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Convergence Theorems for Modified Implicit Iterative Methods with Perturbation for Pseudocontractive Mappings

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Abstract: In this paper, first, we introduce a path for a convex combination of a pseudocontractive type of mappings with a perturbed mapping and prove strong convergence of the proposed path in a real reflexive Banach space having a weakly continuous duality mapping. Second, we propose two modified implicit iterative methods with a perturbed mapping for a continuous pseudocontractive mapping in the same Banach space. Strong convergence theorems for the proposed iterative methods are established. The results in this paper substantially develop and complement the previous well-known results in this area.

Keywords: modified implicit iterative methods with perturbed mapping; pseudocontractive mapping; strongly pseudocontractive mapping; nonexpansive mapping; weakly continuous duality mapping; fixed point

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1. Introduction

Let E be a real Banach space, and let E^* be the dual space of E. Let C be a nonempty closed convex subset of E. Recall that a mapping $f : C \to C$ is called *contractive* if there exists $k \in (0,1)$ such that $\|fx - fy\| \le k\|x - y\|, \ \forall x, y \in C$ and that a mapping $S : C \to C$ is called *nonexpansive* if $\|Sx - Sy\| \le \|x - y\|, \ \forall x, y \in C$.

Let J denote the normalized duality mapping from E into 2^{X^*} defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \ \|f\| = \|x\|\}, \quad x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pair between E and E^*. The mapping $T : C \to C$ is called *pseudocontractive* (respectively, *strong pseudocontractive*), if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \le \|x - y\|^2, \quad \forall x, \ y \in C,$$

(respectively, $\langle Tx - Ty, j(x - y)\rangle \le \beta\|x - y\|^2$ for some $\beta \in (0,1)$).

The class of pseudocontractive mappings is one of the most important classes of mappings in nonlinear analysis, and it has been attracting mathematician's interest. Apart from them being a generalization of nonexpansive mappings, interest in pseudocontractive mappings stems mainly from their firm connection with the class of accretive mappings, where a mapping A with domain $D(A)$ and range $R(A)$ in E is called accretive if the inequality

$$\|x - y\| \le \|x - y + s(Ax - Ay)\|,$$

holds for every $x, \ y \in D(A)$ and for all $s > 0$.

Within the past 50 years or so, many authors have been devoting their study to the existence of zeros of accretive mappings or fixed points of pseudocontractive mappings and several iterative methods for finding zeros of accretive mappings or fixed points of pseudocontractive mappings. We can refer to References [1–14] and the references in therein.

In 2007, Morales [15] introduced the following viscosity iterative method for pseudocontractive mapping:

$$x_t = tfx_t + (1-t)Tx_t, \ \ t \in (0,1), \tag{1}$$

where $T : C \to E$ is a continuous pseudocontractive mapping satisfying the weakly inward condition and $f : C \to C$ is a bounded continuous strongly pseudocontractive mapping. In a reflexive Banach space with a uniformly Gâteaux differentiable norm such that every closed convex bounded subset of C has the fixed point property for nonexpansive self-mappings, he proved the strong convergence of the sequences generated by the iterative method in Equation (1) to a point q in $Fix(T)$ (the set of fixed points of T), where q is the unique solution to the following variational inequality:

$$\langle fq - q, J(p-q) \rangle \leq 0, \quad \forall p \in Fix(T). \tag{2}$$

In 2009, using the method of Reference [16], Ceng et al. [17] introduced the following modified viscosity iterative method and modified implicit viscosity iterative method with a perturbed mapping for a pseudocontractive mapping:

$$x_t = tfx_t + r_tSx_t + (1-t-r_t)Tx_t, \ \ t \in (0,1), \tag{3}$$

where $0 < r_t < 1-t$, $T : C \to C$ is a continuous pseudocontractive mapping, $S : C \to C$ is a nonexpansive mapping, and $f : C \to C$ is a Lipschitz strongly pseudocontractive mapping.

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Ty_n, \\ x_{n+1} = \beta_n f y_n + \gamma_n S y_n + (1-\beta_n-\gamma_n)y_n, \end{cases} \tag{4}$$

and

$$\begin{cases} x_n = \alpha_n y_n + (1-\alpha_n)Ty_n, \\ y_n = \beta_n f x_{n-1} + \gamma_n S x_{n-1} + (1-\beta_n-\gamma_n)x_{n-1}, \end{cases} \tag{5}$$

where $f : C \to C$ is a contractive mapping , $x_0 \in C$ is an arbitrary initial point, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1]$ such that $\lim_{n\to\infty}(\gamma_n/\beta_n) = 0$ and $\beta_n + \gamma_n < 1$. In a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, they proved the strong convergence of the sequences generated by the iterative methods in Equations (3)–(5) to a point q in $Fix(T)$, where q is the unique solution to the variational inequality in Equation (2). Their results developed and improved the corresponding results of Song and Chen [11], Zeng and Yao [16], Xu [18], Xu and Ori [19], and Chen et al. [20].

In this paper, as a continuation of study in this direction, in a reflexive Banach space having a weakly sequentially continuous duality mapping J_φ with gauge function φ, we consider the viscosity iterative methods in Equations (3)–(5) for a continuous pseudocontractive mapping T, a continuous bounded strongly pseudocontractive mapping f, and a nonexpansive mapping S. We establish strong convergence of the sequences generated by proposed iterative methods to a fixed point of the mapping T, which solves a variational inequality related to f. The main results develop and supplement the corresponding results of Song and Chen [11], Morales [15], Ceng et al. [17], and Xu [18] to different Banach space as well as Zeng and Yao [16], Xu and Ori [19], Chen et al. [20], and the references therein.

2. Preliminaries

Throughout the paper, we use the following notations: " $\rightharpoonup$ " for weak convergence, " $\overset{*}{\rightharpoonup}$" for weak* convergence, and " $\to$ " for strong convergence.

Let E be a real Banach space with the norm $\|\cdot\|$, and let E^* be its dual. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. Let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a mapping. We denote the set of fixed points of the mapping T by $Fix(T)$. That is, $Fix(T) := \{x \in C : Tx = x\}$.

Recall that a Banach space E is said to be *smooth* if for each $x \in S_E = \{x \in E : \|x\| = 1\}$, there exists a unique functional $j_x \in E^*$ such that $\langle x, j_x \rangle = \|x\|$ and $\|j_x\| = 1$ and that a Banach space E is said to be *strictly convex* [21] if the following implication holds for $x,\ y \in E$:

$$\|x\| \le 1,\ \ \|y\| \le 1,\ \ \|x-y\| > 0 \ \ \Rightarrow \left\| \frac{x+y}{2} \right\| < 1.$$

By a gauge function, we mean a continuous strictly increasing function φ defined on $\mathbb{R}^+ := [0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. The mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \{f \in E^* : \langle x, f \rangle = \|x\|\|f\|, \|f\| = \varphi(\|x\|)\} \ \text{ for all } x \in E$$

is called the *duality mapping* with gauge function φ. In particular, the duality mapping with gauge function $\varphi(t) = t$ denoted by J is referred to as the *normalized duality mapping*. It is known that a Banach space E is smooth if and only if the normalized duality mapping J is single-valued. The following property of duality mapping is also well-known:

$$J_\varphi(\lambda x) = \text{sign}\,\lambda \left(\frac{\varphi(|\lambda| \cdot \|x\|)}{\|x\|} \right) J(x) \ \text{ for all } x \in E \setminus \{0\},\ \ \lambda \in \mathbb{R}, \tag{6}$$

where $\mathbb{R}$ is the set of all real numbers. The following are some elementary properties of the duality mapping J [21,22]:

(i) For $x \in E$, $J(x)$ is nonempty, bounded, closed, and convex;
(ii) $J(0) = 0$;
(iii) for $x \in E$ and a real α, $J(\alpha x) = \alpha J(x)$;
(iv) for $x, y \in E$, $f \in J(x)$ and $g \in J(y)$, $\langle x - y, f - g \rangle \ge 0$;
(v) for $x, y \in E$, $f \in J(x)$, $\|x\|^2 - \|y\|^2 \ge 2\langle x - y, f \rangle$.

