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Stability Results for Implicit Fractional Pantograph Differential Equations via ϕ-Hilfer Fractional Derivative with a Nonlocal Riemann-Liouville Fractional Integral Condition

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Abstract: This paper presents a class of implicit pantograph fractional differential equation with more general Riemann-Liouville fractional integral condition. A certain class of generalized fractional derivative is used to set the problem. The existence and uniqueness of the problem is obtained using Schaefer's and Banach fixed point theorems. In addition, the Ulam-Hyers and generalized Ulam-Hyers stability of the problem are established. Finally, some examples are given to illustrative the results.

1. Introduction

The fractional-order differential equation is the oldest theory in the field of science and engineering. This theory has been used over the years, as the outcomes were found to be important in the field of economics, control theory and material sciences see [1–4]. Because of the nonlocal property

of fractional-order differential equation, researchers are allowed to select the most appropriate operator and use it in order to get a better description of the complex phenomena in the real world. The generalization of classical calculus are the fractional calculus. Nevertheless, there are various definitions of fractional integrals and derivatives of arbitrary order with different types of operator. Recently, Furati et al. [5] proposed a Hilfer fractional derivatives which interpolates with Riemann-Liouville and Caputo fractional derivatives. These fractional operator provide an extra degree of freedom when choosing the initial condition. Furthermore, models based on this operator provide an excellent results compared with the integer-order derivatives, for example, we refer the interesting reader to see [6–18].

Qualitative analysis of fractional differential equations plays a vital role in the field of fractional differential equations. However, many researchers studied the existence and uniqueness of solution of differential equation with different types of fractional integral and derivatives. More recently, motivated by classical Riemann-Liouville, Caputo fractional derivative, Hilfer-fractional derivative, ψ-Riemann-Liouville integral and ψ-Caputo fractional derivatives, Sousa and Oliveira [19] initiated an interesting fractional differential operator called ψ-Hilfer fractional derivatives, that is a fractional derivative of a function with respect to another function ψ. These fractional derivatives generalized the aforementioned fractional derivatives and integrals. The main advantages of these operator is the freedom of choice of the function ψ and its merge and acquire the properties of the aforementioned fractional operators. Results based on these setting can be found in [18–34]. The Ulam-Hyers stability point of view, is the vital and special type of stability that attracts many researchers in the field of mathematical analysis. Moreover, the Ulam-Hyers and Ulam-Hyers-Rassias stability of linear, implicit and nonlinear fractional differential equations were examined in [17,35–49].

Pantograph differential equations are a special class of delay differential equation arising in deterministic situations and are of the form:

$$\begin{cases} g'(s) = kg(s) + lg(\lambda s), & s \in [0,b],\ b > 0,\ 0 < \lambda < 1, \\ g(0) = g_0. \end{cases} \tag{1}$$

The pantograph is a device used in electric trains to collects electric current from the overload lines. This equation was modeled by Ockendon and Tayler [50]. Pantograph equation play a vital role in physics, pure and applied mathematics, such as control systems, electrodynamics, probability, number theory, and quantum mechanics. Motivated by their importance, a lot of researchers generalized these equation in to various forms and introduced the solvability aspect of such problems both theoretically and numerically, (for more details see [16,51–57] and references therein). However, very few works have been proposed with respect to pantograph fractional differential equations.

In [48], the authors considered an implicit fractional differential equations with nonlocal condition described by:

$$\begin{cases} D_{0^+}^{\alpha,\beta} w(\tau) = f(\tau, w(\tau), D_{0^+}^{\alpha,\beta} w(\tau)), & \tau \in I = [0,T], \\ I_{0^+}^{1-\gamma} w(0) = \sum_{i=1}^{m} c_i w(\eta_i), & \alpha \le \gamma = \alpha + \beta - \alpha\beta, \eta_i \in [0,T], \end{cases} \tag{2}$$

where $D_{0^+}^{\alpha,\beta}(\cdot)$ is the Hilfer fractional derivative of order $(0 < \alpha < 1)$ and type $0 \le \beta \le 1$. The existence and uniqueness results were obtained by applying Schaefer's fixed point theorem and Banach's contraction principle. Moreover, the authors discussed the stability analysis via Gronwall's lemma. Sousa and Oliveira [47] discussed the existence, uniqueness and Ulam-Hyers-Rassias stability for a class of φ-Hilfer fractional differential equations described by:

$$\begin{cases} {}^{H}\mathfrak{D}_{a^+}^{\alpha,\beta;\varphi} g(t) = f(t, g(t), {}^{H}\mathfrak{D}_{a^+}^{\alpha,\beta;\varphi} g(t)), & t \in \mathcal{J} = [a,T], \\ \mathfrak{I}_{a^+}^{1-\gamma;\varphi} g(a) = g_a, & \alpha \le \gamma = \alpha + \beta - \alpha\beta, \quad T > a, \end{cases} \tag{3}$$

where $^H\mathfrak{D}_{0^+}^{\alpha,\beta;\varphi}(\cdot)$ is the φ-Hilfer fractional derivative of order $(0 < \alpha \le 1)$ and operator $(0 \le \beta \le 1)$, $\mathfrak{I}_{0^+}^{1-\gamma;\varphi}(\cdot)$, is the Riemann-Liouville fractional integral of order $1 - \gamma$, with respect to the function φ, $f : [a, T] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. Recently Harikrishman et. al [58] established existence and uniqueness of nonlocal initial value problem for fractional pantograph differential equation involving ψ-Hilfer fractional derivative of the form:

$$\begin{cases} ^H\mathrm{D}_{a^+}^{\alpha,\beta;\psi}v(s) = f(s, v(s), v(\lambda s)), & s \in (a, b], \quad s > a, \quad 0 < \lambda < 1, \\ \mathrm{I}_{a^+}^{1-\gamma;\psi}v(a) = \sum_{j=1}^k c_i v(\tau_j), & \tau_j \in (a, b], \quad \gamma = \alpha + \beta - \alpha\beta, \end{cases} \tag{4}$$

where $^H\mathrm{D}_{a^+}^{\alpha,\beta;\phi}(\cdot)$ is the ψ-Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 \le \beta \le 1$, $\mathrm{I}_{a^+}^{1-\gamma;\psi}(\cdot)$, is the Riemann-Liouville fractional integral of order $1 - \gamma$, with respect to the continuous function ψ such that $\psi'(\cdot) > 0$, $f \in C(t \in (a, b], \mathbb{R}^2, \mathbb{R})$.

Motivated by the papers [21,47,48] and some familiar results on fractional pantograph differential equations [16,52,55,58]. We discuss the existence and uniqueness of the solution of the implicit pantograph fractional differential equations involving ϕ-Hilfer fractional derivatives. Furthermore, the Ulam-Hyers and generalized Ulam-Hyers stability are also discussed. The implicit pantograph fractional differential equations involving ϕ-Hilfer fractional derivatives is of the form

$$\begin{cases} ^H\mathcal{D}_{0^+}^{r,p;\phi}z(t) = f(t, z(t), z(\gamma t), {}^H\mathcal{D}_{0^+}^{r,p;\phi}z(\gamma t)), & t \in \mathcal{J} = (0, T], \ 0 < \gamma < 1, \\ \mathcal{I}_{0^+}^{1-q;\phi}z(0^+) = \sum_{i=1}^m b_i \mathcal{I}_{0^+}^{\rho;\phi}z(\xi_i), & r \le q = r + p - rp, \end{cases} \tag{5}$$

where $^H\mathcal{D}_{0^+}^{r,p;\phi}(\cdot)$ is the generalized ϕ-Hilfer fractional derivatives of order $(0 < r < 1)$ and type $(0 \le p \le 1)$, $\mathcal{I}_{0^+}^{1-q;\phi}(\cdot)$ and $\mathcal{I}_{0^+}^{\rho;\phi}(\cdot)$ are ϕ-Riemann-Lioville fractional integral of order $1 - q$ and $\rho > 0$ respectively with respect to the continuous function ϕ such that $\phi'(\cdot) \ne 0$, $f : (0, T] \times \mathbb{R}^3 \to \mathbb{R}$ is a given continuous function, $T > 0$, $b_i \in \mathbb{R}$ and $\xi_i \in \mathcal{J}$ satisfying $0 < \xi_1 \le \xi_2 \le \cdots \le \xi_m < T$ for $i = 1, 2, \cdots, m$.

As far as we know, to the best of our understanding, results of Ulam-Hyers and generalized Ulam-Hyers stability with respect to the pantograph differential equation are very few and in fact most authors discuss existence and uniqueness, while we study existence, uniqueness and stability analysis for a class of implicit pantograph fractional differential equations with ϕ-Hilfer derivatives and nonlocal Riemann-Liouville fractional integral condition.

This paper contributes to the growth of qualitative analysis of fractional differential equation in particular pantograph fractional differential equation when ϕ-Hilfer fractional derivatives involved and the nonlocal initial condition proposed in this paper generalized the following initial conditions:

- If $\rho \to 0$, the initial condition reduces to multi-point nonlocal condition.
- If $\rho \to 1$, the initial condition coincide with the nonlocal integral condition.
- In physical problems, the nonlocal condition yields an excellent results compared with the initial condition $z(0) = z_0$ [59,60].

