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Schauder-Type Fixed Point Theorem in Generalized Fuzzy Normed Linear Spaces

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Abstract: In the present article, the Schauder-type fixed point theorem for the class of fuzzy continuous, as well as fuzzy compact operators is established in a fuzzy normed linear space (fnls) whose underlying t-norm is left-continuous at $(1,1)$. In the fuzzy setting, the concept of the measure of non-compactness is introduced, and some basic properties of the measure of non-compactness are investigated. Darbo's generalization of the Schauder-type fixed point theorem is developed for the class of ψ-set contractions. This theorem is proven by using the idea of the measure of non-compactness.

Keywords: Schauder fixed point theorem; fuzzy normed linear space; t-norm; measure of non-compactness

MSC: 03B52; 03E72; 46B20; 46B99; 46A19; 03E70; 15A03; 54H25

1. Introduction

In 1930, Schauder established an important theorem in the field of fixed point theory. The theorem stated that "If B is a compact, convex subset of a Banach space X and $f : B \to B$ is a continuous function then f has a fixed point." However, to develop more results in functional analysis, Schauder relaxed the compactness by closedness. The theorem has an enormous influence on the theory of differential equations. At first, the Schauder-type fixed point theorem was applied to Peano's existence theorem for the first order differential equations. After that, many interesting applications of this theorem were given to differential equations. For example, in 2007, Chu and Torres [1] proved the existence of positive solutions to the second order singular differential equations with the help of this fixed point theorem. In 2009, A. F. Dizaji et al. [2] determined the sufficient condition for the existence of periodic solution of the initial value problems, which correspond to the Duffing's oscillator with time varying coefficients as an application of the Schauder-type fixed point theorem. Recently, in 2019, Shengjun Li et al. [3] established the existence of the periodic orbits of rapidly symmetric systems with a repulsive singularity. The line of proof of this existence problem is based on the use of Schauder's fixed point theorem. Moreover, the global existence of the solution for a class of functional equations is also studied using the Schauder fixed point theorem, which arises in various types of neural networks such as the Hopfield neural network, the Cohen–Grossberg neural network, cellular networks, etc. For the references, please see [4–6].

Due to its huge application in real-life problems, much scientific attention has been drawn towards the generalization of this theorem. In 1935, A. N. Tychonoff [7] extended Schauder's theorem to locally convex spaces. In 1950, M. Hukuhara [8] unified both the theorem of Schauder and Tychonoff. In 1955 [9], G. Darbo extended the Schauder theorem to a more general class of mappings, the so-called α-set contractions, which contain compact, as well as continuous mappings. Darbo proved this theorem using the concept of Kuratowski's measure of non-compactness. In 1961 [10], Ky Fan generalized both Schauder's and Tychonoff's theorem for the class of continuous set-valued mappings. In recent years, a significant contribution has been made towards the generalization of Schauder's fixed point theorem. For example, in 2012, R. L. Pouso [11] introduced a new version of Schauder's theorem for the class of discontinuous operators. In 2013, R. P. Agarawal et al. [12] established this theorem in semilinear Banach spaces. In 2016, Wei-Shih Du [13] generalized this theorem in an another direction, i.e., the compactness assumption is replaced by the finite open cover, and the continuity condition is totally removed.

On the other hand, several authors, viz. Xio and Zhu, Bag and Samanta, and Zhang and Guo, have played important roles in the process of the formulation of the Schauder-type fixed point theorem in the fuzzy setting. For the references, please see [14–16]. However, all of them considered the underlying t-norm as the continuous t-norm. Therefore, naturally, a question may arise: Is it possible to prove the Schauder-type fixed point theorem in a fuzzy normed linear space (fnls) w.r.t. the general t-norm?

In this paper, we try to give an affirmative answer to this question.

In this paper, we develop the Schauder-type fixed point theorem for a fuzzy continuous, as well as a fuzzy compact operator in an fnls whose underlying t-norm is left-continuous only at $(1,1)$. We also establish Darbo's generalization of the Schauder-type fixed point theorem in the fuzzy setting for the class of ψ-set contraction mappings using the properties of the measure of non-compactness.

This article is divided into three parts. Section 2 deals with preliminary results, which are used in the subsequent sections. In Section 3, the Schauder-type fixed point theorem for the class of fuzzy continuous, as well as fuzzy compact mappings is established in generalized fnls. In Section 4, the definition of the measure of non-compactness is given, and some basic properties are studied to prove Darbo's generalization of the Schauder-type fixed point theorem.