We say that a Banach space E has a *weakly continuous duality mapping* if there exists a gauge function φ such that the duality mapping J_φ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $J_\varphi(x_n) \overset{*}{\rightharpoonup} J_\varphi(x)$. A duality mapping J_φ is weakly continuous at 0 if J_φ is single-valued and if $x_n \rightharpoonup 0$, $J_\varphi(x_n) \overset{*}{\rightharpoonup} 0$. For example, every l^p space $(1 < p < \infty)$ has a weakly continuous duality mapping with gauge function $\varphi(t) = t^{p-1}$ [21–23]. Set

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau \ \ \text{ for all } t \in \mathbb{R}^+.$$

Then it is known that $J_\varphi(x)$ is the subdifferential of the convex functional $\Phi(\|\cdot\|)$ at x. A Banach space E that has a weakly continuous duality mapping implies that E satisfies Opial's property. This means that whenever $x_n \rightharpoonup x$ and $y \neq x$, we have $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ [21,23].

The following lemma is Lemma 2.1 of Jung [24].

Lemma 1. ([24]) *Let E be a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ. Let $\{x_n\}$ be a bounded sequence of E and $f : E \to E$ be a continuous mapping. Let $g : E \to \mathbb{R}$ be defined by*

$$g(z) = \limsup_{n\to\infty} \langle z - fz, J_\varphi(z - x_n) \rangle$$

for $z \in E$. Then, g is a real valued continuous function on E.

We need the following well-known lemma for the proof of our main result [21,22].

Lemma 2. *Let E be a real Banach space, and let φ be a continuous strictly increasing function on $\mathbb{R}^+$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. Define*

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau \text{ for all } t \in \mathbb{R}^+.$$

Then, the following inequalities hold:

$$\Phi(kt) \leq k\Phi(t), \ \ 0 < k < 1,$$

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, j_\varphi(x+y)\rangle \text{ for all } x, y \in E,$$

where $j_\varphi(x+y) \in J_\varphi(x+y)$.

The following lemma can be found in Reference [18].

Lemma 3. ([18]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n\delta_n, \quad n \geq 0,$$

where $\{\lambda_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1-\lambda_n) = 0$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \lambda_n|\delta_n| < \infty$,

Then, $\lim_{n\to\infty} s_n = 0$.

Let C be a nonempty closed convex subset of a real Banach space E. Recall that $S : C \to C$ is called *accretive* if $I - S$ is pseudocontractive. If $T : C \to C$ is a pseudocontractive mapping, then $I - T$ is accretive. We denote $A = J_1 = (2I - T)^{-1}$. Then, $Fix(A) = Fix(T)$ and the operator $A : R(2I - T) \to C$ is nonexpansive and single-valued, where I denotes the identity mapping.

We also need the following result which can be found in Reference [11].

Lemma 4. ([11]) *Let C be a nonempty closed convex subset of a real Banach space E, and let $T : C \to C$ be a continuous pseudocontractive mapping. We denote $A = (2I - T)^{-1}$.*

(i) The mapping A is nonexpansive self-mapping on C, i.e., for all $x,\ y \in nC$, there holds

$$\|Ax - Ay\| \leq \|x - y\|, \quad \text{and} \quad Ax \in C.$$

(ii) If $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, *then* $\lim_{n\to\infty} \|x_n - Ax_n\| = 0$.

The following Lemmas, which are well-known, can be found in many books in the geometry of Banach spaces (see References [21,23]).

Lemma 5. (Demiclosedness Principle) *Let C be a nonempty closed convex subset of a Banach space E, and let $T : C \to C$ be a nonexpansive mapping. Then, $x_n \rightharpoonup x$ in C and $(I - T)x_n \to y$ imply that $(I - T)x = y$.*

Lemma 6. *If E is a Banach space such that E^* is strictly convex, then E is smooth and any duality mapping is norm-to-weak*-continuous.*

Finally, we need the following result which was given by Deimling [4].

Lemma 7. ([4]) *Let C be a nonempty closed convex subset of a Banach space E, and let $T : C \to C$ be a continuous strong pseudocontractive mapping with a pseudocontractive coefficient $\beta \in (0, 1)$. Then, T has a unique fixed point in C.*

3. Convergence of Path with Perturbed Mapping

As we know, the path convergency plays an important role in proving the convergence of iterative methods to approximate fixed points. In this direction, we first prove the existence of a path for a convex combination of a pseudocontractive type of mappings with a perturbed mapping and boundedness of the path.

Proposition 1. *Let C be a nonempty closed convex subset of a real Banach space E. Let $T : C \to C$ be a continuous pseudocontractive mapping, let $S : C \to C$ be a nonexpansive mapping, and let $f : C \to C$ be a continuous strongly pseudocontractive mapping with a pseudocontractive coefficient $\beta \in (0, 1)$.*

(i) *There exists a unique path $t \mapsto x_t \in C$, $t \in (0, 1)$, satisfying*

$$x_t = tfx_t + r_t Sx_t + (1 - t - r_t)Tx_t, \tag{7}$$

provided $r_t : (0, 1) \to [0, 1 - t)$ is continuous and $\lim_{t\to 0}(r_t/t) = 0$.

(ii) *In particular, if T has a fixed point in C, then the path $\{x_t\}$ is bounded.*

Proof. (i) For each $t \in (0, 1)$, define the mapping $T_{(S,f)} : C \to C$ as follows:

$$T_{(S,f)} = tf + r_t S + (1 - t - r_t)T,$$

where $0 < r_t < 1 - t$ and $\lim_{t\to 0}(r_t/t) = 0$. Then, it is easy to show that the mapping $T_{(S,f)}$ is a continuous strongly pseudocontractive self-mapping of C. Therefore, by Lemma 7, $T_{(S,f)}$ has a unique fixed point in C, i.e., for each given $t \in (0, 1)$, there exists $x_t \in C$ such that

$$x_t = tfx_t + r_t Sx_t + (1 - t - r_t)Tx_t.$$

To show continuity, let $t, t_0 \in (0, 1)$. Then, there exists $j \in J(x_t - x_{t_0})$ such that

$$\begin{aligned}
\langle x_t - x_{t_0}, j\rangle &= \langle tfx_t + r_t Sx_t + (1 - t - r_t)Tx_t - (t_0 f x_{t_0} + r_t Sx_t + (1 - t_0 - r_{t_0})Tx_{t_0}), j\rangle \\
&= t\langle fx_t - fx_{t_0}, j\rangle + (t - t_0)\langle fx_{t_0}, j\rangle + r_t\langle Sx_t - Sx_{t_0}, j\rangle + (r_t - r_{t_0})\langle Sx_{t_0}, j\rangle \\
&\quad + (1 - t - r_t)\langle Tx_t - Tx_{t_0}, j\rangle + ((t - t_0) + (r_t - r_{t_0}))\langle Tx_{t_0}, j\rangle,
\end{aligned}$$