In addition, we notice that the function $f(s, v(s), v(\lambda s))$, $s \in (a, b]$, $0 < \lambda < 1$, defined in Equation (4) is not well-define for some choices of λ.

Therefore, the paper is organized as follows: In Section 2, it recalls some basic and fundamental definitions and lemmas. In Section 3, we prove existence and uniqueness of the proposed problem (5). Ulam-Hyers and generalized Ulam-Hyers stability for the proposed problem were discussed in Section 4. While in Section 5, two examples were given to illustrate the applicability of our results. Lastly, the conclusion part of the paper is given in Section 6.

2. Preliminaries

This section will recall some useful prerequisites facts, definitions and some fundamental lemmas with respect to fractional differential equations.

Throughout the paper, we denote $\mathcal{C}[\mathcal{J}, \mathbb{R}]$ the Banach space of all continuous functions from $\mathcal{J}$ into $\mathbb{R}$ with the norm defined by [1]

$$\|f\| = \sup_{t \in \mathcal{J}} \{|f(t)|\}.$$

The weighted space $\mathcal{C}_{q,\phi}[\mathcal{J}, \mathbb{R}]$ of continuous function f on the interval $[a, T]$ is defined by

$$\mathcal{C}_{q,\phi}[\mathcal{J}, \mathbb{R}] = \{f(t) : (a, T) : (\phi(t) - \phi(0))^q f(t) \in \mathcal{C}[\mathcal{J}, \mathbb{R}]\},$$

with the norm

$$\|f\|_{\mathcal{C}_{q,\phi}[\mathcal{J}, \mathbb{R}]} = \|(\phi(t) - \phi(0))^q f(t)\| = \max |(\phi(t) - \phi(0))^q f(t) : t \in \mathcal{J}|.$$

Moreover, for each $n \in \mathbb{N}$ and $0 \le q < 1$ with $q = r + p - rp$

$$\mathcal{C}_{q,\phi}^n[\mathcal{J}, \mathbb{R}] = \{f^n \in \mathcal{C}_{q;\phi}[\mathcal{J}, \mathbb{R}]\}$$

$$\mathcal{C}_{q;\phi}^{r,p}[\mathcal{J}, \mathbb{R}] = \{f \in \mathcal{C}_{q;\phi}[\mathcal{J}, \mathbb{R}] : \mathcal{D}_{0^+}^{r,p;\phi} \in \mathcal{C}_{q;\phi}[\mathcal{J}, \mathbb{R}]\}.$$

Indeed, for $n = 0$, we have

$$\mathcal{C}_{q;\phi}^0[\mathcal{J}, \mathbb{R}] = \mathcal{C}_{q;\phi}[\mathcal{J}, \mathbb{R}],$$

with the norm

$$\|f\|_{\mathcal{C}_{q;\phi}^n[\mathcal{J}, \mathbb{R}]} = \sum_{k=0}^{n-1} \|f^k\|_{\mathcal{C}[\mathcal{J}, \mathbb{R}]} + \|f^n\|_{\mathcal{C}_{q;\phi}^n[\mathcal{J}, \mathbb{R}]}.$$

Furthermore, we present the following space $\mathcal{C}_{1-q;\phi}^{r,p}[\mathcal{J}, \mathbb{R}]$ and $\mathcal{C}_{1-q;\phi}^q[\mathcal{J}, \mathbb{R}]$ defined as:

$$\mathcal{C}_{1-q;\phi}^{r,p}[\mathcal{J}, \mathbb{R}] = \{f \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}], \mathcal{D}_{0^+}^{r,p;\phi} \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]\}$$

and

$$\mathcal{C}_{1-q;\phi}^q[\mathcal{J}, \mathbb{R}] = \{f \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}], \mathcal{D}_{0^+}^{q;\phi} \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]\}.$$

Clearly, $\mathcal{C}_{1-q;\phi}^q[\mathcal{J}, \mathbb{R}] \subset \mathcal{C}_{1-q;\phi}^{r,p}[\mathcal{J}, \mathbb{R}]$.

Definition 1 ([1]). *Let $(0, b)$ be a finite or infinite interval on the half-axis $\mathbb{R}^+$, and $\phi(\xi) \ge 0$ be monotone function on $(a, b]$ whose $\phi'(\xi)$ is continuous on $(0, b)$. The ϕ-Hilfer Riemann-Liouville fractional integral of order $r \in \mathbb{R}^+$ of function w is defined by*

$$(\mathcal{I}_{0^+}^{r;\phi} w)(\xi) = \frac{1}{\Gamma(r)} \int_{0^+}^{\xi} \phi'(s)(\phi(\xi) - \phi(s)) w(s) ds, \quad \xi > 0, \tag{6}$$

where $\Gamma(\cdot)$ represent the Gamma function.

Definition 2 ([5]). *Let $n - 1 < r < n$, $0 \le p \le 1$. The left-sided Hilfer fractional derivative of order r and parameter p of function w is defined by*

$$\mathcal{D}_{0^+}^{r,p} w(\xi) = \left(\mathcal{I}_{0^+}^{p(n-r)} \mathcal{D}^n \mathcal{I}_{0^+}^{(1-p)(n-r)} w \right)(\xi), \tag{7}$$

where $\mathcal{D}^n = \left(\frac{d}{d\xi} \right)^n$.

The following Definition generalized Euqation (7).

Definition 3 ([19]). *Let $f, \phi \in C^n[\mathcal{J}, \mathbb{R}]$ be two functions such that $\phi(\xi) \geq 0$ and $\phi'(t) \neq 0$ for all $\xi \in [\mathcal{J}, \mathbb{R}]$ and $n - 1 < r < n$ with $n \in \mathbb{N}$. The left-side ϕ-Hilfer fractional derivative of a function w of order r and type $(0 \leq p \leq 1)$ is defined by*

$$^H\mathcal{D}_{0^+}^{r,p;\phi}w(\xi) = \mathcal{I}_{0^+}^{p(n-r);\phi}\left(\frac{1}{\phi'(\xi)}\frac{d}{d\xi}\right)^n \mathcal{I}_{0^+}^{(1-p)(n-r);\phi}w(\xi). \tag{8}$$

The following lemma shows the semigroup properties of ϕ-Hilfer fractional integral and derivative.

Lemma 1 ([5]). *Let $r \geq 0$, $0 \leq p < 1$ and $w \in L^1[\mathcal{J}, \mathbb{R}]$. Then*

$$\mathcal{I}_{0^+}^{r;\phi}\mathcal{I}_{0^+}^{p;\phi}w(\xi) = \mathcal{I}_{0^+}^{r+p;\phi}w(\xi),$$

a.e $\xi \in \mathcal{J}$.
In particular, if $w \in C_{q,\phi}[\mathcal{J}, \mathbb{R}]$ and $w \in C[\mathcal{J}, \mathbb{R}]$, then

$$\mathcal{I}_{0^+}^{r;\phi}\mathcal{I}_{0^+}^{p;\phi}w(\xi) = \mathcal{I}_{0^+}^{r+p;\phi}w(\xi),$$

for all $\xi \in (0, T]$ and

$$^H\mathcal{D}_{0^+}^{r;\phi}\mathcal{I}_{0^+}^{r;\phi}w(\xi) = w(\xi),$$

for all $\xi \in \mathcal{J}$.

The composition of the ϕ-Hilfer fractional integral and derivative operator is given by the following lemmas.

Lemma 2 ([21]). *Let $r \geq 0$, $0 \leq p < 1$ and $q = r + p - rp$. If $w(\xi) \in C_{1-q}^q[\mathcal{J}, \mathbb{R}]$, then*

$$\mathcal{I}_{0^+}^{q;\phi}{}^H\mathcal{D}_{0^+}^{q;\phi}w(\xi) = \mathcal{I}_{0^+}^{r;\phi}{}^H\mathcal{D}_{0^+}^{r,p;\phi}w(\xi)$$

and

$$^H\mathcal{D}_{0^+}^{q;\phi}\mathcal{I}_{0^+}^{r;\phi}w(\xi) =^H \mathcal{D}_{0^+}^{p(1-r);\phi}w(\xi).$$

Lemma 3 ([6,19]). *If $w \in C^n[\mathcal{J}, \mathbb{R}]$ and let $n - 1 < r < n$, $0 \leq p \leq 1$ and $q = r + p - rp$. Then*

$$\mathcal{I}_{0^+}^{r;\phi}{}^H\mathcal{D}_{0^+}^{r,p;\phi}w(\xi) = w(\xi) - \sum_{k=1}^{n}\frac{(\phi(\xi) - \phi(0))^{q-k}}{\Gamma(q-k+1)}w_\phi^{[n-k]}\mathcal{I}_{0^+}^{(1-p)(n-r);\phi}w(0),$$

for all $\xi \in \mathcal{J}$. Moreover, if $0 < r < 1$, we have

$$\mathcal{I}_{0^+}^{r;\phi}{}^H\mathcal{D}_{0^+}^{r,p;\phi}w(\xi) = w(\xi) - \frac{(\phi(\xi) - \phi(0))^{q-1}}{\Gamma(q)}\mathcal{I}_{0^+}^{(1-p)(1-r);\phi}w(0).$$

In addition, if $w \in C_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$ and $\mathcal{I}_{0^+}^{1-q;\phi}w \in C_{1-q;\phi}^1[\mathcal{J}, \mathbb{R}]$, then

$$\mathcal{I}_{0^+}^{q;\phi}{}^H\mathcal{D}_{0^+}^{q;\phi}w(\xi) = w(\xi) - \frac{(\phi(\xi) - \phi(0))^{q-1}}{\Gamma(q)}\mathcal{I}_{0^+}^{(1-q);\phi}w(0),$$

for all $0 < q < 1$ and $t \in \mathcal{J}$.