2. Preliminaries

Definition 1 ([17])**.** *Let X be a linear space over the field $\mathcal{F}$ ($\mathbb{C}$ or $\mathbb{R}$). A fuzzy subset N of $X \times \mathbb{R}$ ($\mathbb{R}$ is the set of all real numbers) is called a fuzzy norm on X if:*

(N1) $\forall t \in \mathbb{R}$ *with* $t \leq 0,\ N(x,t) = 0$;

(N2) $(\forall t \in \mathbb{R},\ t > 0,\ N(x,t) = 1)\ iff\ x = \theta$;

(N3) $\forall t \in \mathbb{R},\ t > 0,\ N(cx,t) = N(x, \frac{t}{|c|})\ if\ c \neq 0$;

(N4) $\forall s,t \in \mathbb{R};\ x,u \in X;\ N(x+u, s+t) \geq N(x,s) * N(u,t)$;

(N5) $N(x,.)$ *is a non-decreasing function of* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x,t) = 1$.

The triplet $(X, N, *)$ is referred to as an fnls.

Throughout the paper, we assume the following conditions:

1. For each $x \neq \theta$, $N(x,t)$ is a left-continuous function w.r.t. t.
2. The t-norm $*$ is left-continuous at one with respect to the first or second component.

Theorem 1 ([17])**.** *Let $(X, N, *)$ be a finite-dimensional fnls in which the underlying t-norm $*$ is continuous at $(1,1)$. Then, a subset A is compact iff A is closed and bounded.*

Lemma 1 ([18])**.** *Let $(X, N, *)$ be an fnls. Then:*

$$\lim_{n\to\infty} N(x_n - x, t) = 1\ \forall t > 0 \Leftrightarrow \lim_{n\to\infty} \wedge \{t > 0 : N(x_n - x, t) > 1 - \alpha\} = 0\ \forall \alpha \in (0,1)$$

Proposition 1 ([18])**.** *Let $(X, N, *)$ be an fnls. Then, the function $M : X \times X \times [0,\infty) \to [0,1]$ defined by $M_N(x,y,t) = N(x-y,t)$ is a fuzzy metric space defined by H. Wu [19]. Thus, the family $\mathcal{B}$ (the collection of* *all* (α, t) *neighborhoods* $B_N(x, \alpha, t)$, $x \in X$, $0 < \alpha < 1$, $t > 0$) *induces a Hausdorff topology* τ *such that* $\mathcal{B}$ *is a base for* τ *and* τ *also satisfies the first countability axiom, where* $B_N(x, \alpha, t) = \{y \in X : N(x - y, t) > 1 - \alpha\}$.

Definition 2 ([20])**.** *A fuzzy metric space* $(X, M, *)$ *is called compact if* (X, τ_M) *is compact.*

Theorem 2 ([20])**.** *A fuzzy metric space* $(X, M, *)$ *is fuzzy totally bounded iff every sequence has a Cauchy subsequence.*

Note 1. *The above result is also true if* $(X, M, *)$ *is the H. Wu-type fuzzy metric space.*

Definition 3 ([21])**.** *Let* $(X, N, *)$ *be an fnls. Let* $\{x_n\}$ *be a sequence in* X. *Then,* $\{x_n\}$ *is said to be convergent if* $\exists\ x \in X$ *such that:*

$$\lim_{n\to\infty} N(x_n - x, t) = 1\ \forall t > 0.$$

In that case, x is called the limit of the sequence $\{x_n\}$ *and is denoted by* $\lim x_n$.

Definition 4 ([21])**.** *A subset A of an fnls is said to be fuzzy bounded if for each* α, $0 < \alpha < 1\ \exists\ t(\alpha) > 0$ *such that* $N(x, t) > 1 - \alpha\ \forall x \in A$.

Definition 5 ([21])**.** *Let* $(X, N, *)$ *be an fnls. A subset F of X is said to be closed if for any sequence* $\{x_n\}$ *in F, it converges to x, i.e.,*

$$\lim_{n\to\infty} N(x_n - x, t) = 1\ \forall t > 0$$

implies that $x \in F$.