and this implies that

$$\begin{aligned}
\|x_t - x_{t_0}\|^2 \le\ & t\beta\|x_t - x_{t_0}\|^2 + |t - t_0|\|fx_{t_0}\|\|x_t - x_{t_0}\| \\
& + r_t\|x_t - x_{t_0}\|^2 + |r_t - r_{t_0}|\|Sx_{t_0}\|\|x_t - x_{t_0}\| \\
& + (1 - t - r_t)\|x_t - x_{t_0}\|^2 + |t - t_0|\|Tx_{t_0}\|\|x_t - x_{t_0}\| + |r_t - r_{t_0}|\|Tx_{t_0}\|\|x_t - x_{t_0}\|.
\end{aligned}$$

and, hence,

$$\begin{aligned}
\|x_t - x_{t_0}\| &\le t\beta\|x_t - x_{t_0}\| + |t - t_0|\|fx_{t_0}\| + |r_t - r_{t_0}|\|Sx_{t_0}\| \\
&\quad + (1 - t - r_t)\|x_t - x_{t_0}\| + |t - t_0|\|Tx_{t_0}\| + |r_t - r_{t_0}|\|Tx_{t_0}\| \\
&= (1 - (1 - \beta)t)\|x_t - x_{t_0}\| + (\|fx_{t_0}\| + \|Tx_{t_0}\|)|t - t_0| + (\|Sx_{t_0}\| + \|Tx_{t_0}\|)|r_t - r_{t_0}|.
\end{aligned}$$

Therefore,

$$\|x_t - x_{t_0}\| \le \frac{\|fx_{t_0}\| + \|Tx_{t_0}\|}{(1 - \beta)t}|t - t_0| + \frac{\|Sx_{t_0}\| + \|Tx_{t_0}\|}{(1 - \beta)t}|r_t - r_{t_0}|,$$

which guarantees continuity.

(ii) By the same argument as in the proof of Theorem 2.1 of Reference [17], we can prove that $\{x_t\}$ defined by Equation (7) is bounded for $t \in (0, t_0)$ for some $t_0 \in (0,1)$, and so we omit its proof. □

The above path of Equation (7) is called the *modified viscosity iterative method with perturbed mapping*, where S is called the perturbed mapping.

The following result gives conditions for existence of a solution of a variational inequality:

$$\langle (I-f)q, J_\varphi(q-p)\rangle \leq 0, \quad \forall p \in Fix(T). \tag{8}$$

Theorem 1. *Let E be a Banach space such that E^* is strictly convex. Let C be a nonempty closed convex subset of a real Banach space E. Let $T : C \to C$ be a continuous pseudocontractive mapping with $Fix(T) \neq \emptyset$, let $S : C \to C$ be a nonexpansive mapping, and let $f : C \to C$ be a continuous strongly pseudocontractive mapping with a pseudocontractive coefficient $\beta \in (0,1)$. Suppose that $\{x_t\}$ defined by Equation (7) converges strongly to a point in $Fix(T)$. If we define $q := \lim_{t\to 0} x_t$, then q is a solution of the variational inequality in Equation (8).*

Proof. First, from Lemma 6, we note that E is smooth and J_φ is norm-to-weak*-continuous.

Since

$$(I-f)x_t = -\frac{1-t-r_t}{t}(I-T)x_t - \frac{r_t}{t}(I-S)x_t,$$

we have for $p \in Fix(T)$

$$\begin{aligned}\langle (I-f)x_t, J_\varphi(x_t-p)\rangle = & -\frac{1-t-r_t}{t}\langle (I-T)x_t-(I-T)p, J_\varphi(x_t-p)\rangle \\ & + \frac{r_t}{t}\langle (S-I)x_t, J_\varphi(x_t-p)\rangle.\end{aligned} \tag{9}$$

Since $I-T$ is accretive and $J(x_t-p)$ is a positive-scalar multiple of $J_\varphi(x_t-p)$ (see Equation (6)), it follow from Equation (9) that

$$\begin{aligned}\langle (I-f)x_t, J_\varphi(x_t-p)\rangle \leq & \frac{r_t}{t}\langle (S-I)x_t, J_\varphi(x_t-p)\rangle \\ \leq & \frac{r_t}{t}\|(S-I)x_t\|\varphi(\|x_t-p\|).\end{aligned} \tag{10}$$

Taking the limit as $t \to 0$, by $\lim_{t\to 0}\frac{r_t}{t} = 0$, we obtain

$$\langle (I-f)q, J_\varphi(q-p)\rangle \leq 0, \quad \forall p \in Fix(T).$$

This completes the proof. □

The following lemma provides conditions under which $\{x_t\}$ defined by Equation (7) converges strongly to a point in $Fix(T)$.

Lemma 8. *Let E be a reflexive smooth Banach space having Opial's property and having some duality mapping J_φ weakly continuous at 0. Let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a continuous pseudocontractive mapping with $Fix(T) \neq \emptyset$, let $S : C \to C$ be a nonexpansive mapping, and let $f : C \to C$ be a continuous bounded strongly pseudocontractive mapping with a pseudocontractive coefficient $\beta \in (0,1)$. Then, $\{x_t\}$ defined by Equation (7) converges strongly to a point in $Fix(T)$ as $t \to 0$.*

Proof. First, from Proposition 1 (ii), we know that $\{x_t : t \in (0, t_0)\}$ is bounded for $t \in (0, t_0)$ for some $t_0 \in (0,1)$.

Since f is a bounded mapping and S is a nonexpansive mapping, $\{fx_t : t \in (0,t_0)\}$ and$\{Sx_t : t \in (0,t_0)\}$ are bounded. Moreover, noting that $x_t = tfx_t + r_tSx_t + (1-t-r_t)Tx_t$, we have

$$Tx_t = \frac{1}{1-t-r_t}x_t - \frac{t}{1-t-r_t}fx_t - \frac{r_t}{1-t-r_t}Sx_t,$$

which implies that

$$\|Tx_t\| \leq \frac{1}{1-t-r_t}\|x_t\| + \frac{t}{1-t-r_t}\|fx_t\| + \frac{r_t}{1-t-r_t}\|Sx_t\|.$$

Thus, we obtain

$$\|Tx_t\| \leq 2\|x_t\| + 2t\|fx_t\| + 2r_t\|Sx_t\|, \ \ \forall t \in (0,t_0)$$

and so $\{Tx_t : t \in (0,t_0)\}$ is bounded. This implies that

$$\lim_{t\to 0}\|x_t - Tx_t\| \leq \lim_{t\to 0} t\|fx_t - Tx_t\| + \lim_{t\to 0} r_t\|Sx_t - Tx_t\| = 0. \tag{11}$$

Now, let $t_m \in (0,t_0)$ for some $t_0 \in (0,1)$ be such that $t_m \to 0$, and let $\{x_m\} := \{x_{t_m}\}$ be a subsequence of $\{x_t\}$. Then,

$$x_m = t_mfx_m + r_mS_m + (1-t_m-r_m)Tx_m.$$

Let $p \in Fix(T)$. Then, we have

$$x_m - p = t_m(fx_m - p) + r_m(Sx_m - p) + (1-t_m-r_m)(Tx_m - Tp)$$

and

$$\begin{aligned}\|x_m - p\|\varphi(\|x_m - p\|) &= \langle x_m - p, J_\varphi(x_m - p)\rangle \\ &\leq t_m\langle fx_m - p, J_\varphi(x_m - p)\rangle + r_m\langle Sx_m - p, J_\varphi(x_m - p)\rangle \\ &\quad + (1-t_m-r_m)\|x_m - p\|\varphi(\|x_m - p\|).\end{aligned}$$