Lemma 4 ([6])**.** *Let $r > 0$, $0 \leq q < 1$ and $w \in \mathcal{C}_{q;\phi}[\mathcal{J}, \mathbb{R}]$. If $r > q$, then $\mathcal{I}_{0^+}^{r;\phi} w \in \mathcal{C}[\mathcal{J}, \mathbb{R}]$ and*

$$\mathcal{I}_{0^+}^{r;\phi} w(0) = \lim_{\xi \to 0} \mathcal{I}_{0^+}^{r;\phi} w(\xi) = 0.$$

Lemma 5 ([21])**.** *Let $r > 0$, $0 \leq p \leq 1$ and $q = r + p - rp$. If $w \in \mathcal{C}_{1-q;\phi}^q[\mathcal{J}, \mathbb{R}]$, then*

$$\mathcal{I}_{0^+}^{q;\phi} \mathcal{D}_{0^+}^{q;\phi} w(\xi) = \mathcal{I}_{0^+}^{r;\phi} \mathcal{D}_{0^+}^{r,p;\phi} w(\xi)$$

and

$$^H\mathcal{D}_{0^+}^{q;\phi} \mathcal{I}_{0^+}^{r;\phi} w(\xi) = \mathcal{D}_{0^+}^{q(1-r);\phi} w(\xi).$$

Lemma 6. *Let $f \in L^1(\mathcal{J})$ such that $\mathcal{D}_{0^+}^{p(1-r);\phi} w \in L^1(\mathcal{J})$ exists, then*

$$\mathcal{D}_{0^+}^{r,p;\phi} \mathcal{I}_{0^+}^{r;\phi} w(\xi) = \mathcal{I}_{0^+}^{p(1-r);\phi} \mathcal{D}_{0^+}^{p(1-r);\phi} w(\xi).$$

Next, we take into account some important properties of ϕ-fractional derivative and integral operator as follows:

Proposition 1 ([1])**.** *Let $\xi > 0$, $r \geq 0$ and $s > 0$. Then, ϕ-fractional integral and derivative of a power function are given by*

$$^H\mathcal{D}_{0^+}^{r;\phi}(\phi(\xi) - \phi(0))^{s-1} = \frac{\Gamma(s)}{\Gamma(s-r)}(\phi(\xi) - \phi(0))^{r+s-1}$$

and

$$\mathcal{I}_{0^+}^{r;\phi}(\phi(\xi) - \phi(0))^{s-1} = \frac{\Gamma(s)}{\Gamma(s+r)}(\phi(\xi) - \phi(0))^{r+s-1}.$$

Furthermore, if $0 < r < 1$, then

$$^H\mathcal{D}_{0^+}^{r;\phi}(\phi(\xi) - \phi(0))^{r-1} = 0.$$

Theorem 1 ([19])**.** *If $w \in \mathcal{C}^1[\mathcal{J}, \mathbb{R}]$, $0 < r < 1$ and $0 \leq p \leq 1$. Then we have the followings:*
(i) $^H\mathcal{D}_{0^+}^{r,p;\phi} \mathcal{I}_{0^+}^{r;\phi} w(\xi) = w(\xi)$.
(ii) $\mathcal{I}_{0^+}^{r;\phi}{}^H\mathcal{D}_{0^+}^{r,p;\phi} w(\xi) = w(\xi) - \frac{(\phi(\xi)-\phi(0))^{q-1}}{\Gamma(q)} \mathcal{I}_{0^+}^{(1-p)(1-r);\phi} w(\xi)$.

Lemma 7 ([6])**.** *Let $h : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ such that for any $z \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$, $h \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$. A function $z \in \mathcal{C}_{1-q;\phi}^q[\mathcal{J}, \mathbb{R}]$ is a solution of the fractional initial value problem:*

$$\begin{cases} ^H D_{0^+}^{r,p;\phi} z(t) = h(t), & 0 < r \leq 1, \quad 0 \leq p \leq 1, \\ I_{0^+}^{1-q,\phi} z(0^+) = z_0 \in \mathbb{R}, & q = r + p - rp, \end{cases}$$

if and only if z satisfies the following integral equation,

$$z(t) = \frac{z_0}{\Gamma(q)}(\phi(t) - \phi(0))^{q-1} + \frac{1}{\Gamma(r)} \int_{0^+}^t \phi'(s)(\phi(t) - \phi(s))^{r-1} h(s) ds.$$

3. Main Results

In this section, we first adopt some techniques from Lemma 7 in order to establish an important mixed-type integral equation of problem (5). Thus, we need the following auxiliary lemma.

Lemma 8. *Let $0 < r < 1$, $0 \le p \le 1$ and $q = r + p - rp$. Suppose $f : \mathcal{J} \times \mathbb{R}^3 \to \mathbb{R}$ is a function such that $f \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$ for any $z \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$. If $z \in \mathcal{C}^q_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$ then z satisfies the problem* (5) *if and only if z satisfies the mixed-type integral equation:*

$$
\begin{aligned}
z(t) =& \frac{\delta\Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)}(\phi(t)-\phi(0))^{q-1} \sum_{i=1}^{m} b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i)-\phi(s))^{\rho+r-1}T_z(s)ds \\
&+ \frac{1}{\Gamma(r)} \int_{0^+}^{t} \phi'(s)(\phi(t)-\phi(s))^{r-1}T_z(s)ds,
\end{aligned}
\tag{9}
$$

where

$$
\delta = \frac{1}{\Gamma(\rho+q) - \displaystyle\sum_{i=1}^{m} b_i(\phi(\xi_i)-\phi(0))^{\rho+q-1}},
\tag{10}
$$

such that $\displaystyle \Gamma(\rho+q) \neq \sum_{i=1}^{m} b_i(\phi(\xi_i)-\phi(0))^{\rho+q-1}$.

For simplicity, we take

$$
T_z(t) = {}^H D_{0^+}^{r,p;\phi}z(t) = f(t, z(t), z(\gamma t), T_z(t)).
\tag{11}
$$

Proof. Suppose $z \in \mathcal{C}^q_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$ is a solution to the problem (5), then, we show that z is also a solution of (5). Indeed, from Lemma 7, we have

$$
z(t) = \frac{(\phi(t)-\phi(0))^{q-1}}{\Gamma(q)}\mathcal{I}_{0^+}^{1-q;\phi}z(0) + \frac{1}{\Gamma(r)}\int_{0^+}^{t}\phi'(s)(\phi(t)-\phi(s))^{r-1}T_z(s)ds.
\tag{12}
$$

Now, if we substitute $t = \xi_i$ and multiply both sides by b_i in Equation (12), we obtain

$$
b_i z(\xi_i) = \frac{(\phi(\xi_i)-\phi(0))^{q-1}}{\Gamma(q)}b_i\mathcal{I}_{0^+}^{1-q;\phi}z(0) + \frac{b_i}{\Gamma(r)}\int_{0^+}^{\xi_i}\phi'(s)(\phi(\xi_i)-\phi(s))^{r-1}T_z(s)ds.
\tag{13}
$$

Next, by applying $\mathcal{I}_0^{\rho;\phi}$ to both sides of Equation (13) and using Lemma 1 and Proposition 1, we get

$$
\begin{aligned}
\mathcal{I}_{0^+}^{\rho;\phi}b_i z(\xi_i) =& \frac{(\phi(\xi_i)-\phi(0))^{\rho+q-1}}{\Gamma(\rho+q)}b_i\mathcal{I}_0^{1-q;\phi}z(0) \\
&+ \frac{b_i}{\Gamma(\rho+r)}\int_{0^+}^{\xi_i}\phi'(s)(\phi(\xi_i)-\phi(s))^{\rho+r-1}T_z(s)ds.
\end{aligned}
\tag{14}
$$

This implies that

$$
\begin{aligned}
\sum_{i=1}^{m}\mathcal{I}_{0^+}^{\rho;\phi}b_i z(\xi_i) =& \frac{1}{\Gamma(\rho+q)}\left(\sum_{i=1}^{m}b_i(\phi(\xi_i)-\phi(0))^{\rho+q-1}\right)\mathcal{I}_{0^+}^{1-q;\phi}z(0) \\
&+ \frac{1}{\Gamma(\rho+r)}\sum_{i=1}^{m}b_i\int_{0^+}^{\xi_i}\phi'(s)(\phi(\xi_i)-\phi(s))^{\rho+r-1}T_z(s)ds.
\end{aligned}
\tag{15}
$$

