \theta - 1)\omega_1(s), \\ \omega(v, a_1) = -\frac{\sqrt{2}}{2}(\sinh \theta + 1)\omega_2(s), \end{cases}$$

where θ *is the angle between* a_1 *and* b_2, $\xi_{1,2}$ *is the angle between* a_1 *and* b_1, *and* $\omega_1(s) = \omega(u, \gamma)$, $\omega_2(s) = \omega(v, \beta)$ *is the angle between* u *and* γ, *between* v *and* β, *respectively.*

Remark 4. *Particularly, when* $\lambda_2 = \lambda_3 = 0$, *then* $a_1 = \pm\alpha$ *which is orthogonal to* u *and* v; *when* $\lambda_2 = 0$, $\lambda_3 \ne 0$, *then* $a_1 = \pm\alpha + \lambda_3\gamma$ *which is orthogonal to* v; *when* $\lambda_2 \ne 0, \lambda_3 = 0$, *then* $a_1 = \pm\alpha + \lambda_2\beta$ *which is orthogonal to* u.

Remark 5. *Obviously, if* $\omega_1(s) = 1$ *or* $\omega_2(s) = 1$, *then* $\omega(\beta, a_1)$ *and* $\omega(\gamma, a_1)$ *are decided by the angles* θ *and* $\xi_{1,2}$, *completely.*

3.3. The Angle between a Lightlike Vector and a Timelike Vector

Let $r(s)$ be a pseudo null curve framed by $\{\alpha, \beta, \gamma\}$, $u = \omega_1(s)\beta$ and $v = \omega_2(s)\gamma$ any two lightlike vectors. Assume $a_2 = \mu_1\alpha + \mu_2\beta + \mu_3\gamma$ be an arbitrary unit timelike vector, where $\mu_i = \mu_i(s)$, $(i = 1, 2, 3)$ are smooth functions and $\mu_1^2 + 2\mu_2\mu_3 = -1$.

From the Frenet frame of $r(s)$, we know that $\langle u, a_2\rangle = \mu_3(s)\omega_1(s), \langle v, a_2\rangle = \mu_2(s)\omega_2(s)$ and $\langle cu, a_2\rangle = c\mu_3(s)\omega_1(s), \langle cv, a_2\rangle = c\mu_2(s)\omega_2(s), (c \in \mathbb{R} - \{0\})$ which satisfy the axiom of angles. Then the angles $\omega(u, a_2)$ and $\omega(v, a_2)$ can be defined by

$$\begin{cases} \omega(u, a_2) = \mu_3(s)\omega_1(s), \\ \omega(v, a_2) = \mu_2(s)\omega_2(s). \end{cases} \tag{23}$$

Considering the Frenet frame of $r(s)$, we can fix two vectors, a spacelike vector $b_1(s) = \frac{\sqrt{2}}{2}(\beta + \gamma)$ and a timelike vector $b_2(s) = \frac{\sqrt{2}}{2}(\beta - \gamma)$. Assume η_1 be the angle between a_2 and b_1, η_2 the angle between a_2 and b_2. Due to b_1 is spacelike, a_2 and b_2 are timelike, from Equations (4) and (5), we know

$$|\langle b_1, a_2\rangle| = \frac{\sqrt{2}}{2}|\mu_2 + \mu_3| = \sinh \eta_1, \ (\eta_1 \geq 0)$$

and

$$|\langle b_2, a_2\rangle| = \frac{\sqrt{2}}{2}|\mu_2 - \mu_3| = \cosh \eta_2, \ (\eta_2 \geq 0).$$

Notice that $\mu_2\mu_3 \leq -\frac{1}{2}$ from $\mu_1^2 + 2\mu_2\mu_3 = -1$. Then we have

1. when $-\frac{1}{2\mu_3} \leq \mu_2 < -\mu_3(\mu_3 < -\frac{\sqrt{2}}{2})$, we get

$$\begin{cases} \mu_2 = -\frac{\sqrt{2}}{2}(\sinh \eta_1 - \cosh \eta_2), \\ \mu_3 = -\frac{\sqrt{2}}{2}(\sinh \eta_1 + \cosh \eta_2); \end{cases} \tag{24}$$