Thus, it follows that

$$\|x_m - p\|\varphi(\|x_m - p\|) \leq \frac{t_m}{t_m + r_m}\langle fx_m - p, J_\varphi(x_m - p)\rangle + \frac{r_m}{t_m + r_m}\langle Sx_m - p, J_\varphi(x_m - p)\rangle. \tag{12}$$

Hence, we get

$$\langle p - fx_m, J_\varphi(x_m - p)\rangle \leq -\frac{t_m + r_m}{t_m}\|x_m - p\|\varphi(\|x_m - p\|) + \frac{r_m}{t_m}\langle Sx_m - p, J_\varphi(x_m - p)\rangle,$$

that is,

$$\langle p - fx_m, J_\varphi(p - x_m)\rangle \geq \frac{t_m + r_m}{t_m}\|x_m - p\|\varphi(\|x_m - p\|) + \frac{r_m}{t_m}\langle p - Sx_m, J_\varphi(x_m - p)\rangle.$$

Therefore, we have

$$\begin{aligned}\langle x_m - fx_m, J_\varphi(p - x_m)\rangle &= \langle x_m - p, J_\varphi(p - x_m)\rangle + \langle p - fx_m, J_\varphi(p - x_m)\rangle \\ &\geq -\|x_m - p\|\varphi(\|x_m - p\|) + \frac{t_m + r_m}{t_m}\|x_m - p\|\varphi(\|x_m - p\|) \\ &\quad + \frac{r_m}{t_m}\langle p - Sx_m, J_\varphi(x_m - p)\rangle \\ &= \frac{r_m}{t_m}\|x_m - p\|\varphi(\|x_m - p\|) + \frac{r_m}{t_m}\langle p - Sx_m, J_\varphi(x_m - p)\rangle.\end{aligned}$$

On the other hand, since $\{x_m\}$ is bounded and E is reflexive, $\{x_m\}$ has a weakly convergent subsequence $\{x_{m_k}\}$, say, $x_{m_k} \rightharpoonup u \in E$. From Equation (11), it follows that

$$\|x_m - Tx_m\| \leq t_m\|fx_m - Tx_m\| + r_m\|Sx_m - Tx_m\| \to 0.$$

From Lemma 4, we know that the mapping $A = (2I - T)^{-1} : C \to C$ is nonexpansive, that $Fix(A) = Fix(T)$, and that $\|x_m - Ax_m\| \to 0$. Thus, by Lemma 5, $u \in Fix(A) = Fix(T)$. Therefore, by Equation (12) and the assumption that J_φ is weakly continuous at 0, we obtain

$$\begin{aligned}\|x_{m_k} - u\|\varphi(\|x_{m_k} - u\|) &\leq \frac{t_{m_k}}{t_{m_k} + r_{m_k}}\langle fx_{m_k} - u, J_\varphi(x_{m_k} - u)\rangle + \frac{r_{m_k}}{t_{m_k} + r_{m_k}}\langle Sx_{m_k} - u, J_\varphi(x_{m_k} - u)\rangle \\ &\leq |\langle fx_{m_k} - u, J_\varphi(x_{m_k} - u)\rangle| + \frac{r_{m_k}}{t_{m_k}}|\langle Sx_{m_k} - u, J_\varphi(x_{m_k} - u)\rangle| \to 0.\end{aligned}$$

Since φ is continuous and strictly increasing, we must have $x_{m_k} \to u$.

Now, we will show that every weakly convergent subsequence of $\{x_m\}$ has the same limit. Suppose that $x_{m_k} \rightharpoonup u$ and $x_{m_j} \rightharpoonup v$. Then, by the above proof, we have $u,\ v \in Fix(T)$ and $x_{m_k} \to u$ and $x_{m_j} \to v$. By Equation (12), we have the following for all $p \in Fix(T)$:

$$\begin{aligned}\|x_{m_k} - p\|\varphi(\|x_{m_k} - p\|) &\leq \frac{t_{m_k}}{t_{m_k} + r_{m_k}}\langle fx_{m_k} - p, J_\varphi(x_{m_k} - p)\rangle + \frac{r_{m_k}}{t_{m_k} + r_{m_k}}\langle Sx_{m_k} - p, J_\varphi(x_{m_k} - p)\rangle \\ &\leq \frac{t_{m_k}}{t_{m_k} + r_{m_k}}\langle fx_{m_k} - p, J_\varphi(x_{m_k} - p)\rangle + \frac{r_{m_k}}{t_{m_k}}|\langle Sx_{m_k} - p, J_\varphi(x_{m_k} - p)\rangle|\end{aligned}$$

and

$$\begin{aligned}\|x_{m_j} - p\|\varphi(\|x_{m_j} - p\|) &\leq \frac{t_{m_j}}{t_{m_j} + r_{m_j}}\langle fx_{m_j} - p, J_\varphi(x_{m_j} - p)\rangle + \frac{r_{m_j}}{t_{m_j} + r_{m_j}}\langle Sx_{m_j} - p, J_\varphi(x_{m_j} - p)\rangle \\ &\leq \frac{t_{m_j}}{t_{m_j} + r_{m_k}}\langle fx_{m_j} - p, J_\varphi(x_{m_k} - p)\rangle + \frac{r_{m_k}}{t_{m_k}}|\langle Sx_{m_k} - p, J_\varphi(x_{m_k} - p)\rangle|.\end{aligned}$$

Taking limits, we get

$$\Phi(\|u - v\|) = \|u - v\|\varphi(\|u - v\|) \leq \langle fu - v, J_\varphi(u - v)\rangle \tag{13}$$

and

$$\Phi(\|v - u\|) = \|v - u\|\varphi(\|v - u\|) \leq \langle fv - u, J_\varphi(v - u)\rangle. \tag{14}$$

Adding up Equations (13) and (14) yields

$$\begin{aligned}2\Phi(\|u - v\|) = 2\|u - v\|\varphi(\|u - v\|) &\leq \|u - v\|\varphi(\|u - v\|) + \langle fu - fv, J_\varphi(u - v)\rangle \\ &\leq (1 + \beta)\|u - v\|\varphi(\|u - v\|) = (1 + \beta)\Phi(\|u - v\|).\end{aligned}$$

Since $\beta \in (0, 1)$, this implies $\Phi(\|u - v\|) \leq 0$, that is, $u = v$. Hence, $\{x_m\}$ is strongly convergent to a point in $Fix(T)$ as $t_m \to 0$.

The same argument shows that, if $t_l \to 0$, then the subsequence $\{x_l\} := \{x_{t_l}\}$ of $\{x_t : t \in (0, t_0)\}$ for some $t_0 \in (0, 1)$ is strongly convergent to the same limit. Thus, as $t \to 0$, $\{x_t\}$ converges strongly to a point in $Fix(T)$. □

Using Theorem 1 and Lemma 8, we show the existence of a unique solution of the variational inequality in Equation (8) in a reflexive Banach space having a weakly continuous duality mapping.