Definition 6 ([21])**.** *Let* $(X, N, *)$ *be an fnls. A subset B of X is said to be the closure of F if for any* $x \in B$, $\exists$, *a sequence* $\{x_n\}$ *in F such that:*

$$\lim_{n\to\infty} N(x_n - x, t) = 1\ \forall t > 0.$$

We denote the set B by $\overline{F}$.

Definition 7 ([21])**.** *Let* $(X, N, *)$ *be an fnls. A subset A of X is said to be compact if any sequence* $\{x_n\}$ *in A has a subsequence converging to an element of A.*

Definition 8 ([21])**.** *A sequence* $\{x_n\}$ *is said to be Cauchy if* $\lim_{n\to\infty} N(x_n - x_{n+p}, t) = 1$, $\forall t > 0$, $p = 1, 2, 3...$

This definition of a Cauchy sequence is equivalent to $\lim_{n,m\to\infty} N(x_n - x_m, t) = 1$, $\forall t > 0$. Throughout the paper, we use this as the definition of the Cauchy sequence.

Lemma 2 ([22])**.** *Let* $(X, N, *)$ *be an fnls. If* $A \subseteq X$ *is fuzzy bounded, then* $\overline{A}$ *is also.*

Definition 9 ([22])**.** *Let* $(X, N_1, *_1)$ *and* $(Y, N_2, *_2)$ *be two fnlss. A linear operator* $T : (X, N_1, *_1) \to (Y, N_2, *_2)$ *is called a fuzzy compact linear operator if for every fuzzy bounded subset M of X, the subset* $T(M)$ *of Y is relatively compact, i.e.,* $\overline{T(M)}$ *is a compact set w.r.t.* τ_{N_2}.

Theorem 3 ([22])**.** *Let* $T : (X, N_1, *_1) \to (Y, N_2, *_2)$ *be a linear operator and* $*_2$ *be continuous at* $(1, 1)$. *Then, T is a fuzzy compact linear operator iff it maps every bounded sequence* $\{x_n\}$ *in* $(X, N_1, *_1)$ *onto a sequence* $\{T(x_n)\}$ *in* $(Y, N_2, *_2)$, *which has a convergent subsequence.*

Lemma 3 ([23]). *A fuzzy metric space $(X, M, *)$ is sequentially compact iff it is compact.*

Note 2. *By Lemma 3, in an fnls, Definition 2 and Definition 7 are equivalent.*

Theorem 4 ([24]). *In an fnls $(X, N, *)$, a subset A of X is fuzzy bounded iff A is bounded in topology τ_N.*

Theorem 5 ([24]). *In an fnls $(X, N, *)$, the following statements are equivalent:*

(i) *A is fuzzy totally bounded.*

(ii) $\forall \alpha \in (0,1),\ \forall t > 0\ \exists\ \{x_1, x_2, \cdots x_n\} \subseteq A : A \subseteq \bigcup_{i=1}^{n} (x_i + B(\theta, \alpha, t))$

Theorem 6 ([24]). *Let $(X, N, *)$ be an fnls and $K \subseteq X$ be a compact set in (X, τ_N). Then, K is fuzzy totally bounded.*

Definition 10 ([25]). *An fnls (X, N) is a fuzzy Banach space if its induced fuzzy metric is complete.*

Definition 11 ([26]). *A subset A of an fnls $(X, N, *)$ is called fuzzy totally bounded if:*

$$\forall \alpha \in (0,1),\ \exists\ \{x_1, x_2, \cdots x_n\} \subseteq X : A \subseteq \bigcup_{i=1}^{n} (x_i + B(\theta, \alpha, \alpha)).$$

Theorem 7 ([26]). *Let $T : (X, N_1) \to (Y, N_2)$ be a mapping where (X, N_1) and (Y, N_2) are fnlss. Then, the following statements are equivalent:*

(i) *T is fuzzy continuous on X.*

(ii) *T is continuous on X.*

(iii) *T maps a fuzzy bounded set to a fuzzy bounded set.*

Theorem 8. *In a fuzzy Banach space $(X, N, *)$, if a subset A of X is fuzzy totally bounded, then it is compact in (X, τ_N).*

Proof. Consider a sequence $\{x_n\}$ in A. By Theorem 2, $\{x_n\}$ has a Cauchy subsequence. Since $(X, N, *)$ is fuzzy Banach space, then the Cauchy subsequence of $\{x_n\}$ is convergent in $(X, N, *)$. Therefore, by Definition 7, A is compact in $(X, N, *)$. □

