mathematics

Article

Quasi (s, r)-Contractive Multi-Valued Operators and Related Fixed Point Theorems

Ei Ei Nyein, Dianlu Tian * and Aung Khaing Zaw

School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China; eieinyein1985@yahoo.com (E.E.N.); akzbee451986@yahoo.com (A.K.Z.)

* Correspondence: 3120170671@bit.edu.cn

Received: 5 September 2019; Accepted: 29 October 2019; Published: 2 January 2020

Abstract: This paper gives the new concepts of quasi (s, r)-contractive multi-valued operators and establishes some related fixed point results for such operators. In addition, an application to certain functional equations arising from dynamic programming is given to illustrate the usage of the obtained results.

Keywords: complete metric space; quasi (s, r)-contractive operator; multi-valued operator; fixed point theorem

1. Introduction and Preliminaries

As it is well known to all, the proverbial Banach contraction mapping principle is a very useful, simple and classical tool in modern mathematics, and has been widely used in many branches of mathematics and physics. Many mathematicians have researched and generalized the Banach contraction mapping principle along different directions, such as the fixed point theorem of fuzzy metric spaces, C^*-algebra valued metric spaces and so on [1–5]. In general the theorem has been extended in two directions. On the one hand, the usual contractive condition is replaced with a weakly contractive condition. On the other hand, the complete metric space is replaced by different types of metric spaces [6–8]. However at present, in order to get an analog result, one always has to equip the powerset of a nonempty set with some suitable metric. One such a metric is a Hausdorff metric. It was Markin [9] who used the Hausdorff metric to study the fixed point theory of the multi-valued contractive mappings for the first time. In 1969, Nadler [10] and Reich [11,12] introduced the fixed point theorems of the multi-valued contractive operators respectively. Recently Popescu [13] gave the concept of the $(s, r)-$contractive multi-valued operator and showed that such an operator is nothing but a weakly Picard operator. Based on [13] Kamran and Hussain [14] introduced the notion of the weakly $(s, r)-$contractive multi-valued operator.

This paper will introduce the concept of quasi (s, r)-contractive multi-valued operator based on the notion and properties of (s, r)-contractive multi-valued operator. Moreover, some fixed point theorems for mappings satisfying the contractive conditions about such an operator are established. In addition, the existence results for a type of functional equations arising in dynamic programming are given as an application.

To begin, let us start from some fundamental definitions and theorems as follows. Details can be seen in [6,10,13,15–20].

Definition 1. *[13] Suppose that (X, d) is a nonempty metric space and $CB(X)$ be the class of all nonempty bounded closed subsets of X. Set*

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}, \ A, B \in CB(X),$$

where $d(x, B) = \inf_{y\in B} d(x, y)$, then $(CB(X), H)$ is a metric space and $H(A, B)$ is called a Hausdorff metric between A and B.

It is easy to see that if (X, d) is a complete metric space, $(CB(X), H)$ is complete as well.

Definition 2. *[10] Let X be a metric space and $T : X \to CB(X)$ be a multi-valued operator. If there exists $k \in [0, 1]$ such that $H(Tx, Ty) \leq k\, d(x, y)$ for all $x, y \in X$, we call T a contractive multi-valued operator.*

Definition 3. *[13] Let (X, d) be a metric space and $T : X \to CB(X)$ be a multi-valued operator. If there exists $r \in [0, 1)$ and $s \geq r$, such that*

$$d(y, Tx) \leq s\, d(x, y) \Rightarrow H(Tx, Ty) \leq rM_T(x, y), \ \forall x, y \in X,$$

where

$$M_T(x, y) = \max\left\{ d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{d(x, Ty) + d(y, Tx)}{2}\right\},$$

then T is called a (s, r)-contractive multi-valued operator.

Theorem 1. *[10] Let (X, d) be a complete metric space and $T : X \to CB(X)$ be an (s, r)-contractive multi-valued operator with $s > r$. Then T has a fixed point, namely, there exists $x \in X$ such that $x \in Tx$.*

Theorem 2. *[10] Let (X, d) be a complete metric space and $T : X \to X$ be an (s, r)-contractive single-valued operator. Then T has a fixed point. Moreover, T has a uniqued fixed point for $s \geq 1$.*

Definition 4. *[15] Let (X, d) be a metric space. The multi-valued map $T : X \to CB(X)$ is said to be a multi-valued quasi-contraction if*

$$H(Tx, Ty) \leq r \max\{ d(x, y),\ d(x, Tx),\ d(y, Ty),\ d(x, Ty),\ d(y, Tx)\}, \ \forall x, y \in X, r \in [0, 1).$$

Theorem 3. *[15] Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a multi-valued quasi-contraction with $r \in [0, \frac{1}{2})$. Then T has a fixed point.*

By using the fact $\frac{d(x,Ty)+ d(y,Tx)}{2} \leq \max\{ d(x, Ty),\ d(y, Tx)\}$, we introduce the new notions which is combined the ideas of Harandi [15], Popescu [13] and Haghi [21] for contractive multi-valued operators.