Theorem 2. *Let E be a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ, and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a continuous pseudocontractive* *mapping such that $Fix(T) \neq \emptyset$, let $S : C \to C$ be a nonexpansive mapping, and let $f : C \to C$ be a continuous bounded strongly pseudocontractive mapping with a pseudocontractive coefficient $\beta \in (0, 1)$. Then, there exists the unique solution in $q \in Fix(T)$ of the variational inequality in Equation (8), where $q := \lim_{t\to\infty} x_t$ with x_t being defined by Equation (7).*

Proof. We notice that the definition of the weak continuity of the duality mapping J_φ implies that E is smooth. Thus, E^* is strictly convex for reflexivity of E. By Lemma 8, $\{x_t\}$ defined by Equation (7) converges strongly to a point q in $Fix(T)$ as $t \to 0$. Hence, by Theorem 1, q is the unique solution of the variational inequality in Equation (8). In fact, suppose that $q, p \in Fix(T)$ satisfy the variational inequality in Equation (8). Then, we have

$$\langle (I - f)q, J_\varphi(q - p)\rangle \le 0 \text{ and } \langle (I - f)p, J_\varphi(p - q)\rangle \le 0.$$

Adding these two inequalities, we have

$$(1 - \beta)\Phi(\|q - p\|) = (1 - \beta)\|q - p\|\varphi(\|q - p\|) \le \langle (I - f)q - (I - f)p, J_\varphi(q - p)\rangle \le 0,$$

and so $q = p$. □

As a direct consequence of Theorem 2, we have the following result.

Corollary 1. ([20, Theorem 3.2]) *Let E be a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ, and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a continuous pseudocontractive mapping such that $Fix(T) \neq \emptyset$, and let $f : C \to C$ be a continuous bounded strongly pseudocontractive mapping with a pseudocontractive coefficient $\beta \in (0, 1)$. Let $\{x_t\}$ be defined by*

$$x_t = tfx_t + (1 - t)Tx_t, \quad \forall t \in (0, 1).$$

Then, as $t \to 0$, x_t converges strongly to a some point of T such that q is the unique solution of the variational inequality in Equation (8).

Proof. Put $S = I$ and $r_t = 0$ for all $t \in (0, 1)$. Then, the result follows immediately from Theorem 2. □

Remark 1. *(1) Theorem 2 develops and supplements Theorem 2.1 of Ceng et al. [17] in the following aspects:*

- *(i) The space is replaced by the space having a weakly continuous duality mapping J_φ with gauge function φ.*
- *(ii) The Lipischiz strongly pseudocontractive mapping f in Theorem 2.1 in Reference [17] is replaced by a bounded continuous strongly pseudocontractive mapping f in Theorem 2.*

(2) Corollary 1 complements Theorem 2.1 of Song and Chen [11] and Corollary 2.2 of Cent et al. [17] by replacing the Lipischiz strongly pseudocontractive mapping f in References [11,17] by the bounded continuous strongly pseudocontractive mapping f in Corollary 3.5 in a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ.

(3) Corollary 1 also develops Theorem 2 of Morales [15] to a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ.

4. Modified Implicit Iterative Methods with Perturbed Mapping

First, we prepare the following result.

Theorem 3. *Let E be a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ, and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a continuous pseudocontractive mapping such that $Fix(T) \neq \emptyset$, let $S : C \to C$ be a nonexpansive mapping, and let $f : C \to C$ be a continuous bounded strongly pseudocontractive mapping with a pseudocontractive coefficient $\beta \in (0, 1)$. Let $\{x_t\}$ be* *defined by Equation (7). If there exists a bounded sequence* $\{x_n\}$ *such that* $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ *and* $q = \lim_{t\to 0} x_t$*, then*

$$\limsup_{n\to\infty}\langle fq - q, J_\varphi(x_n - q)\rangle \le 0.$$

Proof. Using the equality

$$x_t - x_n = (1 - t - r_t)(Tx_t - x_n) + t(fx_t - x_n) + r_t(Sx_t - x_n)$$

and the inequality

$$\langle Tx - Ty, J_\varphi(x - y)\rangle \le \|x - y\|\varphi(\|x - y\|), \quad \forall x,\ y \in C,$$

we derive

$$\begin{aligned}
\|x_t - x_n\|\varphi(\|x_t - x_n\|) &= (1 - t - r_t)\langle Tx_t - x_n, J_\varphi(x_t - x_n)\rangle + t\langle fx_t - x_n, J_\varphi(x_t - x_n)\rangle \\
&\quad + r_t\langle Sx_t - x_n, J_\varphi(x_t - x_n)\rangle \\
&= (1 - t - r_t)(\langle Tx_t - Tx_n, J_\varphi(x_t - x_n)\rangle + \langle Tx_n - x_n, J_\varphi(x_t - x_n)\rangle \\
&\quad t\langle fx_t - x_t, J_\varphi(x_t - x_n)\rangle + t\|x_t - x_n\|\varphi(\|x_t - x_n\|) \\
&\quad + r_t\langle Sx_t - x_t, J_\varphi(x_t - x_n)\rangle + r_t\|x_t - x_n\|\varphi(\|x_t - x_n\|) \\
&\le \|x_t - x_n\|\varphi(\|x_t - x_n\|) + \|Tx_n - x_n\|\varphi(\|x_t - x_n\|) \\
&\quad t\langle fx_t - x_t, J_\varphi(x_t - x_n)\rangle + r_t\|Sx_t - x_n\|\varphi(\|x_t - x_n\|)
\end{aligned}$$

and, hence,

$$\langle x_t - fx_t, J_\varphi(x_t - x_n)\rangle \le \frac{\|Tx_n - x_n\|}{t}\varphi(\|x_t - x_n\|) + \frac{r_t}{t}\|Sx_t - x_t\|\varphi(\|x_t - x_n\|).$$

Therefore, by $\limsup_{n\to\infty}\varphi(\|x_t - x_n\|) < \infty$, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle x_t - fx_t, J_\varphi(x_t - x_n)\rangle &\le \limsup_{n\to\infty}\frac{\|Tx_n - x_n\|}{t}\varphi(\|x_t - x_n\|) \\
&\quad + \limsup_{n\to\infty}\frac{r_t}{t}\|Sx_t - x_t\|\varphi(\|x_t - x_n\|) \\
&= \limsup_{n\to\infty}\frac{r_t}{t}\|Sx_t - x_t\|\varphi(\|x_t - x_n\|) \\
&= \frac{r_t}{t}\|Sx_t - x_t\|\limsup_{n\to\infty}\varphi(\|x_t - x_n\|).
\end{aligned}$$

Thus, noting that $\lim_{t\to 0}\limsup_{n\to\infty}\varphi(\|x_t - x_n\|) < \infty$, by Lemma 1, we conclude

$$\begin{aligned}
\limsup_{n\to\infty}\langle fq - q, J_\varphi(x_n - q)\rangle &= \lim_{t\to 0}\limsup_{n\to\infty}\langle fx_t - x_t, J_\varphi(x_n - x_t)\rangle \\
&\le \lim_{t\to 0}\left[\frac{r_t}{t}\|Sx_t - x_t\|\right]\lim_{t\to 0}\limsup_{n\to\infty}\varphi(\|x_t - x_n\|) \\
&= 0 \times \lim_{t\to 0}\limsup_{n\to\infty}\varphi(\|x_t - x_n\|) = 0.
\end{aligned}$$

This completes the proof. □

Theorem 4. *Let E be a reflexive Banach space having a weakly continuous duality mapping* J_φ *with gauge function* φ*, and let C be a nonempty closed convex subset of E. Let* $T : C \to C$ *be a continuous pseudocontractive mapping such that* $Fix(T) \neq \emptyset$*, let* $S : C \to C$ *be a nonexpansive mapping, and let* $f : C \to C$ *be a contractive* *mapping with a contractive coefficient $k \in (0,1)$. For $x_0 \in C$, let $\{x_n\}$ be defined by the following iterative scheme:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Ty_n \\ x_{n+1} = \beta_n f y_n + \gamma_n S y_n + (1-\beta_n-\gamma_n)y_n, \quad \forall n \geq 0, \end{cases} \tag{15}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\lim_{n\to\infty} \beta_n = 0, \sum_{n=0}^{\infty} \beta_n = \infty$;
(iii) $\lim_{n\to\infty}(\gamma_n/\beta_n) = 0, \ \beta_n + \gamma_n \leq 1, \ \forall n \geq 0$.