2. when $-\mu_3 \leq \mu_2 \leq -\frac{1}{2\mu_3}(\mu_3 \geq \frac{\sqrt{2}}{2})$, we get

$$\begin{cases} \mu_2 = \frac{\sqrt{2}}{2}(\sinh \eta_1 - \cosh \eta_2), \\ \mu_3 = \frac{\sqrt{2}}{2}(\sinh \eta_1 + \cosh \eta_2); \end{cases} \tag{25}$$

3. when $\mu_2 \geq -\mu_3(\mu_3 < -\frac{\sqrt{2}}{2})$ or $\mu_2 \geq -\frac{1}{2\mu_3}(-\frac{\sqrt{2}}{2} \leq \mu_3 < 0)$ we get

$$\begin{cases} \mu_2 = \frac{\sqrt{2}}{2}(\sinh \eta_1 + \cosh \eta_2), \\ \mu_3 = \frac{\sqrt{2}}{2}(\sinh \eta_1 - \cosh \eta_2); \end{cases} \tag{26}$$

4. when $\mu_2 < -\mu_3(\mu_3 \geq \frac{\sqrt{2}}{2})$ or $\mu_2 \leq -\frac{1}{2\mu_3}(0 < \mu_3 < \frac{\sqrt{2}}{2})$, we get

$$\begin{cases} \mu_2 = -\frac{\sqrt{2}}{2}(\sinh \eta_1 + \cosh \eta_2), \\ \mu_3 = -\frac{\sqrt{2}}{2}(\sinh \eta_1 - \cosh \eta_2). \end{cases} \tag{27}$$

Taking Equations (24)–(27) into Equation (23), we have the following definition.

Definition 8. *Let $r(s)$ be a pseudo null curve framed by $\{\alpha, \beta, \gamma\}$ in $\mathbb{E}_1^3$, $u = \omega_1(s)\beta$ and $v = \omega_2(s)\gamma$ any two lightlike vectors; $b_1 = \frac{\sqrt{2}}{2}(\beta + \gamma)$ a unit spacelike vector and $b_2 = \frac{\sqrt{2}}{2}(\beta - \gamma)$ a unit timelike vector; $a_2 = \mu_1\alpha + \mu_2\beta + \mu_3\gamma$, $\mu_i = \mu_i(s), (i = 1, 2, 3)$ an arbitrary unit timelike vector. Then the angles $\omega(u, a_2)$ and $\omega(v, a_2)$ can be defined explicitly as*

1. *when $-\frac{1}{2\mu_3} \leq \mu_2 < -\mu_3(\mu_3 < -\frac{\sqrt{2}}{2})$, we get*

$$\begin{cases} \omega(u, a_2) = -\frac{\sqrt{2}}{2}(\sinh \eta_1 + \cosh \eta_2)\omega_1(s), \\ \omega(v, a_2) = -\frac{\sqrt{2}}{2}(\sinh \eta_1 - \cosh \eta_2)\omega_2(s); \end{cases}$$

2. *when* $-\mu_3 \leq \mu_2 \leq -\frac{1}{2\mu_3}(\mu_3 \geq \frac{\sqrt{2}}{2})$, *we get*

$$\begin{cases} \omega(u, a_2) = \frac{\sqrt{2}}{2}(\sinh \eta_1 + \cosh \eta_2)\omega_1(s), \\ \omega(v, a_2) = \frac{\sqrt{2}}{2}(\sinh \eta_1 - \cosh \eta_2)\omega_2(s); \end{cases}$$

3. *when* $\mu_2 \geq -\mu_3(\mu_3 < -\frac{\sqrt{2}}{2})$ *or* $\mu_2 \geq -\frac{1}{2\mu_3}(-\frac{\sqrt{2}}{2} \leq \mu_3 < 0)$, *we get*

$$\begin{cases} \omega(u, a_2) = \frac{\sqrt{2}}{2}(\sinh \eta_1 - \cosh \eta_2)\omega_1(s), \\ \omega(v, a_2) = \frac{\sqrt{2}}{2}(\sinh \eta_1 + \cosh \eta_2)\omega_2(s); \end{cases}$$