2. Main Results

Illuminated by the concept of (s, r)-contractive multi-valued operator, this section will introduce a new operator, namely, the quasi (s, r)-contractive multi-valued operator and give some related fixed point theorems.

Definition 5. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued operator. If there exist $r \in [0,1)$ and $s \geq r$ such that*

$$d(y,Tx) \leq s\, d(x,y) \Rightarrow H(Tx,Ty) \leq rM^*(x,y),\ \forall x,y \in X,$$

where

$$M^*(x,y) = \max\{\, d(x,y),\ d(x,Tx),\ d(y,Ty),\ d(x,Ty),\ d(y,Tx)\}\,,$$

then T is called a quasi (s,r)-contractive multi-valued operator on X.

The following theorem generalizes the result of [13] to the setting of complete metric space.

Theorem 4. *Suppose that (X,d) is a complete metric space and $T : X \to CB(X)$ is a quasi (s,r)-contractive multi-valued operator with $r \in [0,1)$ and $s > r$. Then T has a fixed point.*

Proof. Let $u_0 \in X$ and $u_1 \in Tu_0$.
If $u_1 = u_0$, then u_0 is a fixed point of T. Let $u_1 \neq u_0$. Take $u_2 \in Tu_1$ such that $d(u_1,u_2) \leq qH(Tu_0,Tu_1)$, where $q > 1$ with $2qr_1 < 1$ and $0 \leq r < r_1 < 1$.

Since $d(u_1,Tu_0) = 0 \leq s\, d(u_1,u_0)$, by our hypothesis

$$H(Tu_0,Tu_1) \leq r \max\{\, d(u_0,u_1),\ d(u_0,Tu_0),\ d(u_1,Tu_1),\ d(u_0,Tu_1),\ d(u_1,Tu_0)\},$$

$$\begin{aligned}
d(u_1,u_2) \leq qH(Tu_0,Tu_1) &\leq qr \max\{\, d(u_0,u_1),\ d(u_0,Tu_0),\ d(u_1,Tu_1),\ d(u_0,Tu_1),\ d(u_1,Tu_0)\} \\
&\leq qr_1 \max\{\, d(u_0,u_1),\ d(u_1,u_2),\ d(u_0,u_2)\} \\
&\leq qr_1 \max\{\, d(u_0,u_1),\ d(u_1,u_2),\ d(u_0,u_1) + d(u_1,u_2)\} \\
&= qr_1[\, d(u_0,u_1) +\ d(u_1,u_2)] \\
&= 2qr_1[\frac{d(u_0,u_1) +\ d(u_1,u_2)}{2}] \\
&\leq 2qr_1 \max\{\, d(u_0,u_1),\ d(u_1,u_2)\},
\end{aligned}$$

where, $2qr_1 < 1$.
Case(i) : If $\max\{\, d(u_0,u_1),\ d(u_1,u_2)\} =\ d(u_1,u_2)$, then $d(u_2,u_1) = 0$.
So u_1 is a fixed point of T since $u_1 = u_2 \in Tu_1$.
Case(ii) : If $\max\{\, d(u_0,u_1),\ d(u_1,u_2)\} =\ d(u_0,u_1)$, then we have

$$d(u_1,u_2) \leq 2qr_1\, d(u_0,u_1).$$

Thus one can construct a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$ with

$$d(u_{n+1},u_{n+2}) \leq 2qr_1\, d(u_n,u_{n+1}),\ \ 2qr_1 < 1, \forall n \geq 0,$$

whenever,

$$d(u_{n+1},Tu_n) \leq s\, d(u_{n+1},u_n),$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} d(u_{n+1}, u_{n+2}) &= d(u_1, u_2) + d(u_2, u_3) + d(u_3, u_4) + \cdots \\
&\leq 2qr_1\, d(u_0, u_1) + 2qr_1\, d(u_1, u_2) + 2qr_1\, d(u_2, u_3) + \cdots \\
&\leq [2qr_1 + (2qr_1)^2 + (2qr_1)^3 + \cdots]\, d(u_0, u_1) \\
&= \frac{2qr_1}{1 - 2qr_1}\, d(u_0, u_1), \ 2qr_1 < 1 \\
&< \infty.
\end{aligned}
$$

It means $\{u_n\}$ in X is a Cauchy sequence and $\lim_{p\to\infty} u_n = u^*$ in X since $(X, d)$is a complete metric space.