Then, $\{x_n\}$ converges strongly to a fixed point x^* of T, which is the unique solution of the following variational inequality

$$\langle (I-f)x^*, J_\varphi(x^*-p)\rangle \leq 0, \quad \forall p \in Fix(T). \tag{16}$$

Proof. First, put $z_t = tfz_t + r_t S z_t + (1-t-r_t)Tz_t$. Then, it follows from Theorem 2 that, as $t \to 0$, z_t converges strongly to some fixed point x^* of T such that x^* is the unique solution in $Fix(T)$ to the variational inequality in Equation (16).

Now, we divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. To this end, let $p \in Fix(T)$. Then, we have

$$\begin{aligned} \|y_n - p\|\varphi(\|y_n-p\|) &= \langle \alpha_n x_n + (1-\alpha_n)Ty_n - p, J_\varphi(y_n-p)\rangle \\ &\leq (1-\alpha_n)\langle Ty_n - Tp, J_\varphi(y_n-p)\rangle + \alpha_n\|x_n-p\|\varphi(\|y_n-p\|) \\ &\leq (1-\alpha_n)\|y_n-p\|\varphi(\|y_n-p\|) + \alpha_n\|x_n-p\|\varphi(\|y_n-p\|) \end{aligned}$$

and, hence,

$$\|y_n - p\| \leq \|x_n - p\|, \quad \forall n \geq 0.$$

Thus, we obtain

$$\begin{aligned} \|x_{n+1}-p\| &\leq \beta_n\|fy_n-p\| + \gamma_n\|Sy_n-p\| + (1-\beta_n-\gamma_n)\|y_n-p\| \\ &\leq \beta_n(\|fy_n - fp\| + \|fp-p\|) + \gamma_n(\|Sy_n-Sp\| + \|Sp-p\|) \\ &\quad + (1-\beta_n-\gamma_n)\|x_n-p\| \\ &\leq \beta_n k\|y_n-p\| + \beta_n\|fp-p\| + \gamma_n\|y_n-p\| + \gamma_n\|Sp-p\| \\ &\quad + (1-\beta_n-\gamma_n)\|x_n-p\| \\ &\leq \beta_n k\|x_n-p\| + \beta_n\|fp-p\| + \gamma_n\|x_n-p\| + \gamma_n\|Sp-p\| \\ &\quad + (1-\beta_n-\gamma_n)\|x_n-p\| \\ &= (1-(1-k)\beta_n)\|x_n-p\| + \beta_n\|fp-p\| + \gamma_n\|Sp-p\|. \end{aligned} \tag{17}$$

Since $\lim_{n\to\infty}(\gamma_n/\beta_n) = 0$, we may assume without loss of generality that $\gamma_n \leq \beta_n$ for all $n > 0$. Therefore, it follows from Equation (17) that

$$\begin{aligned} \|x_{n+1}-p\| &\leq (1-(1-k)\beta_n)\|x_n-p\| + (1-k)\beta_n \cdot \frac{1}{1-k}(\|fp-p\| + \|Sp-p\|) \\ &\leq \max\left\{\|x_n-p\|, \frac{1}{1-k}(\|fp-p\| + \|Sp-p\|)\right\}. \end{aligned}$$

By induction, we derive

$$\|x_n - p\| \leq \max\left\{\|x_0-p\|, \frac{1}{1-k}(\|fp-p\| + \|Sp-p\|)\right\}, \quad \forall n \geq 0.$$

This show that $\{x_n\}$ is bounded and so is $\{y_n\}$.

Step 2. We show that $\{fy_n\}$, $\{Sy_n\}$, and $\{Ty_n\}$ are bounded. Indeed, observe that

$$\|fy_n\| \le \|fy_n - fp\| + \|fp\| \le k\|y_n - p\| + \|fp\|$$

and

$$\|Sy_n\| \le \|Sy_n - Sp\| + \|Sp\| \le \|y_n - p\| + \|Sp\|.$$

Thus, $\{fy_n\}$ and $\{Sy_n\}$ are bounded. Since $\lim_{n\to\infty} \alpha_n = 0$, there exist $n_0 \ge 0$ and $a \in (0,1)$ such that $\alpha_n \le a$ for all $n \ge n_0$. Noting that $y_n = \alpha_n x_n + (1-\alpha_n)Ty_n$, we have

$$Ty_n = \frac{1}{1-\alpha_n} y_n - \frac{\alpha_n}{1-\alpha_n} x_n$$

and so

$$\|Ty_n\| \le \frac{1}{1-\alpha_n}\|y_n\| + \frac{\alpha_n}{1-\alpha_n}\|x_n\| \le \frac{1}{1-a}\|y_n\| + \frac{a}{1-a}\|x_n\|.$$

Consequently, the sequence $\{Ty_n\}$ is also bounded.

Step 3. We show that $\limsup_{n\to\infty}\langle fx^* - x^*, J_\varphi(y_n - x^*)\rangle \le 0$. In fact, from condition (i) and boundedness of $\{x_n\}$ and $\{Ty_n\}$, we get

$$\|y_n - Ty_n\| = \alpha_n\|x_n - Ty_n\| \to 0 \quad (n \to \infty). \tag{18}$$

Thus, it follows from Equation (18) and Theorem 3 that $\limsup_{n\to\infty}\langle fx^* - x^*, J_\varphi(y_n - x^*)\rangle \le 0$.