Definition 12 ([27]). *(Fuzzy continuous) A mapping T from (X, N_1) to (Y, N_2) is said to be fuzzy continuous at $x_0 \in X$ if for given $\epsilon > 0$, $\alpha \in (0,1)\ \exists\ \delta(\alpha, \epsilon) > 0$, $\beta(\alpha, \epsilon) \in (0,1)$ such that $\forall x \in X$:*

$$N_1(x - x_0, \delta) > \beta \implies N_2(Tx - Tx_0, \epsilon) > \alpha$$

If T is fuzzy continuous at each,
$x \in X$, then T is fuzzy continuous on X.

Definition 13 ([27]). *(Sequentially fuzzy continuous) A mapping T from (X, N_1) to (Y, N_2) is said to be sequentially fuzzy continuous at $x_0 \in X$ if for any sequence $\{x_n\}$, $x_n \in X$ with $x_n \to x_0$ implies $Tx_n \to Tx_0$, i.e., $\lim_{n\to\infty} N_1(x_n - x_0, t) = 1\ \forall t > 0, \implies \lim_{n\to\infty} N_2(Tx_n - Tx_0, t) = 1\ \forall t > 0$*

Theorem 9 ([27]). *Let $T : (X, N_1) \to (Y, N_2)$ be a mapping where (X, N_1) and (Y, N_2) are fnlss. Then, T is fuzzy continuous iff it is sequentially fuzzy continuous.*

Note 3. *From Definition 9, it is clear that if T is a fuzzy compact linear operator, then T maps bounded sets of X to bounded sets of Y by Theorem 4. Thus, T is a continuous mapping from (X, τ_{N_1}) to (Y, τ_{N_2}).*

3. Schauder-Type Fixed Point Theorem

In this section, we first define the uniformly fuzzy convergence and pointwise fuzzy convergence for a sequence of functions and investigate the relation between them. After that, we propound three types of Schauder-type fixed point theorems for the fuzzy compact class, as well as the fuzzy continuous linear operator in a generalized fnls and try to prove them.

Definition 14. *Let $f_n : (X, N_1, *_1) \to (Y, N_2, *_2)$ be a family of functions.*

(i) *$\{f_n\}$ is said to be uniformly fuzzy convergent to a function f on a subset A of X if for each $\alpha \in (0,1)$,*

$$\lim_{n\to\infty} \bigvee_{x\in A} \wedge\{t > 0 : N_2(f_n(x) - f(x), t) > 1 - \alpha\} = 0$$

i.e., for each $\alpha \in (0,1)$ and for each $\epsilon > 0$ $\exists$ $N_0(\alpha, \epsilon) \in \mathbb{N}$ such that:

$$\bigvee_{x\in A} \wedge \{t > 0 : N_2(f_n(x) - f(x), t) > 1 - \alpha\} < \epsilon \; \forall n \geq N_0$$

(ii) *$\{f_n\}$ is said to be pointwise fuzzy convergent to a function f on a subset A of X if for each $\alpha \in (0,1)$, for each $x \in Y$ $\exists$ $N_0(\alpha, \epsilon, x) \in \mathbb{N}$ such that:*

$$\wedge\{t > 0 : N_2(f_n(x) - f(x), t) > 1 - \alpha\} < \epsilon \; \forall n \geq N_0$$

From the definition, it is obvious that (i) implies (ii), but (ii) does not imply (i).We verify this by the following example.

Example 1. *Let us consider a real nls (normed linear space) $(\mathbb{R}, \| \|)$, where $\mathbb{R}$ is the set of all real numbers and $\|x\| = |x|$, $\forall x \in \mathbb{R}$. Define two functions as follows:*

$$N_1(x,t) = \begin{cases} \frac{t}{t+\|x\|}, & t > 0 \\ 0, & t \leq 0 \end{cases} \qquad N_2(x,t) = \begin{cases} 1, & t \geq \|x\| \\ 0, & t < \|x\| \end{cases}$$

Define $f_n : (\mathbb{R}, N_1, \wedge) \to (\mathbb{R}, N_2, \wedge)$ by $f_n(x) = x^n$. Now, if we consider $f_n : [0,1] \to [0,1]$, then f_n is pointwise fuzzy convergent, but not uniformly fuzzy convergent.