We now show that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $d(u^*, Tu_{n_k}) \leq s\, d(u^*, u_{n_k}), \ \forall k \in \mathbb{N}$.

Indeed, if there is a positive integer $\mathbb{N}$ such that

$$d(u^*, Tu_n) > s\, d(u^*, u_n), \ \forall n \geq \mathbb{N}.$$

This implies

$$d(u^*, u_{n+1}) > s\, d(u^*, u_n), \ \forall n \geq \mathbb{N}.$$

Using induction, one can obtain that for all $n \geq \mathbb{N}$, $p \geq 1$,

$$d(u^*, u_{n+p}) > s^p\, d(u^*, u_n).$$

Futhermore,

$$
\begin{aligned}
d(u_n, u_{n+p}) &\leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + \cdots + d(u_{n+p-1}, u_{n+p}) \\
&\leq d(u_n, u_{n+1}) + 2qr_1\, d(u_n, u_{n+1}) + \cdots + (2qr_1)^{p-1}\, d(u_n, u_{n+1}) \\
&= (\frac{1 - (2qr_1)^p}{1 - 2qr_1})\, d(u_n, u_{n+1}), \forall n \geq \mathbb{N}, p \geq 1.
\end{aligned}
$$

Set $p \to \infty$, then we have

$$d(u^*, u_n) \leq \frac{1}{1 - 2qr_1}\, d(u_n, u_{n+1}), \forall n \geq 0.$$

So

$$
\begin{aligned}
d(u^*, u_{n+p}) &\leq \frac{1}{1 - 2qr_1}\, d(u_{n+p}, u_{n+p+1}) \\
&\leq \frac{r_1^p}{1 - 2qr_1}\, d(u_n, u_{n+1}).
\end{aligned}
$$

But $d(u^*, u_{n+p}) > s^p\, d(u^*, u_n)$, so

$$s^p d(u^*, u_n) < \ d(u^*, u_{n+p}) \leq \frac{(r_1)^p}{1 - 2qr_1}\, d(u_n, u_{n+1}),$$

and

$$d(u^*, u_n) \leq \frac{(r_1)^p}{s^p(1 - 2qr_1)}\, d(u_n, u_{n+1}) = \frac{(\frac{r_1}{s})^p}{1 - 2qr_1}\, d(u_n, u_{n+1}),$$

set $p \to \infty$, we have

$$d(u^*, u_n) \leq \lim_{p\to\infty} \frac{(\frac{r_1}{s})^p}{1-2qr_1}\, d(u_n, u_{n+1}) = 0.$$

It implies that $u^* = u_n$. This is contradict to $d(u^*, u_{n+p}) > s^p\, d(u^*, u_n)$. Therefore there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$d(u^*, Tu_{n_k}) \leq s\, d(u^*, u_{n_k}),\ \forall k \in \mathbb{N}.$$

By hypothesis, one has

$$H(Tu^*, Tu_{n_k}) \leq r \max\left\{d(u^*, u_{n_k}),\ d(u^*, Tu^*),\ d(u_{n_k}, Tu_{n_k}),\ d(u^*, Tu_{n_k}),\ d(u_{n_k}, Tu^*)\right\}.$$

Therefore,

$$d(u_{n_{k+1}}, Tu^*) \leq r \max\left\{d(u^*, u_{n_k}),\ d(u^*, Tu^*),\ d(u_{n_k}, u_{n_{k+1}}),\ d(u^*, u_{n_{k+1}}),\ d(u_{n_k}, Tu^*)\right\}.$$

Letting $k \to \infty$, we get

$$\begin{aligned} d(u^*, Tu^*) &\leq r \max\{\, d(u^*, Tu^*),\ d(u^*, Tu^*)\}, \\ d(u^*, Tu^*) &\leq r\, d(u^*, Tu^*), \end{aligned}$$

where it implies that $d(u^*, Tu^*) = 0$. Hence $u^* \in Tu^*$ and u^* is a fixed point of T. This completes the proof.
□

The following example shows that under the condition of Theorem 4 the fixed point may not be unique.