Inserting the initial condition: $\mathcal{I}_{0^+}^{1-q;\phi}z(0^+) = \sum_{i=1}^{m} \mathcal{I}_{0^+}^{\rho;\phi} b_i z(\xi_i)$ in Equation (15) we have

$$
\begin{aligned}
\mathcal{I}_{0^+}^{1-q;\phi}z(0) =& \frac{1}{\Gamma(\rho+q)} \left(\sum_{i=1}^{m} b_i (\phi(\xi_i) - \phi(0))^{\rho+q-1} \right) \mathcal{I}_{0^+}^{1-q;\phi}z(0) \\
& + \frac{1}{\Gamma(\rho+r)} \sum_{i=1}^{m} b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s)ds,
\end{aligned}
\tag{16}
$$

which implies that

$$
\begin{aligned}
\frac{1}{\Gamma(\rho+r)} \sum_{i=1}^{m} b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s)ds \\
= \left(1 - \frac{1}{\Gamma(\rho+q)} \sum_{i=1}^{m} b_i (\phi(\xi_i) - \phi(0))^{\rho+q-1} \right) \mathcal{I}_{0^+}^{1-q;\phi}z(0) \\
= \frac{1}{\delta\Gamma(\rho+q)} \mathcal{I}_{0^+}^{1-q;\phi}z(0).
\end{aligned}
\tag{17}
$$

Thus,
$$
\mathcal{I}_{0^+}^{1-q;\phi}z(0) = \frac{\delta\Gamma(\rho+q)}{\Gamma(\rho+r)} \sum_{i=1}^{m} b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s)ds.
\tag{18}
$$

Hence, the result follows by putting Equation (18) in Equation (12). This implies that $z(t)$ satisfies Equation (9).

Conversely, suppose that $z \in \mathcal{C}_{1-q;\phi}^{q}$ satisfies the mixed-type integral Equation (9), then, we show that z satisfies Equation (5). Applying $\mathcal{D}_{0^+}^{q;\phi}$ to both sides of Equation (9) and using Lemma 2 and Proposition 1, we get

$$
\begin{aligned}
\mathcal{D}_{0^+}^{q;\phi}z(t) =& \mathcal{D}_{0^+}^{q;\phi} \left(\frac{\delta\Gamma(\rho+q)}{\Gamma(q)\Gamma(r)} (\phi(t) - \phi(0))^{q-1} \sum_{i=1}^{m} b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s)ds \right) \\
& + \mathcal{D}_{0^+}^{q;\phi} \left(\frac{1}{\Gamma(r)} \int_{0^+}^{t} \phi'(s)(\phi(t) - \phi(s))^{r-1} T_z(s)ds \right) \\
=& \mathcal{D}_{0^+}^{p(1-r);\phi} f(t, z(t), z(\gamma t), \mathcal{D}_{0^+}^{r,p;\phi}z(\gamma t)).
\end{aligned}
\tag{19}
$$

Since $\mathcal{D}_{0^+}^{r,p;\phi}z \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$, then by definition of $\mathcal{C}_{1-q;\phi}^{q}[\mathcal{J}, \mathbb{R}]$ and make use of Equation (19), we have
$$
\mathcal{D}_{0^+}^{p(1-r);\phi} f = D\mathcal{I}_{0^+}^{1-p(1-r);\phi} f \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}].
$$

For every $f \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$ and Lemma 3, we can see that $\mathcal{I}_{0^+}^{1-p(1-r);\phi} f \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$, which implies that $\mathcal{I}_{0^+}^{1-p(1-r);\phi} f \in \mathcal{C}_{1-q;\phi}^{1}[\mathcal{J}, \mathbb{R}]$ from the definition of $\mathcal{C}_{q;\phi}^{n}[\mathcal{J}, \mathbb{R}]$. Applying $\mathcal{I}_{0^+}^{p(1-r);\phi}$ on both sides of Equation (19) and using Lemma 3, we have

$$
\begin{aligned}
\mathcal{I}_{0^+}^{p(1-r);\phi} \mathcal{D}_{0^+}^{q;\phi}z(t) =& \mathcal{I}_{0^+}^{p(1-r);\phi} \mathcal{D}_{0^+}^{p(1-r);\phi} T_z(t) \\
=& T_z(t) - \frac{\left(\mathcal{I}_{0^+}^{1-p(1-r);\phi} T_z \right)(0^+)}{\Gamma(p(1-r))} (\phi(t) - \phi(0))^{p(r-1)-1} \\
=& T_z(t) = f(t, z(t), z(\gamma t), \mathcal{D}_{0^+}^{r,p;\phi}z(\gamma t)).
\end{aligned}
\tag{20}
$$

Finally, we show that if $z \in \mathcal{C}^q_{1-q}[\mathcal{J}, \mathbb{R}]$ satisfies Equation (9), it also satisfies the initial condition. Thus, by applying $\mathcal{I}^{1-q;\phi}_{0^+}$ to both sides of Equation (9) and using Lemma 1 and Proposition 1, we obtain

$$
\begin{aligned}
\mathcal{I}^{1-q;\phi}_{0^+} & z(t) \\
&= \mathcal{I}^{1-q;\phi}_{0^+} \left(\frac{\delta \Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)} (\phi(t) - \phi(0))^{q-1} \sum_{i=1}^m b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s) ds \right) \\
&\quad + \mathcal{I}^{1-q;\phi}_{0^+} \left(\frac{1}{\Gamma(r)} \int_{0^+}^t \phi'(s)(\phi(t) - \phi(s))^{r-1} T_z(s) ds \right) \\
&= \frac{\delta \Gamma(\rho+q)}{\Gamma(\rho+r)} \sum_{i=1}^m b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s) ds + \mathcal{I}^{1-p(1-r);\phi}_{0^+} T_z(t).
\end{aligned}
\tag{21}
$$

Using Lemma 4 and the fact that $1 - q < 1 - p(1 - r)$, then taking limit as $t \to 0$ in Equation (21) yields

$$
\mathcal{I}^{1-q;\phi}_{0^+} z(0^+) = \frac{\delta \Gamma(\rho+q)}{\Gamma(\rho+r)} \sum_{i=1}^m b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s) ds.
\tag{22}
$$

Now, substituting $t = \xi_i$ and multiplying through by b_i in Equation (9), we get

$$
\begin{aligned}
b_i z(\xi_i) =& \frac{\delta \Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)} b_i (\phi(\xi_i) - \phi(0))^{q-1} \sum_{i=1}^m b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s) ds \\
&+ \mathcal{I}^{1-q;\phi}_{0^+} \frac{b_i}{\Gamma(r)} \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{r-1} T_z(s) ds.
\end{aligned}
\tag{23}
$$

Applying $\mathcal{I}^{\rho;\phi}_{0^+}$ to both sides of Equation (23), we obtain

$$
\begin{aligned}
\mathcal{I}^{\rho;\phi}_{0^+} b_i z(\xi_i) =& \frac{\delta b_i (\phi(\xi_i) - \phi(0))^{\rho+q-1}}{\Gamma(\rho+r)} \sum_{i=1}^m b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s) ds \\
&+ \frac{b_i}{\Gamma(\rho+r)} \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s) ds,
\end{aligned}
\tag{24}
$$

which implies

$$
\begin{aligned}
\sum_{i=1}^m & b_i \mathcal{I}^{\rho;\phi}_{0^+} z(\xi_i) \\
&= \frac{\delta}{\Gamma(\rho+r)} \sum_{i=1}^m b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s) ds \sum_{i=1}^m b_i (\phi(\xi_i) - \phi(0))^{\rho+q-1} \\
&\quad + \frac{1}{\Gamma(\rho+r)} \sum_{i=1}^m b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s) ds \\
&= \frac{1}{\Gamma(\rho+r)} \sum_{i=1}^m b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s) ds \times \\
&\qquad \left(1 + \delta \sum_{i=1}^m b_i (\phi(\xi_i) - \phi(0))^{\rho+q-1} \right)
\end{aligned}
\tag{25}
$$

and

$$
\mathcal{I}^{1-q;\phi}_{0^+} z(0^+) = \frac{\delta \Gamma(\rho+q)}{\Gamma(\rho+r)} \sum_{i=1}^m b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s) ds.
\tag{26}
$$

Therefore, in view of Equations (22) and (26), we have

$$\mathcal{I}_{0^+}^{1-q;\phi}z(0^+) = \sum_{i=1}^{m} b_i \mathcal{I}_{0^+}^{\rho;\phi} z(\xi_i). \tag{27}$$

□

3.1. Existence Result Via Schaefer'S Fixed Point Theorem

This subsection will provide the proof of the existence results of Equation (5) using Schaefer's fixed point theorem.