Lemma 4. *Let f be self-mapping defined on a fuzzy Banach space $(X, N, *)$ and f also be a fuzzy compact linear operator on a subset M of X. Then, there exists a sequence of continuous mappings $\{f_n\}$ such that:*

(i) *$\{f_n\}$ is uniformly fuzzy convergent to f.*
(ii) *$\{f_n(M)\}$ generates a finite-dimensional subspace of X.*

Proof. Since f is a fuzzy compact linear operator, thus the set $\{\overline{f(x); x \in M}\}$ is a fuzzy compact set, i.e., $\{\overline{f(x); x \in M}\}$ is a compact set w.r.t. τ_N. Now, by Theorem 6, $\{\overline{f(x); x \in M}\}$ is fuzzy totally bounded. Let $\alpha_0 \in (0,1)$ and $\{t_n\}$ be a strictly decreasing sequence that tends to 0. Then, for each t_n, we can find a finite No. of elements $y_1^n, y_2^n, \cdots y_m^n \in f(M)$ such that:

$$\begin{aligned} f(M) &\subseteq \bigcup_{i=1}^{m} y_i^n + B_N(\theta, \alpha_0, t_n) \\ &\implies N(f(x) - y_i^n, t_n) > 1 - \alpha_0 \; \forall x \in M \\ &\implies \wedge\{t > 0 : N(f(x) - y_i^n, t) > 1 - \alpha_0\} \leq t_n \end{aligned} \tag{1}$$

We now define f_n on $f(M)$ for each $y \in f(M)$, by:

$$f_n(y) = \sum_{i=1}^{m} g_i^n(y) y_i^n / \sum_{i=1}^{m} g_i^n(y)$$
$$where\ g_i^n(y) = max\{0, t_n - \wedge\{t > 0 : N(y - y_i^n, t) > 1 - \alpha_0\}\}$$

Since the family $\wedge\{t > 0 : N(x,t) > 1 - \alpha\}$ is a continuous function on X for each $\alpha \in (0,1)$ and f is continuous on M by Note 3, so $g_i^n(x)$ is continuous on M. Thus, each f_n is a continuous function on $f(M)$. Now,

$$\begin{aligned}
&\wedge \{t > 0 : N(f_n(y) - y, t) > 1 - \alpha_0\} \\
&= \wedge\{t > 0 : N(\sum_{i=1}^{m} g_i^n(y) y_i^n / \sum_{i=1}^{m} g_i^n(y) - y, t) > 1 - \alpha_0\} \\
&= \frac{1}{\sum_{i=1}^{m} g_i^n(y)} \wedge \{t > 0 : N(\sum_{i=1}^{m} g_i^n(y)\{y_i^n - y\}, t) > 1 - \alpha_0\} \\
&= \frac{\sum_{i=1}^{m} g_i^n(y)}{\sum_{i=1}^{m} g_i^n(y)} \wedge \{\frac{t}{\sum_{i=1}^{m} g_i^n(y)} > 0 : N(y_i^n - y, \frac{t}{\sum_{i=1}^{m} g_i^n(y)}) > 1 - \alpha_0\} \\
&= \wedge\{t' > 0 : N(y_i^n - y, t') > 1 - \alpha_0\} \leq t_n
\end{aligned} \tag{2}$$

Now, define $\tilde{f}_n : M \to M$ by $\tilde{f}_n(x) = f_n(f(x))$. Thus, by Inequality (1),

$$\begin{aligned}
&\wedge \{t > 0 : N(f_n(f(x)) - f(x), t) > 1 - \alpha_0\} \leq t_n \ \forall x \in \ M \\
&\implies \bigvee_{x \in M} \wedge \{t > 0 : N(f_n(f(x)) - f(x), t) > 1 - \alpha_0\} \leq t_n
\end{aligned}$$

Thus, $\lim_{n\to\infty} \bigvee_{x \in M} \wedge\{t > 0 : N(f_n(f(x)) - f(x), t) > 1 - \alpha_0\} = 0.$

Since $\alpha_0 \in (0,1)$ is arbitrary, then the above relation is true for each $\alpha_0 \in (0,1)$. Thus, $\tilde{f}_n$ uniformly fuzzy converges to f. Condition (ii) is automatically valid by the construction of f_n. □