Example 1. *Let $X = [1, \infty)$ with $d(x, y) = |x - y|$ for all $x, y \in X$. Define $T : X \to CB(X)$ by*

$$Tx = \left[3, 3 + \frac{x}{4}\right],\ x \in X.$$

Consider

$$H(Tx, Ty) = \frac{1}{4}|x - y| = \frac{1}{4}\, d(x, y),$$

where we choose $r = \frac{1}{3} \in [0, 1)$, $s = \frac{1}{2} > r$. Then the conditions of Theorem 4 are fulfilled. It is clear that the points 3 and 4 are both fixed points of T which implies that the fixed points are not unique.

It is necessary for us to consider when the fixed point of the quasi (s, r)-contractive multi-valued operator is unique.

Corollary 1. *Let (X, d) be a complete metric space and $T : X \to X$ be a quasi (s, r)-contractive single-valued operator with $r \in [0, 1)$ and $s \geq 1$. Then T has a unique fixed point.*

Proof. Suppose u^* and v^* are fixed points of T and $u^* \neq v^*$.
Then

$$d(v^*, Tu^*) = d(v^*, u^*) \leq s\, d(v^*, u^*),\quad s \geq 1.$$

Using the hypothesis,

$$\begin{aligned} H(Tu^*, Tv^*)) &\leq r \max\{ d(u^*, v^*),\ d(u^*, Tu^*),\ d(v^*, Tv^*),\ d(u^*, Tv^*),\ d(v^*, Tu^*)\} \\ &\leq r \max\{ d(u^*, v^*),\ d(v^*, v^*),\ d(u^*, u^*)\} \\ &= r\, d(u^*, v^*). \end{aligned}$$

But $H(Tu^*, Tv^*) = d(u^*, v^*)$.
So, $d(u^*, v^*) \leq r\, d(u^*, v^*)$, $r < 1$.
It implies $d(u^*, v^*) = 0$ and $u^* = v^*$ which leads to a contradiction. □

The following is another result about the quasi (s, r)-contractive multi-valued operator.

Theorem 5. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued operator. Suppose that there exist constants $r, s \in [0, 1)$ with $s > r$ such that*

$$\frac{1}{1+r}\, d(x, Tx) \leq d(x, y) \leq \frac{1}{1-s}\, d(Tx, x) \Rightarrow H(Tx, Ty) \leq rM^*(x, y),$$

where

$$M^*(x, y) = \max\{ d(x, y),\ d(x, Tx),\ d(y, Tx),\ d(x, Ty),\ d(y, Tx)\},$$

then T has a fixed point.

Proof. Let $s_1 \in [0, 1)$ such that $0 \leq r < s_1 < s < 1$. Let $u_0 \in X$ and $u_1 \in Tu_0$ such that

$$d(u_0, u_1) \leq \frac{1-s_1}{1-s}\, d(u_0, Tu_0).$$

If $u_1 = u_0$, then u_0 is a fixed point of T. Let $u_1 \neq u_0$.
Then we obtain

$$\frac{1}{1+r} d(u_0, Tu_0) \leq d(u_0, u_1) \leq \frac{1}{1-s_1} d(u_0, u_1) \leq \frac{1}{1-s}\, d(u_0, Tu_0).$$

By our hypothesis, we get

$$H(Tu_0, Tu_1) \leq r \max\{ d(u_0, u_1),\ d(u_0, Tu_0),\ d(u_1, Tu_1),\ d(u_0, Tu_1),\ d(u_1, Tu_0)\},$$

where $r \in [0, 1)$.
Take $u_2 \in Tu_1$ such that $d(u_1, u_2) \leq qH(Tu_0, Tu_1)$, where $q > 1$ with $2qr_1 < 1$ and $0 \leq r < r_1 < 1$.
Therefore

$$\begin{aligned} d(u_1, u_2) &\leq qr_1 \max\{ d(u_0, u_1),\ d(u_0, Tu_1),\ d(u_1, Tu_1),\ d(u_0, Tu_1),\ d(u_1, Tu_0)\} \\ &\leq qr_1 \max\{ d(u_0, u_1),\ d(u_1, u_2),\ d(u_0, u_2)\} \\ &\leq qr_1 \max\{ d(u_0, u_1),\ d(u_1, u_2),\ d(u_0, u_1) + d(u_1, u_2)\} \\ &= 2qr_1 \{ \frac{d(u_0, u_1) + d(u_1, u_2)}{2} \} \\ &\leq 2qr_1 \max\{ d(u_0, u_1),\ d(u_1, u_2)\}. \end{aligned}$$