Step 4. We show that $\limsup_{n\to\infty}\langle fx^* - x^*, J_\varphi(x_{n+1} - x^*)\rangle \le 0$. Indeed, by Equations (15) and (18), we have

$$\begin{aligned}\|x_{n+1} - y_n\| &= \|\beta_n fy_n + \gamma_n Sy_n + (1-\beta_n-\gamma_n)y_n - (\alpha_n x_n + (1-\alpha_n)Ty_n)\| \\ &\le \alpha_n\|x_n - Ty_n\| + \beta_n\|fy_n - y_n\| + \gamma_n\|Sy_n - y_n\| + \|y_n - Ty_n\| \to 0 \ \ (n\to\infty).\end{aligned}$$

Since the duality mapping J_φ is single-valued and weakly continuous, we have

$$\lim_{n\to\infty}\langle fx^* - x^*, J_\varphi(x_{n+1} - x^*) - J_\varphi(y_n - x^*)\rangle = 0.$$

Therefore, we obtain from step 3 that

$$\begin{aligned}\limsup_{n\to\infty}\langle fx^* - x^*, J_\varphi(x_{n+1} - x^*)\rangle &\le \limsup_{n\to\infty}\langle fx^* - x^*, J_\varphi(y_n - x^*)\rangle \\ &\quad + \limsup_{n\to\infty}\langle fx^* - x^*, J_\varphi(x_{n+1} - x^*) - J_\varphi(y_n - x^*)\rangle \\ &= \limsup_{n\to\infty}\langle fx^* - x^*, J_\varphi(y_n - x^*)\rangle \le 0.\end{aligned}$$

Step 5. We show that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. In fact, it follows from Equation (15) that

$$\begin{aligned}x_{n+1} - x^* = {} &\beta_n(fy_n - fx^*) + \gamma_n(Sy_n - Sx^*) + (1-\beta_n-\gamma_n)(y_n - x^*) \\ &+ \beta_n(fx^* - x^*) + \gamma_n(Sx^* - x^*).\end{aligned}$$

Therefore, using inequalities $\|y_n - x^*\| \le \|x_n - x^*\|$, $\|fx - fy\| \le k\|x - y\|$, and $\|Sx - Sy\| \le \|x - y\|$ and using Lemma 2, we have

$$\begin{aligned}
\Phi(\|x_{n+1}-x^*\|) \leq{}& \Phi(\|\beta_n(fy_n - fx^*) + \gamma_n(Sy_n - Sx^*) + (1-\beta_n-\gamma_n)(y_n - x^*)\|) \\
&+ \beta_n\langle fx^* - x^*, J_\varphi(x_{n+1}-x^*)\rangle + \gamma_n\langle Sx^* - x^*, J_\varphi(x_{n+1}-x^*)\rangle \\
\leq{}& \Phi(\beta_n k\|y_n - x^*\| + \gamma_n\|y_n - x^*\| + (1-\beta_n-\gamma_n)\|y_n - x^*\|) \\
&+ \beta_n\langle fx^* - x^*, J_\varphi(x_{n+1}-x^*)\rangle + \gamma_n\langle Sx^* - x^*, J_\varphi(x_{n+1}-x^*)\rangle \\
\leq{}& \Phi((1-(1-k)\beta_n)\|x_n - x^*\|) \\
&+ \beta_n\langle fx^* - x^*, J_\varphi(x_{n+1}-x^*)\rangle + \gamma_n\langle Sx^* - x^*, J_\varphi(x_{n+1}-x^*)\rangle \\
\leq{}& (1-(1-k)\beta_n)\Phi(\|x_n - x^*\|) + \beta_n\langle fx^* - x^*, J_\varphi(x_{n+1}-x^*)\rangle \\
&+ \gamma_n\|Sx^* - x^*\|\varphi(\|x_{n+1}-x^*\|) \\
\leq{}& (1-\lambda_n)\Phi(\|x_n - x^*\|) + \lambda_n\delta_n,
\end{aligned} \tag{19}$$

where $\lambda_n = (1-k)\beta_n$ and

$$\delta_n = \frac{1}{1-k}\left[\langle fx^* - x^*, J_\varphi(x_{n+1}-x^*)\rangle + \frac{\gamma_n}{\beta_n}\|Sx^* - x^*\|\varphi(\|x_{n+1}-x^*\|)\right].$$

From conditions (ii) and (iii) and from step 4, it is easily seen that $\sum_{n=0}^{\infty}\lambda_n = \infty$ and $\limsup_{n\to\infty}\delta_n \leq 0$. Thus, applying Lemma 3 to Equation (19), we conclude that $\lim_{n\to\infty}\Phi(\|x_n - x^*\|) = 0$ and, hence, $\lim_{n\to\infty}\|x_n - x^*\| = 0$. This completes the proof. □

Theorem 5. *Let E be a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ, and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a continuous pseudocontractive mapping such that $Fix(T) \neq \emptyset$, let $S : C \to C$ be a nonexpansive mapping, and let $f : C \to C$ be a contractive mapping with a contractive coefficient $k \in (0,1)$. For $x_0 \in C$, let $\{x_n\}$ be defined by the following iterative scheme:*

$$\begin{cases} x_n = \alpha_n y_n + (1-\alpha_n)Tx_n \\ y_n = \beta_n fx_{n-1} + \gamma_n Sx_{n-1} + (1-\beta_n-\gamma_n)x_{n-1}, \quad \forall n \geq 0, \end{cases} \tag{20}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty}\beta_n = \infty$;
(iii) $\lim_{n\to\infty}(\gamma_n/\beta_n) = 0$, $\beta_n + \gamma_n \leq 1$, $\forall n \geq 0$.

Then, $\{x_n\}$ converges strongly to a fixed point x^* of T, which is the unique solution of the variational inequality in Equation (16).

Proof. First, as in Theorem 4, we put $z_t = tfz_t + r_tSz_t + (1-t-r_t)Tz_t$. Then, from Theorem 2, it follows that, as $t \to 0$, z_t converges strongly to some fixed point x^* of T such that x^* is the unique solution in $Fix(T)$ to the variational inequality in Equation (16).

Now, we divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. To this end, let $p \in Fix(T)$. Then, by Equation (20), we have

$$\begin{aligned}
\|x_n - p\|\varphi(\|x_n - p\|) ={}& \langle \alpha_n y_n + (1-\alpha_n)Tx_n - p, J_\varphi(x_n - p)\rangle \\
\leq{}& (1-\alpha_n)\langle Tx_n - Tp, J_\varphi(x_n - p)\rangle + \alpha_n\|y_n - p\|\varphi(\|x_n - p\|) \\
\leq{}& (1-\alpha_n)\|x_n - p\|\varphi(\|x_n - p\|) + \alpha_n\|y_n - p\|\varphi(\|y_n - p\|)
\end{aligned}$$

and, hence,

$$\|x_n - p\| \leq \|y_n - p\|, \quad \forall n \geq 0.$$

Thus, we obtain

$$
\begin{aligned}
\|x_n - p\| &\le \|y_n - p\| \\
&\le \beta_n\|fx_{n-1} - p\| + \gamma_n\|Sx_{n-1} - p\| + (1-\beta_n-\gamma_n)\|x_{n-1} - p\| \\
&\le \beta_n(\|fx_{n-1} - fp\| + \|fp - p\|) + \gamma_n(\|Sx_{n-1} - Sp\| + \|Sp - p\|) \\
&\quad + (1-\beta_n-\gamma_n)\|x_{n-1} - p\| \\
&\le \beta_n k\|x_{n-1} - p\| + \beta_n\|fp - p\| + \gamma_n\|x_{n-1} - p\| + \gamma_n\|Sp - p\| \\
&\quad + (1-\beta_n-\gamma_n)\|x_{n-1} - p\| \\
&= (1-(1-k)\beta_n)\|x_{n-1} - p\| + \beta_n\|fp - p\| + \gamma_n\|Sp - p\|.
\end{aligned} \tag{21}
$$

Since $\lim_{n\to\infty}(\gamma_n/\beta_n) = 0$, we may assume without loss of generality that $\gamma_n \le \beta_n$ for all $n > 0$. Therefore, it follows from Equation (21) that

$$
\begin{aligned}
\|x_n - p\| &\le (1-(1-k)\beta_n)\|x_{n-1} - p\| + (1-k)\beta_n \cdot \frac{1}{1-k}(\|fp - p\| + \|Sp - p\|) \\
&\le \max\left\{\|x_{n-1} - p\|, \frac{1}{1-k}(\|fp - p\| + \|Sp - p\|)\right\}.
\end{aligned}
$$

By induction, we derive

$$
\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{1}{1-k}(\|fp - p\| + \|Sp - p\|)\right\}, \quad \forall n \ge 0.
$$

This show that $\{x_n\}$ is bounded and so is $\{y_n\}$.