Theorem 2 ([61])**.** *Let* $\mathcal{A} : \mathcal{X} \to \mathcal{X}$ *be a completely continuous operator. Suppose that the set* $\mathcal{E}(\mathcal{A}) = \{p \in \mathcal{X} : p = \varrho \mathcal{A} p, \text{ for some } \varrho \in [0, 1]\}$ *is bounded, then* $\mathcal{A}$ *has a fixed point.*

Thus we need the following assumptions:
(A_1) Let $f : \mathcal{J} \times \mathbb{R}^3 \to \mathbb{R}$ be a function such that $f \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$ for any $z \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$.
(A_2) There exist $k, l, m, n \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$ with $k^* = \sup\limits_{t \in \mathcal{J}} |k(t)| < 1$ such that

$$|f(t, u, v, w)| \leq k(t) + l(t)|x| + m(t)|y| + n(t)|z|, \quad t \in \mathcal{J}, \quad u, v, w \in \mathbb{R}.$$

Theorem 3. *Let* $0 < r < 1, 0 \leq p \leq 1$ *and* $q = r + p - rp$. *Suppose that the assumptions* (A_1) *and* (A_2) *are satisfied. Then there exist at least one solution of the problem* (5) *in the space* $\mathcal{C}_{1-q;\phi}^{r,p}[\mathcal{J}, \mathbb{R}]$.

Proof. Define the operator $F : \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}] \to \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$ by

$$\begin{aligned}
(Fz)(t) &= \frac{\delta \Gamma(\rho + q)}{\Gamma(q)\Gamma(\rho + r)} (\phi(t) - \phi(0))^{q-1} \sum_{i=1}^{m} b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s) ds \\
&+ \frac{1}{\Gamma(r)} \int_{0^+}^{t} \phi'(s)(\phi(t) - \phi(s))^{r-1} T_z(s) ds,
\end{aligned} \tag{28}$$

then, clearly the operator F is well-defined. The proof is given in the following steps: Step 1: the operator F is continuous. Let z_n be a sequence such that $z_n \to z$ in $\mathcal{C}_{1-q,\phi}[\mathcal{J}, \mathbb{R}]$. Then for each $t \in \mathcal{J}$, we have

$$\begin{aligned}
&|((Fz_n)(t) - (Fz)(t))(\phi(t) - \phi(0))^{1-q}| \\
&\leq \frac{|\delta| \Gamma(\rho + q)}{\Gamma(q)\Gamma(\rho + r)} \sum_{i=1}^{m} b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} |T_{z_n}(s) - T_z(s)| ds \\
&+ \frac{1}{\Gamma(r)} (\phi(t) - \phi(0))^{1-q} \int_{0^+}^{t} \phi'(s)(\phi(t) - \phi(s))^{r-1} |T_{z_n}(s) - T_z(s)| ds \\
&\leq \frac{|\delta| \Gamma(\rho + q)}{\Gamma(q)\Gamma(\rho + r)} \mathcal{B}(q, \rho + r) \sum_{i=1}^{m} b_i (\phi(\xi_i) - \phi(s))^{\rho+r+q-1} \|T_{z_n}(\cdot) - T_z(\cdot)\|_{\mathcal{C}_{1-q;\phi}} \\
&+ \frac{\mathcal{B}(q, r)}{\Gamma(r)} (\phi(T) - \phi(0))^r \|T_{z_n}(\cdot) - T_z(\cdot)\|_{\mathcal{C}_{1-q;\phi}} \\
&\leq \left[\frac{|\delta| \Gamma(\rho + q)}{\Gamma(q)\Gamma(\rho + r)} \mathcal{B}(q, \rho + r) \sum_{i=1}^{m} b_i (\phi(\xi_i) - \phi(s))^{\rho+r+q-1} \right. \\
&\left. + \frac{\mathcal{B}(q, r)}{\Gamma(r)} (\phi(T) - \phi(0))^r \right] \|T_{z_n}(\cdot) - T_z(\cdot)\|_{\mathcal{C}_{1-q;\phi}}.
\end{aligned} \tag{29}$$

Since f is continuous, this implies that T_z is also continuous. Therefore, we have

$$\|T_{z_n} - T_z\|_{\mathcal{C}_{1-q;\phi}} \to 0, \quad \text{as} \quad n \to \infty.$$

Step 2: F maps bounded sets into bounded sets in $\mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$.

Indeed, it suffices to show that for any $\kappa > 0$, there exist a $\mu > 0$ such that for any $z \in \mathbf{B}_\kappa = \{z \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}] : \|z\| \le \kappa\}$, thus we have $\|F(z)\|_{\mathcal{C}_{1-q;\phi}} \le \mu$.

For simplicity, we put

$$E_1 = \frac{|\delta|\Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)} \sum_{i=1}^{m} b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1}|T_z(s)|ds \tag{30}$$

and

$$E_2 = \frac{1}{\Gamma(r)} (\phi(t) - \phi(0))^{1-q} \int_{0^+}^{t} \phi'(s)(\phi(t) - \phi(s))^{r-1}|T_z(s)|ds. \tag{31}$$

It follows from assumption (A_2) that

$$\begin{aligned} |T_z(t)| &= |f(t, z(t), z(\gamma t), T_z(t))| \\ &\le k(t) + l(t)|z| + m(t)|z| + n(t)|T_z(t)| \\ &\le \frac{k^* + (l^* + m^*)|z(t)|}{1 - n^*}. \end{aligned} \tag{32}$$

Thus, in view of Equations (30)–(32), we get

$$\begin{aligned} E_1 &\le \frac{|\delta|\Gamma(\rho+q)}{\Gamma(q)(1-n^*)} \sum_{i=1}^{m} b_i \left(\frac{k^*}{\Gamma(\rho+r+1)} (\phi(\xi_i) - \phi(0))^{\rho+r} \right. \\ &\quad \left. + (l^* + m^*) \frac{(\phi(\xi_i) - \phi(0))^{\rho+r+q-1}}{\Gamma(\rho+r)} \mathcal{B}(q, \rho+r)\|z\|_{\mathcal{C}_{1-q;\phi}} \right) \\ E_2 &\le \frac{1}{(1-n^*)} \left(\frac{k^*}{\Gamma(r+1)} (\phi(T) - \phi(0))^{\rho+r-q+1} \right. \\ &\quad \left. + \frac{(l^* + m^*)\mathcal{B}(q,r)}{\Gamma(r)} (\phi(T) - \phi(0))^r \|z\|_{\mathcal{C}_{1-q;\phi}} \right). \end{aligned}$$

This implies that,

$$\begin{aligned} |(Fz)(t)&((\phi(t)-\phi(0))^{q-1})| \\ &\le \frac{k^*}{(1-n^*)} \left[\frac{|\delta|\Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r+1)} \sum_{i=1}^{m} b_i(\phi(\xi_i) - \phi(0))^{\rho+r} \right. \\ &\quad \left. + \frac{k^*}{\Gamma(r+1)} (\phi(T) - \phi(0))^{\rho+r-q+1} \right] \\ &\quad + \frac{(l^* + m^*)}{(1-n^*)} \left[\frac{|\delta|\Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)} \mathcal{B}(q, \rho+r) \sum_{i=1}^{m} b_i(\phi(\xi_i) - \phi(0))^{\rho+r+q-1} \right. \\ &\quad \left. + \frac{\mathcal{B}(q,r)}{\Gamma(r)} (\phi(T) - \phi(0))^r \right] \|z\|_{\mathcal{C}_{1-q;\phi}} \\ &= \mu. \end{aligned} \tag{33}$$

Step 3: F maps bounded sets into equicontinuous set of $\mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$. Let $t_1, t_2 \in \mathcal{J}$ such that $t_1 \geq t_2$ and B_κ be a bounded set of $\mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$ as defined in Step 2. Let $z \in B_\kappa$, then

$$
\begin{aligned}
&\left| ((\phi(t_1) - \phi(a))^{q-1})(Fz)(t_1) - ((\phi(t_2) - \phi(0))^{q-1})(Fz)(t_2) \right| \\
&\leq \left| \frac{1}{\Gamma(r)} (\phi(t_1) - \phi(0))^{1-q} \int_{0+}^{t_1} \phi'(s)(\phi(t_1) - \phi(s))^{r-1} T_z(s) ds \right. \\
&\quad \left. - \frac{1}{\Gamma(r)} (\phi(t_2) - \phi(0))^{1-q} \int_{0+}^{t_2} \phi'(s)(\phi(t_2) - \phi(s))^{r-1} T_z(s) ds \right| \\
&\leq \frac{1}{\Gamma(r)} \left| \int_{0+}^{t_1} \phi'(s) \left[((\phi(t_1) - \phi(0))^{q-1})(\phi(t_1) - \phi(s))^{r-1} \right. \right. \\
&\quad \left. \left. - ((\phi(t_2) - \phi(0))^{q-1})(\phi(t_2) - \phi(s))^{r-1} T_z(s) ds \right] \right| \\
&\quad + \left| \frac{(\phi(t_2) - \phi(0))^{q-1}}{\Gamma(r)} \int_{t_1}^{t_2} \phi'(s)(\phi(t_1) - \phi(0))^{r-1} T_z(s) ds \right| \\
&\to 0, \quad as \quad t_1 \to t_2.
\end{aligned}
\tag{34}
$$

Thus, steps 1–3, together with the Arzela–Ascoli theorem, show that the operator F is completely continuous.

Step 4: a priori bounds.