Case(i) : If $\max\{ d(u_0, u_1),\ d(u_1, u_2)\} = d(u_1, u_2)$, then

$$d(u_2, u_1) \leq 2qr_1\, d(u_1, u_2).$$

It implies that $d(u_1, u_2) = 0$ and so u_1 is a fixed point of T.
Case(ii) : If $\max\{ d(u_0, u_1), d(u_1, u_2)\} = d(u_0, u_1)$, then

$$d(u_1, u_2) \leq 2qr_1\, d(u_0, u_1), \;\; 2qr_1 < 1.$$

Thus, one can construct a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$d(u_{n+1}, u_{n+2}) \leq (2qr)^{n+1}\, d(u_{n+1}, u_n),$$

with

$$d(u_n, u_{n+1}) \leq \frac{1-s_1}{1-s}\, d(u_n, Tu_n), \;\; \forall n \geq 0$$

$$\begin{aligned}
\sum_{n=1}^{\infty} d(u_n, u_{n+1}) &= d(u_0, u_1) + d(u_1, u_2) + d(u_2, u_3) + \cdots \\
&\leq [1 + 2qr_1 + (2qr_1)^2 + \cdots]\, d(u_0, u_1) \\
&= \frac{1}{1-2qr_1}\, d(u_0, u_1) \\
&< \infty.
\end{aligned}$$

Then we obtain the sequence $\{u_n\}$ in X is a Cauchy and $u_n \to u^*$ in X, since X is a complete meric space.

Since

$$\begin{aligned}
d(u_{n+p}, u_n) &\leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + \cdots + d(u_{n+p-1}, u_{n+p}) \\
&\leq [1 + 2qr_1 + (2qr_1)^2 + \cdots + (2qr_1)^{p-1}]\, d(u_n, u_{n+1}) \\
&= \frac{1-(2qr_1)^p}{1-2qr_1}\, d(u_n, u_{n+1}), \;\; \forall n \geq \mathbb{N}, \;\; p \geq 1.
\end{aligned}$$

$$\begin{aligned}
\lim_{p\to\infty} d(u_{n+p}, u_n) &\leq \lim_{p\to\infty} (\frac{1-(2qr_1)^p}{1-2qr_1})\, d(u_n, u_{n+1}) \\
d(u^*, u_n) &\leq \frac{1}{1-2qr_1} d(u_n, u_n+1), \;\; \forall n \geq 0. \\
&\leq \frac{1}{1-r_1}\, d(u_n, u_n+1) \\
&\leq \frac{1}{1-r}\, d(u_n, u_n+1)
\end{aligned}$$

Since

$$d(u_n, u_{n+1}) \leq \frac{1-s_1}{1-s}\, d(u_n, Tu_n)$$

it follows that

$$d(u^*, u_n) \leq \frac{1}{1-s}\, d(u_n, Tu_n), \;\; \forall n \geq 0.$$

Now we have to show that

$$\frac{1}{1+r}\, d(u_n, Tu_n) \leq d(u^*, u_n).$$

Assume that there is a positive integer $\mathbb{N}$ such that

$$d(u^*, u_n) < \frac{1}{1+r}\, d(u_n, Tu_n), \;\; \forall n \geq \mathbb{N}.$$

Then we have

$$\begin{aligned} d(u_n, u_{n+1}) &\leq d(u_n, u^*) + d(u^*, u_{n+1}) \\ &< \frac{1}{1+r}\,[\,d(u_n, Tu_n) + d(u_{n+1}, Tu_{n+1})] \\ &\leq \frac{1}{1+r}\,[\,d(u_n, Tu_n) + 2qr_1\, d(u_n, u_{n+1})] \\ d(u_n, u_{n+1}) &< \frac{1}{(1+r)(1-2qr_1)}\, d(u_n, Tu_n) \\ &< d(u_n, Tu_n) \end{aligned}$$

which is impossible.