4. *when* $\mu_2 < -\mu_3(\mu_3 \geq \frac{\sqrt{2}}{2})$ *or* $\mu_2 \leq -\frac{1}{2\mu_3}(0 < \mu_3 < \frac{\sqrt{2}}{2})$, *we get*

$$\begin{cases} \omega(u, a_2) = -\frac{\sqrt{2}}{2}(\sinh \eta_1 - \cosh \eta_2)\omega_1(s), \\ \omega(v, a_2) = -\frac{\sqrt{2}}{2}(\sinh \eta_1 + \cosh \eta_2)\omega_2(s), \end{cases}$$

where η_1 *is the angle between* a_2 *and* b_1, η_2 *is the angle between* a_2 *and* b_2, *and* $\omega_1(s) = \omega(u, \gamma)$, $\omega_2(s) = \omega(v, \beta)$ *is the angle between* u *and* γ, *between* v *and* β, *respectively.*

Remark 6. *Obviously, if* $\omega_1(s) = 1$ *or* $\omega_2(s) = 1$, *then* $\omega(\beta, a_2)$, $\omega(\gamma, a_2)$ *are decided by the angles* η_1 *and* η_2, *completely.*

Example 1. *Let* $r(s)$ *be a pseudo null curve framed by* $\{\alpha, \beta, \gamma\}$ *and* $u = \sin s\beta$, $v = \cos s\gamma$ *two lightlike vectors. Then according to Definition 6, the angle between* u *and* v *is*

$$\omega(u, v) = \sin s \cos s.$$

Example 2. *Let* $r(s)$ *be a pseudo null curve framed by* $\{\alpha, \beta, \gamma\}$ *and* $u = e^s\beta$ *a lightlike vector. There is a unit spacelike vector* $x = x_1\alpha + x_2\beta + x_3\gamma$, $x_i = x_i(s)$, $(i = 1, 2, 3)$ *whose intersection angles with* $b_1 = \frac{\sqrt{2}}{2}(\beta + \gamma)$ *and* $b_2 = \frac{\sqrt{2}}{2}(\beta - \gamma)$ *are all* $\frac{\pi}{4}$. *From Definition 7, the angle* $\omega(u, x)$ *can be expressed as follows:*

1. *when* x *and* b_1 *span a timelike subspace,* $\omega_1(s) = e^s$ *and* $\xi_1 = \theta = \frac{\pi}{4}$, *we have*

   - *if* $\frac{1}{2x_3} \leq x_2 < -\sqrt{2} - x_3(x_3 < -\frac{\sqrt{2}}{2})$, *then*

$$\omega(u, x) = -\frac{\sqrt{2}}{2}(\cosh \frac{\pi}{4} + \sinh \frac{\pi}{4})e^s = -\frac{\sqrt{2}}{2}e^{s+\frac{\pi}{4}};$$

   - *if* $\sqrt{2} - x_3 < x_2 \leq \frac{1}{2x_3}(x_3 > \frac{\sqrt{2}}{2})$, *then*

$$\omega(u, x) = \frac{\sqrt{2}}{2}(\cosh \frac{\pi}{4} + \sinh \frac{\pi}{4})e^s = \frac{\sqrt{2}}{2}e^{s+\frac{\pi}{4}};$$

   - *if* $\sqrt{2} - x_3 < x_2 \leq \frac{1}{2x_3}(0 < x_3 < \frac{\sqrt{2}}{2})$ *or* $x_2 > \sqrt{2} - x_3(x_3 \leq 0)$, *then*

$$\omega(u, x) = \frac{\sqrt{2}}{2}(\cosh \frac{\pi}{4} - \sinh \frac{\pi}{4})e^s = \frac{\sqrt{2}}{2}e^{s-\frac{\pi}{4}};$$