Step 2. We show that $\{fx_n\}$, $\{Sx_n\}$, and $\{Tx_n\}$ are bounded. Indeed, observe that

$$
\|fx_n\| \le \|fx_n - fp\| + \|fp\| \le k\|x_n - p\| + \|fp\|
$$

and

$$
\|Sx_n\| \le \|Sx_n - Sp\| + \|Sp\| \le \|x_n - p\| + \|Sp\|.
$$

Thus, $\{fx_n\}$ and $\{Sx_n\}$ are bounded. Since $\lim_{n\to\infty}\alpha_n = 0$, there exist $n_0 \ge 0$ and $a \in (0,1)$ such that $\alpha_n \le a$ for all $n \ge n_0$. Noting that $x_n = \alpha_n y_n + (1-\alpha_n)Tx_n$, we have

$$
Tx_n = \frac{1}{1-\alpha_n}x_n - \frac{\alpha_n}{1-\alpha_n}y_n
$$

and so

$$
\|Tx_n\| \le \frac{1}{1-\alpha_n}\|x_n\| + \frac{\alpha_n}{1-\alpha_n}\|y_n\| \le \frac{1}{1-a}\|x_n\| + \frac{a}{1-a}\|y_n\|.
$$

Consequently, the sequence $\{Tx_n\}$ is also bounded.

Step 3. We show that $\limsup_{n\to\infty}\langle fx^* - x^*, J_\varphi(x_n - x^*)\rangle \le 0$. In fact, from condition (i) and boundedness of $\{x_n\}$ and $\{Tx_n\}$, we get

$$
\|x_n - Tx_n\| = \alpha_n\|y_n - Tx_n\| \to 0 \quad (n \to \infty). \tag{22}
$$

Thus, it follows from Equation (22) and Theorem 3 that $\limsup_{n\to\infty}\langle fx^* - x^*, J_\varphi(x_n - x^*)\rangle \le 0$.

Step 4. We show that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. In fact, using the equality

$$
\begin{aligned}
x_n - x^* &= \alpha_n[\beta_n(fx_{n-1} - fx^*) + \gamma_n(Sx_{n-1} - Sx^*) + (1-\beta_n-\gamma_n)(x_{n-1} - x^*)] \\
&\quad + \alpha_n[\beta_n(fx^* - x^*) + \gamma_n(Sx^* - x^*)] + (1-\alpha_n)(Tx_n - x^*)
\end{aligned}
$$

by Equation (20) and the inequalities $\langle Tx - Ty, J_\varphi(x-y)\rangle \le \|x-y\|\varphi(\|x-y\|) = \Phi(\|x-y\|)$, $\|fx - fy\| \le k\|x-y\|$, and $\|Sx - Sy\| \le \|x-y\|$, from Lemma 2, we derive

$$\begin{aligned}\Phi(\|x_n - x^*\|) &= \Phi(\alpha_n\|\beta_n(fx_{n-1} - fx^*) + \gamma_n(Sx_{n-1} - Sx^*) + (1-\beta_n-\gamma_n)(x_{n-1} - x^*)\|) \\ &\quad + \alpha_n\beta_n\langle fx^* - x^*, J_\varphi(x_n - x^*)\rangle + \alpha_n\gamma_n\langle Sx^* - x^*, J_\varphi(x_n - x^*)\rangle \\ &\quad + (1-\alpha_n)\langle Tx_n - x^*, J_\varphi(x_n - x^*)\rangle \\ &\le \alpha_n\Phi(\beta_n k\|x_{n-1} - x^*\| + \gamma_n\|x_{n-1} - x^*\| + (1-\beta_n-\gamma_n)\|x_{n-1} - x^*\|) \\ &\quad + \alpha_n\beta_n\langle fx^* - x^*, J_\varphi(x_n - x^*)\rangle + \alpha_n\gamma_n\langle Sx^* - x^*, J_\varphi(x_n - x^*)\rangle \\ &\quad + (1-\alpha_n)\|x_n - x^*\|\varphi(\|x_n - x^*\|) \\ &\le \alpha_n(1-(1-k)\beta_n)\Phi(\|x_{n-1} - x^*\|) \\ &\quad + \alpha_n\beta_n\langle fx^* - x^*, J_\varphi(x_n - x^*)\rangle + \alpha_n\gamma_n\|Sx^* - x^*\|\varphi(\|x_n - x^*\|) \\ &\quad + (1-\alpha_n)\Phi(\|x_n - x^*\|).\end{aligned} \tag{23}$$

By Equation (23), we obtain

$$\begin{aligned}\Phi(\|x_n - x^*\|) &\le (1-(1-k)\beta_n)\Phi(\|x_{n-1} - x^*\|) + \beta_n\langle fx^* - x^*, J_\varphi(x_n - x^*)\rangle \\ &\quad + \gamma_n\|Sx^* - x^*\|\varphi(\|x_n - x^*\|) \\ &\le (1-(1-k)\beta_n)\|x_{n-1} - x^*\| + \beta_n\langle fx^* - x^*, J_\varphi(x_n - x^*)\rangle + \gamma_n\|Sx^* - x^*\|M,\end{aligned} \tag{24}$$

where $M > 0$ is a constant such that $\varphi(\|x_n - x^*\|) \le M$ for all $n \ge 1$. Put $\lambda_n = (1-k)\beta_n$ and

$$\delta_n = \frac{1}{1-k}\left[\langle fx^* - x^*, J_\varphi(x_n - x^*\rangle + \frac{\gamma_n}{\beta_n}\|Sx^* - x^*\|M\right].$$

From conditions (ii) and (iii) and from step 3, it easily seen that $\sum_{n=0}^{\infty}\lambda_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Since Equation (24) reduces to

$$\Phi(\|x_n - x^*\|) \le (1-\lambda_n)\Phi(\|x_{n-1} - x^*\|) + \lambda_n\delta_n, \tag{25}$$

applying Lemma 3 to Equation (25), we conclude that $\lim_{n\to\infty}\Phi(\|x_n - x^*\|) = 0$ and, hence, $\lim_{n\to\infty}\|x_n - x^*\| = 0$. This completes the proof. □

Remark 2. *(1) Theorem 3 develops Theorem 2.3 of Ceng et al. [17] in the following aspects:*

- *(i) The space is replaced by the space having a weakly continuous duality mapping J_φ with gauge function φ.*
- *(ii) The Lipischiz strongly pseudocontractive mapping f in Theorem 2.3 in Reference [17] is replaced by a bounded continuous strongly pseudocontractive mapping f in Theorem 3.*

(2) Theorem 4 complements Theorem 3.1 as well as Theorem 3.4 of Ceng et al. [17] in a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ.

(3) Theorem 5 also means that Theorem 3.2 as well as Theorem 3.5 of Ceng et al. [17] hold in a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ.

(4) Whenever $S = I$ and $\gamma_n = 0$ for all $n \ge 0$ in Theorem 5, it is easily seen that Theorem 3.1 Theorem 3.4 of Song and Chen [11] hold in a reflexive Banach space which has a weakly continuous duality mapping J_φ with gauge function φ.

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