So there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ in X such that

$$d(u^*, u_{n_k}) \geq \frac{1}{1+r}\, d(Tu_{n_k}, u_{n_k}), \;\; \forall k \geq \mathbb{N}.$$

Since

$$d(u^*, u_n) \leq \frac{1}{1-s}\, d(u_n, Tu_n), \;\; \forall n \geq 0,$$

and using the hypothesis, we obtain

$$H(Tu^*, Tu_{n_k}) \leq rM^*(u^*, u_{n_k}).$$

Thus

$$\begin{aligned} d(u_{n_k+1}, Tu^*) &\leq r \max\left\{\, d(u^*, u_{n_k}),\; d(u_{n_k}, Tu_{n_k}),\; d(u^*, Tu^*),\; d(u_{n_k}, Tu^*),\; d(u^*, Tu_{n_k})\right\} \\ \lim_{k\to\infty} d(u_{n_k}, Tu^*) &\leq \lim_{k\to\infty} r \max\left\{\, d(u^*, u_{n_k}),\; d(u_{n_k}, Tu_{n_k}),\; d(u^*, Tu^*),\; d(u_{n_k}, Tu^*),\; d(u^*, Tu_{n_k})\right\} \\ d(u^*, Tu^*) &\leq r \max\{\, d(u^*, u^*),\; d(u^*, Tu^*)\} \\ d(u^*, Tu^*) &\leq r\, d(u^*, Tu^*). \end{aligned}$$

It implies that $u^* \in Tu^*$ and u^* is a fixed point of T. □

Corollary 2. *Let (X, d) be a complete metric space and $T : X \to X$ be a quasi (s, r)-contractive single-valued mapping. Assume that there exist $r \in [0, 1)$ such that $\forall x, y \in X$*

$$\frac{1}{1+r}\, d(x, Tx) \leq\, d(x, y) \leq \frac{1}{1-r}\, d(x, Tx) \Rightarrow H(Tx, Ty) \leq rM^*(x, y),$$

where

$$M^*(x, y) = \max\{\, d(x, y),\; d(x, Tx),\; d(y, Ty),\; d(x, Ty),\; d(y, Tx)\}\,.$$

Then there exists a fixed point of T.

Proof. Let $u_0 \in X$ and $u_1 = Tu_0$. Take $u_2 \in Tu_1$ for $r \in [0, 1)$.
It is claim that

$$\frac{1}{1+r}\, d(u_0, Tu_0) \leq\, d(u_0, u_1) \leq \frac{1}{1-r}\, d(u_0, Tu_0),$$

and thus, by assumption of Theorem 5, we obtain

$$d(u_1, u_2) \leq H(Tu_0, Tu_1) \leq 2qr_1\, d(u_0, u_1), \;\; q > 1 \;\; with \;\; 2qr_1 < 1, \;\; 0 \leq r < r_1 < 1$$

One can construct a sequence $\{u_n\}$ in X with $u_{n+1} = Tu_n$ such that

$$d(u_{n+1}, u_{n+2}) \leq 2qr_1\, d(u_n, u_{n+1}).$$

Then the sequence $\{u_n\}$ in X is a Cauchy sequence and $u_n \rightarrow u^*$ in X since X is a complete meric space.

We can prove that

$$d(u^*, u_n) \leq \frac{1}{1-r}\, d(u_n, u_{n+1}), \ \ \forall n \geq 0$$

and there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ in X such that

$$d(u_{n_k}, u^*) \geq \frac{1}{1+r}\, d(u_{n_k+1}, u_{n_k}),$$

hold for $k \geq \mathbb{N}$. so

$$\begin{aligned} d(u_{n_k+1}, Tu^*) &\leq r \max\{\, du^*, u_{n_k}),\ d(u_{n_k}, Tu_{n_k}),\ d(u^*, Tu^*),\ d(u_{n_k}, Tu^*),\ d(u^*, Tu_{n_k})\} \\ &\leq r \max\{\, d(u^*, u_{n_k}),\ d(u_{n_k}, u_{n_k+1}),\ d(u^*, Tu^*),\ d(u_{n_k+1}, u^*),\ d(Tu^*, u_{n_k+1})\} \\ \lim_{k\to\infty} d(u_{n_{k+1}}, Tu^*) &\leq \lim_{k\to\infty} r \max\{\, d(u^*, u_{n_k}),\ d(u_{n_k}, u_{n_k+1}),\ d(u^*, Tu^*),\ d(u_{n_k+1}, u^*),\ d(Tu^*, u_{n_k+1})\} \\ d(u^*, Tu^*) &\leq r\, d(u^*, Tu^*). \end{aligned}$$

so $d(u^*, Tu^*) = 0$ and hence $Tu^* = u^*$.