   - *if* $\frac{1}{2x_3} \leq x_2 < -\sqrt{2} - x_3(-\frac{\sqrt{2}}{2} < x_3 < 0)$ *or* $x_2 < -\sqrt{2} - x_3(x_3 \geq 0)$, *then*

$$\omega(u, x) = -\frac{\sqrt{2}}{2}(\cosh \frac{\pi}{4} - \sinh \frac{\pi}{4})e^s = -\frac{\sqrt{2}}{2}e^{s-\frac{\pi}{4}};$$

2. *when x and* b_1 *span a spacelike subspace,* $\omega_1(s) = e^s$ *and* $\xi_2 = \theta = \frac{\pi}{4}$*, we have*

   - *if* $x_3 \leq x_2 < -x_3(-\frac{\sqrt{2}}{2} < x_3 < 0)$ *or* $-\sqrt{2} - x_3 < x_2 < -x_3(x_3 \leq -\frac{\sqrt{2}}{2})$*, then*

$$\omega(u,x) = -\frac{\sqrt{2}}{2}(\cos\frac{\pi}{4} + \sinh\frac{\pi}{4})e^s = -\frac{1}{2}e^s - \frac{\sqrt{2}}{4}(e^{s+\frac{\pi}{4}} - e^{s-\frac{\pi}{4}});$$

   - *if* $-x_3 \leq x_2 < x_3(0 < x_3 < \frac{\sqrt{2}}{2})$ *or* $-x_3 \leq x_2 < \sqrt{2} - x_3(x_3 \geq \frac{\sqrt{2}}{2})$*, then*

$$\omega(u,x) = \frac{\sqrt{2}}{2}(\cos\frac{\pi}{4} + \sinh\frac{\pi}{4})e^s = \frac{1}{2}e^s + \frac{\sqrt{2}}{4}(e^{s+\frac{\pi}{4}} - e^{s-\frac{\pi}{4}});$$

   - *if* $x_3 \leq x_2 < \sqrt{2} - x_3(0 \leq x_3 < \frac{\sqrt{2}}{2})$ *or* $-x_3 \leq x_2 < \sqrt{2} - x_3(x_3 < 0)$*, then*

$$\omega(u,x) = \frac{\sqrt{2}}{2}(\cos\frac{\pi}{4} - \sinh\frac{\pi}{4})e^s = \frac{1}{2}e^s - \frac{\sqrt{2}}{4}(e^{s+\frac{\pi}{4}} - e^{s-\frac{\pi}{4}});$$

   - *if* $-\sqrt{2} - x_3 < x_2 < x_3(-\frac{\sqrt{2}}{2} < x_3 < 0)$ *or* $-\sqrt{2} - x_3 < x_2 < -x_3(x_3 \geq 0)$*, then*

$$\omega(u,x) = -\frac{\sqrt{2}}{2}(\cos\frac{\pi}{4} - \sinh\frac{\pi}{4})e^s = -\frac{1}{2}e^s + \frac{\sqrt{2}}{4}(e^{s+\frac{\pi}{4}} - e^{s-\frac{\pi}{4}});$$

3. *when x and* b_1 *span a lightlike subspace,* $\omega_1(s) = e^s$ *and* $\theta = \frac{\pi}{4}$*, we have*

   - *if* $x_2 = -\sqrt{2} - x_3(x_3 \leq -\frac{\sqrt{2}}{2})$*, then*

$$\omega(u,x) = -\frac{\sqrt{2}}{2}(\sinh\theta + 1)e^s = -\frac{\sqrt{2}}{2}e^s - \frac{\sqrt{2}}{4}(e^{s+\frac{\pi}{4}} - e^{s-\frac{\pi}{4}});$$

   - *if* $x_2 = \sqrt{2} - x_3(x_3 > \frac{\sqrt{2}}{2})$*, then*

$$\omega(u,x) = \frac{\sqrt{2}}{2}(\sinh\theta + 1)e^s = \frac{\sqrt{2}}{2}e^s + \frac{\sqrt{2}}{4}(e^{s+\frac{\pi}{4}} - e^{s-\frac{\pi}{4}});$$