Remark 1. *In Lemma 4, each $\{f_n\}$ contains a fixed point, say x_n. This can be shown in the following way:*

Now, the sequence $\{f_n\}$, which is uniformly fuzzy convergent to f, is of the form:

$$f_n(x) = \sum_{i=1}^{m} g_i^n(f(x)) y_i^n / \sum_{i=1}^{m} g_i^n(f(x))$$
$$where\ g_i^n(f(x)) = max\{0, t_n - \wedge\{t > 0 : N(f(x) - y_i^n, t) > 1 - \alpha_0\}\}$$

Now, if we choose $C_n = \overline{Co}\{y_i^n\}_{i=1}^m$ (convex closure of $\{y_i^n\}_{i=1}^m$), $Y_n = Span\{y_i^n\}_{i=1}^m$, then C_n is a closed, bounded, convex subset of the finite-dimensional subspace Y_n of X and $f_n(C_n) \subseteq C_n$ (by the definition of f_n). Each f_n is continuous. Now, by the Brouwer fixed point theorem, $\exists$ a point $x_n \in C_n$ such that $f_n(x_n) = x_n$.

Remark 2. *If $\{f_n\}$ is uniformly fuzzy convergent to f on X, then for each $x \in X$,*

$$\lim_{n\to\infty} N_2(f_n(x) - f(x), t) = 1 \ \forall t > 0.$$

Proof. Since $\{f_n\}$ is uniformly fuzzy convergent to f, then $\{f_n\}$ pointwise fuzzy converges to f. Thus:

$$\lim_{n\to\infty} \wedge \{t > 0 : N_2(f_n(x) - f(x), t) > 1 - \alpha\} = 0 \ \forall x \in X, \ \forall \alpha \in (0,1).$$

Now, from Lemma 1, the required result follows immediately. □

Lemma 5. *Let $\{T_n\}$ be a sequence of fuzzy compact linear operators defined on $E \subseteq X$, where $(X, N, *)$ is an fnls. Again, $\{T_n\}$ is uniformly fuzzy convergent on E. Then, the set $\tilde{E} = \bigcup_{i=1}^{\infty} \overline{T_i E}$ is a fuzzy compact set, i.e., compact w.r.t. the topology τ_N.*

Proof. We show that $\tilde{E}$ is fuzzy totally bounded. Then, by Theorem 8, the assertion of the lemma is automatically valid. Let $\epsilon > 0$ be an arbitrary No. and $\alpha_0 \in (0,1)$ be given. Then, by the left-continuity of $*$ at $(1,1)$ $\exists\, \beta_0 \in (0,1)$ such that:

$$(1-\beta_0) * (1-\beta_0) * (1-\beta_0) > 1-\alpha_0$$

Since $\{T_n\}$ uniformly fuzzy converges to T, then $\exists N_0(\epsilon/4, \beta_0) \in \mathbb{N}$ such that:

$$\begin{aligned} &\bigvee_{x\in E} \wedge\{t>0 : N(T_n(x)-T(x),t) > 1-\beta_0\} < \epsilon/4\ \forall n \geq N_0 \\ &\implies \wedge\{t>0 : N(T_n(x)-T(x),t) > 1-\beta_0\} < \epsilon/4\ \forall n \geq N_0,\ \forall x \in E \\ &\implies N(T_n(x)-T(x),\epsilon/4) > 1-\beta_0\ \forall n \geq N_0,\ \forall x \in E \end{aligned} \tag{3}$$

Again, the sets $\overline{T_0 E}, \overline{T_1 E}, \cdots, \overline{T_{N_0} E}$ are fuzzy compact sets, i.e., compact w.r.t τ_N by the definition of the fuzzy compact linear operator. Therefore, $\bigcup_{i=0}^{N_0} \overline{T_i E}$ is compact w.r.t. τ_N. By Theorem 6, $\bigcup_{i=0}^{N_0} \overline{T_i E}$ is fuzzy totally bounded. Now, by the definition of the fuzzy total boundedness, we can find $y_1, y_2, \cdots, y_n$ such that:

$$T_j(x) \in \bigcup_{i=0}^{n} B_N(y_i, \epsilon/2, \beta_0)\ \forall\ T_j(x) \in \bigcup_{i=0}^{N_0} \overline{T_i E}$$