It implies that u^* is a fixed point of T. □

3. Application

In this section, we discuss the existence and uniqueness of solutions of a functional equation by using Theorem 4.

We give the basic notation to use in the section. Let X and Y be Banach spaces and $U \subset X$, $V \subset Y$.

Let $B(U)$ denote the set of all bounded functions on U. If the metric $d_B : B(U) \times B(U) \longrightarrow [0, \infty)$ is defined by $d_B(h, k) = sup_{x \in U}|h(x) - k(x)|$, then $(B(U), d_B)$ is a complete metric space.

Assume that U and V are the state and decision spaces respectively.

Then the problem of dynamic programming reduces to the problem of solving the functional equation:

$$f(x) = \sup_{y \in V} H(x, y, f(\tau(x, y))),$$

where $\tau : U \times V \rightarrow U$ represents the transformation of the process and $f(x)$ represents the optimal return function with initial functional

$$f(x) = \sup_{y \in V}\{g(x, y) + G(x, y, f(\tau(x, y)))\},\ (x \in U),$$

where $g : U \times V \rightarrow \mathbb{R}$ and $G : U \times V \times \mathbb{R} \rightarrow \mathbb{R}$ are bounded functions.

Define $T : B(U) \rightarrow B(U)$ by

$$T(h(x)) = \sup_{y \in V}\{g(x, y) + G(x, y, f(\tau(x, y)))\},\ \forall h \in B(U), x \in V.$$

Then the following result is grated to find the existence and uniqueness of a solution of the classic functional equation by using theorem.

Theorem 6. *Assume that there exist $r \in [0,1)$, $s > r$ such that for all $(x,y) \in U \times V$, $h,k \in B(U)$ and $t \in U$. If the inequality*

$$d_B(k(t), Th(t)) \leq s d_B(k(t), h(t)) \Rightarrow d_B(G(x,h,h(t)), G(x,y,k(t))) \leq rM^*(h(t),k(t)),$$

where

$$M^*(h(t),k(t)) = \max\{\, d_B(h(t),k(t)),\ d_B(h(t),Th(t)),\ d_B(k(t),Tk(t)),\ d_B(h(t),Tk(t)),\ d_B(k(t),Th(t))\}.$$

Then the functional equation (*) has a bounded solution. Moreover, if $s \geq 1$, then the solution is unique.

Proof. Let $l_1, l_2 \in B(U)$ and $x \in U$. Take $y_1, y_2 \in V$. Let ϵ be a positive real number such that

$$Tl_1(x) < g(x,y_1) + G(x,y_1,l_1(\tau_1)) + \epsilon, \tag{1}$$

$$Tl_2(x) < g(x,y_2) + G(x,y_2,l_2(\tau_2)) + \epsilon, \tag{2}$$

where $\tau_i = \tau_i(x,y_i), i \in \{1,2\}$.

By the definition of T, we have

$$Tl_1(x) \geq g(x,y_2) + G(x,y_2,l_1(\tau_2)), \tag{3}$$

$$Tl_2(x) \geq g(x,y_1) + G(x,y_1,l_2(\tau_1)). \tag{4}$$

Assume that $d_B(l_2(x), Tl_1(x)) \leq s d_B(l_2(x), l_1(x))$. That is, $|l_2(x) - Tl_1(x)| \leq s|l_2(x) - l_1(x)|$.

So, by using Equations (1) and (4), we obtain

$$\begin{aligned} Tl_1(x) - Tl_2(x) &< G(x,y_1,h_1(\tau_1)) - G(x,y_1,h_2(\tau_1)) + \epsilon \\ &= d_B(G(x,y_1,h_1(\tau_1)), G(x,y_1,h_2(\tau_1)) + \epsilon. \end{aligned}$$

Similarly, from Equations (2) and (3), we obtain

$$Tl_2(x) - Tl_1(x) < rM^*(l_1(x), l_2(x)) + \epsilon.$$

Thus

$$|T(l_1(x)) - T(l_2(x))| < rM^*(l_1(x), l_2(x)) + \epsilon.$$

That is, $d_B(Tl_1(x), Tl_2(x)) \leq rM^*(l_1(x), l_2(x))$.