It is enough to show that the set $\chi = \{z \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}] : z = \sigma(Fz), 0 < \sigma < 1\}$ is bounded. Now, let $z \in \chi, z = \sigma(Fz)$ for some $0 < \sigma < 1$. Thus for each $t \in \mathcal{J}$, we obtain

$$
\begin{aligned}
z(t) = \sigma & \left[\frac{\delta \Gamma(\rho + q)}{\Gamma(q) \Gamma(\rho + r)} (\phi(t) - \phi(0))^{q-1} \sum_{i=1}^{m} b_i \int_{0+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho + r - 1} T_z(s) ds \right. \\
& \left. + \frac{1}{\Gamma(r)} \int_{0+}^{t} \phi'(s)(\phi(t) - \phi(s))^{r-1} T_z(s) ds \right].
\end{aligned}
$$

It follows from assumption (A_2), that for every $t \in \mathcal{J}$,

$$
\begin{aligned}
|z(t)(\phi(t) - \phi(0))^{1-q}| &\leq |(Fz)(t)(\phi(t) - \phi(0))^{1-q}| \\
&\leq \frac{k^*}{(1 - n^*)} \left[\frac{|\delta| \Gamma(\rho + q)}{\Gamma(q) \Gamma(\rho + r + 1)} \sum_{i=1}^{m} b_i (\phi(\xi_i) - \phi(0))^{\rho + r} \right. \\
&\quad + \frac{k^*}{\Gamma(r + 1)} (\phi(T) - \phi(0))^{\rho + r - q + 1} \bigg] \\
&\quad + \frac{(l^* + m^*)}{(1 - n^*)} \left[\frac{|\delta| \Gamma(\rho + q)}{\Gamma(q) \Gamma(\rho + r)} \mathcal{B}(q, \rho + r) \sum_{i=1}^{m} b_i (\phi(\xi_i) - \phi(0))^{\rho + r + q - 1} \right. \\
&\quad + \frac{\mathcal{B}(q, r)}{\Gamma(r)} (\phi(T) - \phi(0))^{r} \bigg] \|z\|_{\mathcal{C}_{1-q;\phi}} \\
&< \infty.
\end{aligned}
\tag{35}
$$

This shows that the set χ is bounded. Hence, by the Schaefer's fixed point theorem, problem (5) has at least one solution. $\square$

3.2. Existence Result Via Banach Contraction Principle

Now, we prove the uniqueness of problem (5) by means of Banach contraction principle. Therefore, the following hypotheses are needed.

(A_3) There exist constants $K, L > 0$ such that

$$
|f(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w})| \leq K(|u - \bar{u}| + |v - \bar{v}|) + L|w - \bar{w}|
$$

for any $u, v, w, \bar{u}, \bar{v}, \bar{w} \in \mathbb{R}$ and $t \in \mathcal{J}$.

(A_4) Suppose that

$$\left(\frac{2K}{1-L} \right) \Omega < 1,$$

where

$$\Omega = \frac{|\delta| \Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)} \mathcal{B}(q, \rho+r) \sum_{i=1}^{m} b_i (\phi(\xi_i) - \phi(0))^{\rho+r+q-1} + \frac{\mathcal{B}(q,r)}{\Gamma(r)} (\phi(T) - \phi(0))^r. \tag{36}$$

Theorem 4. *Let $0 < r < 1$, $0 \le p \le 1$ and $q = r + p - rp$. Suppose that the hypotheses (A_1), (A_3) and (A_4) are satisfied. Then, problem (5) has a unique solution in the space $\mathcal{C}_{1-q;\phi}^{r,p}[\mathcal{J}, \mathbb{R}]$.*

Proof. Define the operator $F : \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}] \to \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$ by

$$
\begin{aligned}
(Fz)(t) = {} & \frac{\delta \Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)} (\phi(t) - \phi(0))^{q-1} \sum_{i=1}^{m} b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} T_z(s) ds \\
& + \frac{1}{\Gamma(r)} \int_{0^+}^{t} \phi'(s)(\phi(t) - \phi(s))^{r-1} T_z(s) ds,
\end{aligned}
\tag{37}
$$

then, clearly the operator F is well-defined. Let $z_1, z_2 \in \mathcal{C}_{1-q;\phi}^{r,p}[\mathcal{J}, \mathbb{R}]$ and $t \in \mathcal{J}$, then, we have

$$
\begin{aligned}
& |((Fz_1)(t) - (Fz_2)(t))(\phi(t) - \phi(0))^{1-q}| \\
& \le \frac{|\delta| \Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)} \sum_{i=1}^{m} b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} |T_{z_1}(s) - T_{z_2}(s)| ds \\
& \quad + \frac{1}{\Gamma(r)} (\phi(t) - \phi(0))^{1-q} \int_{0^+}^{t} \phi'(s)(\phi(t) - \phi(s))^{r-1} |T_{z_1}(s) - T_{z_2}(s)| ds
\end{aligned}
\tag{38}
$$

and

$$
\begin{aligned}
|T_{z_1}(t) - T_{z_2}(t)| & = |f(t, z_1(t), z_1(\gamma t)), T_{z_1}(t) - f(t, z_2(t), z_2(\gamma t), T_{z_2}(t))| \\
& \le K(|z_1(t) - z_2(t)| + |z_1(\gamma t) - z_2(\gamma t)|) + L|(T_{z_1})(t) - (T_{z_2})(t)| \\
& \le \left(\frac{2K}{1-L} \right) |z_1(t) - z_2(t)|.
\end{aligned}
\tag{39}
$$

Thus, by substituting Equation (39) in Equation (38), we obtain

$$
\begin{aligned}
& |((Fz_1)(t) - (Fz_2)(t))(\phi(t) - \phi(0))^{1-q}| \\
& \le \frac{|\delta| \Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)} \sum_{i=1}^{m} b_i \left(\frac{2K}{(1-L)} \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i) - \phi(s))^{\rho+r-1} ds \right) \|z_1(t) - z_2(t)\|_{\mathcal{C}_{1-q;\phi}} \\
& \quad + \frac{1}{\Gamma(r)} (\phi(t) - \phi(0))^{1-q} \frac{2K}{(1-L)} \left(\int_{0^+}^{t} \phi'(s)(\phi(t) - \phi(s))^{r-1} ds \right) \|z_1(t) - z_2(t)\|_{\mathcal{C}_{1-q;\phi}} \\
& \le \frac{2K}{(1-L)} \left(\frac{|\delta| \Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)} \mathcal{B}(q, \rho+r) \sum_{i=1}^{m} b_i (\phi(\xi_i) - \phi(0))^{\rho+r+q-1} \right. \\
& \quad \left. + \frac{\mathcal{B}(q,r)}{\Gamma(r)} (\phi(T) - \phi(0))^r \right) \|z_1(t) - z_2(t)\|_{\mathcal{C}_{1-q;\phi}}.
\end{aligned}
\tag{40}
$$

Also,

$$
\begin{aligned}
\|(Fz_1) - (Fz_2)\|_{\mathcal{C}_{1-q;\phi}} \le \frac{2K}{(1-L)} & \left(\frac{|\delta|\Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)}\mathcal{B}(q,\rho+r)\sum_{i=1}^{m} b_i(\phi(\xi_i) - \phi(0))^{\rho+r+q-1} \right. \\
& \left. + \frac{\mathcal{B}(q,r)}{\Gamma(r)}(\phi(T) - \phi(0))^r \right) \|z_1(t) - z_2(t)\|_{\mathcal{C}_{1-q;\phi}}.
\end{aligned}
\tag{41}
$$

It follows from hypotheses (A_4) that F is a contraction map. Therefore, by Banach contraction principle, we can conclude that problem (5) has a unique solution. $\square$

4. Ulam-Hyers Stabilty

Two types of Ulam stability for (5) are discussed in this section, namely Ulam-Hyers and generalized Ulam-Hyers stability.

Definition 4. *Problem* (5) *is said to be Ulam-Hyers stable if there exists* $\omega \in \mathbb{R}_+ \backslash \{0\}$, *such that for each* $\epsilon > 0$ *and solution* $x \in \mathcal{C}_{1-q;\phi}^1[\mathcal{J}, \mathbb{R}]$ *of the inequality*

$$
|{}^H\mathcal{D}_{0^+}^{r,p;\phi}x(t) - f(t, x(t), x(\gamma t), {}^H\mathcal{D}_{0^+}^{r,p;\phi}x(\gamma t))| \le \epsilon, \quad t \in \mathcal{J},
\tag{42}
$$

there exists a solution $z \in \mathcal{C}_{1-q;\phi}^1[\mathcal{J}, \mathbb{R}]$ *of equation* (5), *such that*

$$
|x(t) - z(t)| \le \omega\epsilon, \quad t \in \mathcal{J}.
$$

Definition 5. *Problem* (5) *is said to be generalized Ulam-Hyers stable if there exist* $\Phi \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, $\Phi_f(0) = 0$, *such that for each solution* $x \in \mathcal{C}_{1-q;\phi}^1[\mathcal{J}, \mathbb{R}]$ *of the* (42), *there exists a solution* $z \in \mathcal{C}_{1-q;\phi}^1[\mathcal{J}, \mathbb{R}]$ *of Equation* (5), *such that*

$$
|x(t) - z(t)| \le \Phi_f\epsilon, \quad t \in \mathcal{J}.
$$

Remark 1. *A function* $x \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$ *is a solution of the inequality* (42), *if and only if there exist a function* $g \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$ *such that:*

(i) $|g(t)| \le \epsilon, \quad t \in \mathcal{J}$.