Now, for any $T_m(x) \in \bigcup_{i=0}^{\infty} \overline{T_i E}$, if $m \leq N_0$, we have:

$$T_m(x) \in \bigcup_{i=0}^{n} B_N(y_i, \epsilon/2, \beta_0) \subseteq \bigcup_{i=0}^{n} B_N(y_i, \epsilon, \alpha_0) \tag{4}$$

If $m > N_0$, then:

$$\begin{aligned} N(T_m(x)-y_i,\epsilon) &\geq N(T_m(x)-T(x),\epsilon/4) * N(T(x)-T_{N_0}(x),\epsilon/4) * N(T_{N_0}(x)-y_i,\epsilon/2) \\ &\geq (1-\beta_0) * (1-\beta_0) * (1-\beta_0) > (1-\alpha_0) \\ \therefore T_m(x) &\in \bigcup_{i=0}^{n} B_N(y_i,\epsilon,\alpha_0) \end{aligned} \tag{5}$$

Thus, $\bigcup_{i=0}^{\infty} \overline{T_i E}$ is fuzzy totally bounded. This completes the proof. □

Lemma 6. *Let T be a continuous self-mapping on $(X, N, *)$ and $\dim T(X) < \infty$. Then, T is a fuzzy compact linear operator.*

Proof. Let $\{y_n\}$ be a fuzzy bounded sequence. Then, by Theorem 7, $\{Ty_n\}$ is a fuzzy bounded sequence. Again, the range set of $\{Ty_n\}$ say $R(Ty_n : n \in \mathbb{N})$ is fuzzy bounded. Now, by Lemma 2, $\overline{R(Ty_n : n \in \mathbb{N})}$ is fuzzy bounded. Since $T(X)$ is finite-dimensional, thus $\overline{R(Ty_n : n \in \mathbb{N})}$ is fuzzy compact. Therefore, $\{Ty_n\}$ has a fuzzy convergent subsequence. Thus, T is a fuzzy compact linear operator by Theorem 3. □

Lemma 7. *Let $(X, N, *)$ be an fnls. For each $\alpha \in (0, 1)$ $\exists$ $\beta \in (0, \alpha/3)$ such that:*

$$B(\theta, \beta, \beta) + B(\theta, \beta, \beta) + B(\theta, \beta, \beta) \subseteq B(\theta, \alpha, \alpha)$$

Proof. Suppose $\alpha_0 \in (0, 1)$. Then, $\alpha_0/3$ also belongs to $(0, 1)$. By the left-continuity of '$*$' at $(1, 1)$, $\exists\beta_0 \in (0, \alpha/3]$ such that:

$$(1 - \beta_0) * (1 - \beta_0) * (1 - \beta_0) > 1 - \alpha_0/3$$

Let $y \in B(\theta, \beta_0, \beta_0) + B(\theta, \beta_0, \beta_0) + B(\theta, \beta_0, \beta_0)$. Thus, $y = y_1 + y_2 + y_3$, where $y_1, y_2, y_3 \in B(\theta, \beta_0, \beta_0)$. Now:

$$\begin{aligned}
&N(y, \alpha_0)\\
&\geq N(y_1, \alpha_0/3) * N(y_2, \alpha_0/3) * N(y_3, \alpha_0/3)\\
&\geq N(y_1, \beta) * N(y_2, \beta) * N(y_3, \beta)\\
&\geq (1 - \beta) * (1 - \beta) * (1 - \beta)\\
&> (1 - \alpha_0)\\
&\therefore y \in B(\theta, \alpha_0, \alpha_0)
\end{aligned}$$

This completes the proof. □

Theorem 10. *(Schauder-type fixed point theorem) Let $(X, N, *)$ be an fnls, C be a bounded, closed, convex subset in X w.r.t. τ_N, and $f : C \to C$ be a fuzzy compact linear operator. Then, there exists a point $x_0 \in C$ such that $f(x_0) = x_0$.*

Proof. Since f is a fuzzy compact linear operator, then by Lemma 4 and Remark 1, there exists a sequence of continuous mappings $\{f_n\}$, which is uniformly fuzzy convergent to f, and each $\{f_n\}$ contains a fixed point, say x_n, i.e., $f_n(x_n) = x_n$, $\forall n \in \mathbb{N}$.