   - *if* $x_2 = \sqrt{2} - x_3(x_3 \leq \frac{\sqrt{2}}{2})$*, then*

$$\omega(u,x) = -\frac{\sqrt{2}}{2}(\sinh\theta - 1)e^s = \frac{\sqrt{2}}{2}e^s - \frac{\sqrt{2}}{4}(e^{s+\frac{\pi}{4}} - e^{s-\frac{\pi}{4}});$$

   - *if* $x_2 = -\sqrt{2} - x_3(x_3 > -\frac{\sqrt{2}}{2})$*, then*

$$\omega(u,x) = \frac{\sqrt{2}}{2}(\sinh\theta - 1)e^s = -\frac{\sqrt{2}}{2}e^s + \frac{\sqrt{2}}{4}(e^{s+\frac{\pi}{4}} - e^{s-\frac{\pi}{4}}).$$

Example 3. *Let* $r(s)$ *be a pseudo null curve framed by* $\{\alpha, \beta, \gamma\}$ *and* $v = \cos s\gamma$ *a lightlike vector. There is a unit timelike vector* $x = x_1\alpha + x_2\beta + x_3\gamma, x_i = x_i(s), (i = 1,2,3)$ *whose intersection angles with* $b_1 = \frac{\sqrt{2}}{2}(\beta + \gamma)$ *and* $b_2 = \frac{\sqrt{2}}{2}(\beta - \gamma)$ *are all* $\frac{\pi}{4}$*. From Definition 8 and* $\omega_2(s) = \cos s$*,* $\eta_1 = \eta_2 = \frac{\pi}{4}$*, the angle* $\omega(v,x)$ *can be expressed as follows:*

1. *when* $-\frac{1}{2x_3} \leq x_2 < -x_3(x_3 < -\frac{\sqrt{2}}{2})$*, we have*

$$\omega(v,x) = -\frac{\sqrt{2}}{2}(\sinh\frac{\pi}{4} - \cosh\frac{\pi}{4})\cos s = \frac{\sqrt{2}}{2}e^{-\frac{\pi}{4}}\cos s;$$

2. *when* $-x_3 \leq x_2 \leq -\frac{1}{2x_3}(x_3 \geq \frac{\sqrt{2}}{2})$, *we have*

$$\omega(v,x) = \frac{\sqrt{2}}{2}(\sinh\frac{\pi}{4} - \cosh\frac{\pi}{4})\cos s = -\frac{\sqrt{2}}{2}e^{-\frac{\pi}{4}}\cos s;$$

3. *when* $x_2 \geq -x_3(x_3 < -\frac{\sqrt{2}}{2})$ *or* $x_2 \geq -\frac{1}{2x_3}(-\frac{\sqrt{2}}{2} \leq x_3 < 0)$, *we have*

$$\omega(v,x) = \frac{\sqrt{2}}{2}(\sinh\frac{\pi}{4} + \cosh\frac{\pi}{4})\cos s = \frac{\sqrt{2}}{2}e^{\frac{\pi}{4}}\cos s;$$

4. *when* $x_2 < -x_3(x_3 \geq \frac{\sqrt{2}}{2})$ *or* $x_2 \leq -\frac{1}{2x_3}(0 < x_3 < \frac{\sqrt{2}}{2})$, *we have*

$$\omega(v,x) = -\frac{\sqrt{2}}{2}(\sinh\frac{\pi}{4} + \cosh\frac{\pi}{4})\cos s = -\frac{\sqrt{2}}{2}e^{\frac{\pi}{4}}\cos s.$$

Author Contributions: J.Q., X.T. and J.L. set up the problem and computed the details. Y.H.K. checked and polished the draft. All authors have read and agreed to the published version of the manuscript.

Funding: The first author was supported by NSFC (No. 11801065) and the fourth author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea Government (MSIP) (2016R1A2B1006974).

Acknowledgments: We thank H. Liu of Northeastern University and the referee for the careful review and the valuable comments to improve the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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