(ii) ${}^H\mathcal{D}_{0^+}^{r,p;\phi}x(t) = f(t, x(t), x(\gamma t), {}^H\mathcal{D}_{0^+}^{r,p;\phi}x(\gamma t)) + g(t), \quad t \in \mathcal{J}$.

Lemma 9. *Let* $0 < r < 1$, $0 \le p \le 1$, *if a function* $x \in \mathcal{C}_{1-q;\phi}[\mathcal{J}, \mathbb{R}]$ *is a solution of the inequality* (42), *then* x *is a solution of the following integral inequality*

$$
\left| x(t) - A_x - \frac{1}{\Gamma(r)}\int_{0^+}^t \phi'(s)(\phi(t) - \phi(s))^{r-1}T_x ds \right| \le \Omega\epsilon.
\tag{43}
$$

Proof. Clearly it follow from Remark 1 that

$$
\begin{aligned}
{}^H\mathcal{D}_{0^+}^{r,p;\phi}x(t) &= f(t, x(t), x(\gamma t), {}^H\mathcal{D}_{0^+}^{r,p;\phi}x(\gamma t)) + g(t) \\
&= T_x(t) + g(t),
\end{aligned}
$$

and

$$
x(t) = \frac{\delta\Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)}(\phi(t)-\phi(0))^{q-1}\sum_{i=1}^{m}b_i\left(\int_{0^+}^{\xi_i}\phi'(s)(\phi(\xi_i)-\phi(s))^{\rho+r-1}T_x(s)ds\right.
$$

$$
\left.+\int_{0^+}^{\xi_i}\phi'(s)(\phi(\xi_i)-\phi(s))^{\rho+r-1}g(s)ds\right)+\frac{1}{\Gamma(r)}\int_{0^+}^{t}\phi'(s)(\phi(t)-\phi(s))^{r-1}T_x(s)ds \tag{44}
$$

$$
+\frac{1}{\Gamma(r)}\int_{0^+}^{t}\phi'(s)(\phi(t)-\phi(s))^{r-1}g(s)ds.
$$

Hence

$$
\left|x(t)-A_x-\frac{1}{\Gamma(r)}\int_{0^+}^{t}\phi'(s)(\phi(t)-\phi(s))^{r-1}T_xds\right|
$$

$$
=\left|\frac{\delta\Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)}(\phi(t)-\phi(0))^{q-1}\sum_{i=1}^{m}b_i\int_{0^+}^{\xi_i}\phi'(s)(\phi(\xi_i)-\phi(s))^{\rho+r-1}g(s)ds\right.
$$

$$
\left.+\frac{1}{\Gamma(r)}\int_{0^+}^{t}\phi'(s)(\phi(t)-\phi(s))^{r-1}g(s)ds\right| \tag{45}
$$

$$
\leq\frac{|\delta|\Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)}(\phi(t)-\phi(0))^{q-1}\sum_{i=1}^{m}|b_i|\int_{0^+}^{\xi_i}\phi'(s)(\phi(\xi_i)-\phi(s))^{\rho+r-1}|g(s)|ds
$$

$$
+\frac{1}{\Gamma(r)}\int_{0^+}^{t}\phi'(s)(\phi(t)-\phi(s))^{r-1}|g(s)|ds
$$

$$
\leq\Omega\epsilon.
$$

□

Theorem 5. *Suppose that the hypotheses* (A_1), (A_3) *and* (A_4) *are satisfied. Then problem* (5) *is both Ulam-Hyers and generalized Ulam-Hyers stable on* $\mathcal{J}$.

Proof. Let $\epsilon > 0$ and $x \in \mathcal{C}_{1-q;\phi}[\mathcal{J},\mathbb{R}]$ be a function which satisfies the inequality (42) and let $z \in \mathcal{C}_{1-q;\phi}[\mathcal{J},\mathbb{R}]$ be a unique solution of the following implicit fractional pantograph differential equation

$$
{}^{H}\mathcal{D}_{0^+}^{r,p;\phi}z(t) = f(t,z(t),z(\gamma t),{}^{H}\mathcal{D}_{0^+}^{r,p;\phi}z(\gamma t))| \leq \epsilon, \quad t \in \mathcal{J}, \quad 0 < r < 1, 0 \leq p \leq 1,
$$

$$
\mathcal{I}_{0^+}^{1-q;\phi}z(0^+) = \mathcal{I}_{0^+}^{1-q;\phi}z(0^+) = \sum_{i=1}^{m}b_i\mathcal{I}_{0^+}^{\rho;\phi}z(\xi_i), \quad \xi_i \in (0,T], \quad q = r+p-rp.
$$

Using Lemma 9, we have

$$
z(t) = A_z - \frac{1}{\Gamma(r)}\int_{0^+}^{t}\phi'(s)(\phi(t)-\phi(s))^{r-1}T_z(s)ds,
$$

where

$$
A_z = \frac{\delta\Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)}(\phi(t)-\phi(a))^{q-1}\sum_{i=1}^{m}b_i\int_{0^+}^{\xi_i}\phi'(s)(\phi(\xi_i)-\phi(s))^{\rho+r-1}T_z(s)ds.
$$

Clearly, if $z(\xi_i) = x(\xi_i)$ and $\mathcal{I}_{0^+}^{1-q;\phi} z(0^+) = \mathcal{I}_{0^+}^{1-q;\phi} z(0^+)$, we get $A_z = A_x$ and that

$$
|A_z - A_x|
$$
$$
= \frac{|\delta|\Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)}(\phi(t)-\phi(0))^{q-1} \sum_{i=1}^{m} b_i \int_{0^+}^{\xi_i} \phi'(s)(\phi(\xi_i)-\phi(s))^{\rho+r-1} |T_z(s) - T_x(s)| ds
$$
$$
\leq \frac{|\delta|\Gamma(\rho+q)}{\Gamma(q)\Gamma(\rho+r)}(\phi(t)-\phi(0))^{q-1} \left(\frac{2K}{1-L}\right) \sum_{i=1}^{m} b_i \mathcal{I}_{0^+}^{\rho+r;\phi} |z(s) - x(s)|(\xi_i)
$$
$$
= 0.
$$

Now for any $t \in \mathcal{J}$ and Lemma 9, we have

$$
|x(t) - z(t)| = \left| x(t) - A_x - \frac{1}{\Gamma(r)} \int_{0^+}^{t} \phi'(s)(\phi(t)-\phi(s))^{r-1} T_x(s) ds \right|
$$
$$
+ \frac{1}{\Gamma(r)} \int_{0^+}^{t} \phi'(s)(\phi(t)-\phi(s))^{r-1} |T_x(s) - T_z(s)| ds
$$
$$
\leq \left| x(t) - A_x - \frac{1}{\Gamma(r)} \int_{0^+}^{t} \phi'(s)(\phi(t)-\phi(s))^{r-1} T_x(s) ds \right|
$$
$$
+ \left(\frac{2K}{1-L}\right) \frac{1}{\Gamma(r)} \int_{0^+}^{t} \phi'(s)(\phi(t)-\phi(s))^{r-1} |x(s) - z(s)| ds
$$
$$
\leq \Omega\epsilon + \left(\frac{2K}{1-L}\right) \frac{\mathcal{B}(r,q)(\phi(T)-\phi(0))^r}{\Gamma(r)} |x(t) - z(t)| ds.
$$

Thus,

$$
|x(t) - z(t)| \leq \omega\epsilon,
$$

where

$$
\omega = \frac{\Omega(1-L)\Gamma(r)}{(1-L)\Gamma(r) - 2K(\phi(T)-\phi(0))^r \mathcal{B}(r,q)}.
$$

Therefore, problem (5) is Ulam-Hyers stable. Moreover, if we set $\Phi_f(\epsilon) = \omega\epsilon$ such that $\Phi_f(0) = 0$, then problem (5) is generalized Ulam-Hyers stable. □

5. Examples

Example 1. *Consider the implicit fractional pantograph differential equation which involves ϕ-Hilfer fractional derivative of the following form:*

$$
\begin{cases}
{}^H\mathcal{D}_{0^+}^{\frac{2}{3},\frac{1}{2};t} z(t) = \dfrac{1}{3(5^{2t}+5)[1+|z(t)|+|z(\frac{1}{2}t)|+|{}^H\mathcal{D}_{1^+}^{\frac{2}{3},\frac{1}{2};t} z(\frac{1}{2}t)|]}, & t \in \mathcal{J} = (0,2], \\
\mathcal{I}_{0^+}^{1-\frac{5}{6};t} z(0) = 3\mathcal{I}_{0^+}^{\frac{1}{2};t} z(\frac{3}{2}), & \frac{2}{3} \leq \frac{5}{6} = \frac{2}{3} + (\frac{1}{2}) - (\frac{2}{3})(\frac{1}{2}).
\end{cases} \tag{46}
$$

By comparing (5) with (46), we have:
$r = \frac{2}{3}$, $p = \gamma = \rho = \frac{1}{2}$, $q = \frac{5}{6}$, $T = 2$ and $\phi(\cdot) = t$. Also from the initial condition we can easily see that $b_1 = 3$ since $m = 1$, $\xi_1 = \frac{3}{2} \in \mathcal{J}$ and $f : \mathcal{J} \times \mathbb{R}^3 \to \mathbb{R}$ is a function defined by

$$
f(t,u,v,w) = \frac{1}{3(5^{2t}+5)(1+|u|+|v|+|w|)}, t \in \mathcal{J}, \quad u,v,w \in \mathbb{R}_+.
$$

Obviously, f is continuous and for all $u, v, w, \bar{u}, \bar{v}, \bar{w} \in \mathbb{R}_+$ and $t \in \mathcal{J}$, we have
$|f(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w})| \leq \frac{1}{90}\left(|u - \bar{u}| + |v - \bar{v}| + |w - \bar{w}|\right)$. *Thus, it follows that conditions (A_1) and (A_3) are true with $K = L = \frac{1}{90}$. Therefore, by simple calculation, we get $|\delta| \approx 0.3935$ and*

$$\left(\frac{2K}{1-L}\right)\Omega \approx 0.0642 < 1.$$

Since, all the assumptions of Theorem 4 are satisfied. Then problem (5) has a unique solution on $\mathcal{J}$. However, we can also find out that $\Omega \approx 2.8551 > 0$ and $\omega = 2.9321 > 0$. Hence, by Theorem 5, problem (5) is both Ulam-Hyers and also generalized Ulam-Hyers stable.