So, we get that

$$d_B(l_2(x), Tl_1(x)) \leq s\, d_B(l_2(x), l_1(x)),$$

implies

$d_B(Tl_1(x), Tl_2(x))$

$\leq r \max\{\, d_B(l_1(x), l_2(x),\ d_B(l_1(x), Tl_1(x)),\ d_B(l_2(x), Tl_2(x)),\ d_B(l_1(x), Tl_2(x)),\ d_B(l_2(x), Tl_1(x))\},$

It can be seen that all conditions of Theorem 4 are satisfied for T and hence it is proved. □

Author Contributions: E.E.N. prepared the original draft, D.T. review end edited the manuscript and A.K.Z. reviewed the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ma, Z.H.; Jiang, L.N.; Sun, H.K. C^*-algebra-valued metric spaces and related fixed point theorems. *Fixed Point Theory Appl.* **2014**, *2014*, 206. [CrossRef]
2. Ma, Z.H.; Jiang, L.N. C^*-algebra-valued b-metric spaces and related fixed point theorems. *Fixed Point Theory Appl.* **2015**, *2015*, 111. [CrossRef]
3. Yao, Y.H.; Postolache, M.; Qin, X.L. Iterative algorithms for the proximal split feasibility problem. *UPB Sci. Bull. Ser. A Appl. Math. Phys.* **2018**, *80*, 37–44.
4. Yao, Y.H.; Qin, X.L.; Yao, J.C. Self-adaptive step-sizes choice for split common fixed point problems. *J. Nonlinear Convex Anal.* **2018**, *11*, 1959–1969.
5. Tian, D.L.; Shi, L.Y.; Chen, R.D. Strong convergence theorems for split inclusion problems in Hilbert spaces. *J. Fixed Point Theory Appl.* **2017**, *19*, 1501–1514. [CrossRef]
6. Dugundji, J.; Granas, A. *Fixed Point Theory*; Springer: Berlin, Germany, 2003.
7. Rus, I.A. Basic problems of the metric fixed point theory revisited (II). *Stud. Univ. Babes Bolyai* **1991**, *36*, 81–89.
8. Shen, C.C.; Jiang, L.N.; Ma, Z.H. C*-algebra-valued G-metric spaces and related fixed point theorems. *J. Funct. Space Math. Sci.* **2008**, *24*, 401–416 [CrossRef]
9. Markin, J.T. Continuous dependence of fixed point sets. *Proc. Am. Math. Soc.* **1973**, *38*, 545-547. [CrossRef]
10. Nadler, S.B., Jr. Multi-valued contraction mappings. *Pac. J. Math.* **1969**, *30* , 475–488. [CrossRef]
11. Reich, S.; Zaslavski, A.J. *Genericity in Nonlinear Analysis*; Springer: New York, NY, USA, 2014.
12. Reich, S. Fixed points of contractive functions. *Boll. Un. Mat. Ital.* **1972**, *5*, 26–42.
13. Popescu, O. A new type of contractive multivalued operators. *Bull. Sci. Math.* **2013**, *137*, 30–44. [CrossRef]
14. Kamran,T.; Hussain, S. Weakly (s, r)-contractive multi-valued operators. *Rend. Circ. Mat. Palermo* **2015**, *64*, 475–482. [CrossRef]
15. Amini-Harandi, A. Fixed point theory for set-valued quasi-contraction maps in metric spaces. *Appl. Math. Lett.* **2011**, *24*, 1791–1794. [CrossRef]
16. Covitz, H.; Nadler, S.B., Jr. Multi-valued contraction mappings in generalized metric spaces. *Israel J. Math.* **1970**, *8*, 5–11. [CrossRef]
17. Daffer, P.Z.; Kaneko, H. Fixed points of generalized contractive multi-valued mappings. *J. Math. Anal. Appl.* **1995**, *192*, 655–666. [CrossRef]
18. Dhompongsa, S.; Yingtaweesittikul, H. Fixed points for multivalued mappings and the metric completeness. *J. Fixed Point Theory Appl.* **2009**, *2009*, 972395. [CrossRef]
19. Mizoguchi, N.; Takahashi, W. Fixed point theorems for multivalued mappings on complete metric spaces. *J. Math. Anal. Appl.* **1989**, *141*, 177–188. [CrossRef]
20. Petrusel, A. Generalized multivalued contractions. *Nonlinear Anal.* **2001**, *47*, 649–659. [CrossRef]
21. Haghi, R.H.; Rezapour, S.; Shahzad, N. On fixed points of quasi-contraction type multifunctions. *Appl. Math. Lett.* **2012**, *25*, 843–846. [CrossRef]