Example 2. *Consider the implicit fractional pantograph differential equation which involves ϕ-Hilfer fractional derivative of form:*

$$\begin{cases} {}^H\mathcal{D}_{0^+}^{\frac{1}{3}, \frac{2}{3}; \sqrt{t}} z(t) = \dfrac{2 + |z(t)| + |z(\frac{3}{2}t)| + \left|{}^H\mathcal{D}_{0^+}^{\frac{1}{3}, \frac{2}{3}; \sqrt{t}} z(\frac{3}{2}t)\right|}{95 e^{2t} \cos 2t \left(1 + |z(t)| + |z(\frac{3}{2}t)| + \left|{}^H\mathcal{D}_{0^+}^{\frac{1}{3}, \frac{2}{3}; \sqrt{t}} z(\frac{3}{2}t)\right|\right)}, & t \in \mathcal{J} = (0, 1], \\ \mathcal{I}_{0^+}^{1 - \frac{7}{9}; \sqrt{t}} z(0) = z(\frac{1}{2}) + 3z(\frac{4}{5}), & \frac{2}{3} < \frac{7}{9} = \frac{1}{3} + (\frac{2}{3}) - (\frac{1}{3})(\frac{2}{3}). \end{cases} \tag{47}$$

By comparing Equation (47) with Equation (5), we obtain that:
$r = \frac{1}{3}, p = \frac{2}{3}, q = \frac{7}{9}, \rho = 0, \gamma = \frac{3}{2}, T = 1$ *and* $\phi(\cdot) = \sqrt{t}$. *Also we can easily see that $b_1 = 1, b_2 = 3$ since $m = 2, \xi_1 = \frac{1}{2}, \xi_2 = \frac{4}{5} \in \mathcal{J}$ and $f : \mathcal{J} \times \mathbb{R}^3 \to \mathbb{R}$ is a function defined by*

$$f(t, u, v, w) = \frac{2 + |u| + |v| + |w|}{95 e^{2t} \cos 2t \left(1 + |u| + |v| + |w|\right)}, \quad t \in \mathcal{J}, \quad u, v, w \in \mathbb{R}_+.$$

Thus, f is continuous and we can see that, for all $u, v, w, \bar{u}, \bar{v}, \bar{w} \in \mathbb{R}_+$ and $t \in \mathcal{J}$,
$|f(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w})| \leq \frac{1}{95}\left(|u - \bar{u}| + |v - \bar{v}| + |w - \bar{w}|\right).$
So assumptions (A_1) and (A_3) are fulfilled with $K = L = \frac{1}{95}$. Furthermore,

$$|f(t, u, v, w)| \leq \frac{1}{95 e^{2t} \cos 2t} (2 + |u| + |v| + |w|), \quad t \in \mathcal{J}.$$

The above implies that (A_2) is true with $k(t) = \frac{2}{95 e^{2t} \cos 2t}, l(t) = m(t) = n(t) = \frac{1}{95 e^{2t} \cos 2t}$ and $k^* = \frac{2}{95}, l^* = m^* = n^* = \frac{1}{95}$. Therefore, all the hypotheses of Theorem 4 are satisfied, which means that problem (5) has at least one solution on $\mathcal{J}$. Moreover, by using the same procedure as in example 5.2, we obtain, that $|\delta| \approx 1.1025, \Omega \approx 3.6662 > 0$ and

$$\left(\frac{2K}{1-L}\right)\Omega \approx 0.0782 < 1.$$

Thus, all the hypotheses of Theorem 4 holds. Hence, problem (5) has a unique solution on $\mathcal{J}$.

Example 3. *Consider the implicit fractional pantograph differential equation which involves ϕ-Hilfer fractional derivative of the following form:*

$$\begin{cases} {}^H\mathcal{D}_{0^+}^{\frac{1}{2}, \frac{1}{3}; t} z(t) = \dfrac{1}{4^{t+3}\left[1 + |z(t)| + |z(\frac{1}{6}t)| + |{}^H\mathcal{D}_{0^+}^{\frac{1}{2}, \frac{1}{3}; t} z(\frac{1}{6}t)|\right]}, & t \in \mathcal{J} = (0, 3], \\ \mathcal{I}_{0^+}^{1 - q; t} z(0) = \sqrt{2}\,\mathcal{I}_{0^+}^{\frac{2}{5}; t} z(2) + \sqrt{5}\,\mathcal{I}_{0^+}^{\frac{2}{5}; t} z(\frac{5}{2}), & q = \frac{1}{2} + (\frac{1}{3}) - (\frac{1}{2})(\frac{1}{3}). \end{cases} \tag{48}$$

By comparing Equation (5) with Equation (48), we get the followings values:
$r = \frac{1}{2}$, $p = \frac{1}{3}$ $\gamma = \frac{1}{6}$ $\rho = \frac{2}{5}$, $q = \frac{2}{3}$, $T = 3$ and $\phi(\cdot) = t$. Also from the initial condition we can easily see that $b_1 = \sqrt{2}$ $b_1 = \sqrt{5}$ since $m = 2$, $\xi_1 = 2$ $\xi_2 = \frac{5}{2}$ and $f : \mathcal{J} \times \mathbb{R}^3 \to \mathbb{R}$ is a function defined by

$$f(t, u, v, w) = \frac{1}{4^{t+3}(1 + |u| + |v| + |w|)}, t \in \mathcal{J}, \quad u, v, w \in \mathbb{R}_+.$$

Thus, f is continuous and for all $u, v, w, \bar{u}, \bar{v}, \bar{w} \in \mathbb{R}_+$ and $t \in \mathcal{J}$, yields
$|f(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w})| \leq \frac{1}{64} \left(|u - \bar{u}| + |v - \bar{v}| + |w - \bar{w}| \right)$. Hence, it follows that conditions (A_1) and (A_3) are true with $K = L = \frac{1}{90}$. Therefore, by substitution these values, we get $|\delta| \approx 0.3456$, $\Omega \approx 7.4535 > 0$ and

$$\left(\frac{2K}{1 - L} \right) \Omega \approx 0.2366 < 1,$$

which implies that, all the assumptions of Theorem 4 are satisfied. Thus, problem (5) has a unique solution on $\mathcal{J}$.

6. Conclusions

In our study, Firstly, we established the equivalence between problem (5) and the Volterra integral equation. Secondly, Banach and Schaefer's fixed point theorems were used to establish the existence and uniqueness solutions for implicit fractional pantograph differential equation which involves ϕ-Hilfer fractional derivatives. Based on ϕ-Hilfer fractional derivatives, we found that the stability of Ulam-Hyers and generalized Ulam-Hyers allowed on the implicit fractional pantograph differential equation, supplemented with a nonlocal Riemann-Liouville condition. In addition, examples were given to illustrate our main results. Moreover, it worthy to mention the following remarks:

- If $\rho \to 0$ and $\phi(t) = t$, we obtain the results of [48] and [52]. Furthermore, if $\rho \to 0$ we obtain the Ulam-Hyers and generalized Ulam-Hyers stability for the implicit fractional pantograph differential equations with ϕ-Hilfer fractional derivatives [52,58] and if $q = 0$ we obtain [51].
- If $\rho \to 1$, the nonlocal Riemann-Liouville integral condition reduces to a nonlocal integral condition which plays an important role in computational fluid dynamics, ill-posed problems and mathematical models [62].
- If $\rho \to 0$, the initial condition reduces to multi-point nonlocal condition.
- If $t \in [a, b]$ as defined in paper [58], the function $f(t, x(t), x(\lambda t))$ is not well-defined for some choice of $0 < \lambda < 1$. Thus, our results modify and improve the above cited remarks and can be considered as the development of the qualitative analysis of fractional differential equations. The study of Ulam-Hyers stability in the frame of ϕ-Hilfer fractional derivative with a generalized nonlocal boundary condition proposed in this paper and other coupled system will be presented in the near future.

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