Since each $x_n \in \tilde{C}$, then by Lemma 5, $\{x_n\}$ has a fuzzy convergent subsequence, say $\{x_{n_k}\}$, i.e., $x_{n_k} \to x_0$. Now, for any $t > 0$, $\alpha_0 \in (0, 1)$ with
$(1 - \beta_0) * (1 - \beta_0) * (1 - \beta_0) > 1 - \alpha_0$, we have:

$$\begin{aligned}
N(f(x_0) - x_0, t) \geq & N(f(x_0) - f_{n_k}(x_0), t/4) * N(f_{n_k}(x_0) - f_{n_k}(x_{n_k}), t/4) *\\
& N(f_{n_k}(x_{n_k}) - x_{n_k}, t/4) * N(x_{n_k} - x_0, t/4)
\end{aligned} \tag{6}$$

Since $\{f_{n_k}\}$ uniformly fuzzy converges to f, then by Remark 2, for $x_0 \in X$,

$$\lim_{k\to\infty} N(f(x_0) - f_{n_k}(x_0), t/4) = 1$$

Again, each $\{f_{n_k}\}$ is continuous, so $x_{n_k} \to x_0 \implies f_{n_k}(x_{n_k}) \to f_{n_k}(x_0)$,

$$i.e., \lim_{k\to\infty} N(f_{n_k}(x_0) - f_{n_k}(x_0), t/4) = 1$$

Taking $lim\ k \to \infty$ in both sides of Inequality 6, we get,

$$\begin{aligned}
&N(f(x_0) - x_0, t) \geq 1\\
&\Rightarrow N(f(x_0) - x_0, t) = 1\ \forall t > 0\\
&\Rightarrow f(x_0) = x_0\ (by\ N2)
\end{aligned}$$

This completes the proof. □

Theorem 11. *Let $(X, N, *)$ be an fnls. Let C be a convex, compact subset of X and f be a continuous operator from C into C. Then, there exists $x_0 \in C$ such that $f(x_0) = x_0$.*

Proof. Since C is compact w.r.t. τ_N, thus by Theorem 6, C is fuzzy totally bounded. Now, consider a strictly decreasing sequence $\{\alpha_n\}$ with $\alpha_n \to 0$, then $\exists \{x_n^1, x_n^2, \cdots x_n^m\} \subseteq C$ such that,

$$C \subseteq \bigcup_{i=1}^{m} \{x_n^i\} + B(\theta, \alpha_n, \alpha_n)$$

Now, define a family of functions such that:

$$f_n(x) = \frac{\sum_{i=1}^{m} \beta_n^i(x) x_n^i}{\sum_{i=1}^{m} \beta_n^i(x)}$$

where $\sum_{i=1}^{m} \beta_n^i(x) = max\{0, \alpha_n - \wedge\{t > 0 : N(f(x) - x_n^i, t) > 1 - \alpha_n\}\}$

Let $x \in C$ and $\alpha_0 \in (0, 1)$.

Since $\alpha_n \to 0$, so $\exists\, N_0 \in \mathbb{N}$ such that $\alpha_n < \alpha_0\ \forall n \geq N_0$. Now:

$$\begin{aligned}
&\wedge \{t > 0 : N(f_n(x) - f(x), t) > 1 - \alpha_0\} \\
&= \wedge\{t > 0 : N(\frac{\sum_{i=1}^{m} \beta_n^i(x) x_n^i}{\sum_{i=1}^{m} \beta_n^i(x)} - f(x), t) > 1 - \alpha_0\} \\
&= \wedge\{t > 0 : N(x_n^i - f(x), t) > 1 - \alpha_0\} \\
&\leq \wedge\{t > 0 : N(x_n^i - f(x), t) > 1 - \alpha_n\} \leq \alpha_n\ \forall n \geq\ N_0 \\
&\Rightarrow \bigvee_{x \in C} \wedge \{t > 0 : N(f_n(x) - f(x), t) > 1 - \alpha_0\} \leq \alpha_n\ \forall n \geq\ N_0 \\
&\Rightarrow \bigvee_{x \in C} \wedge \{t > 0 : N(f_n(x) - f(x), t) > 1 - \alpha_0\} \to 0\ as\ n \to \infty
\end{aligned}$$

Since $\alpha_0 \in (0, 1)$ is arbitrary, thus $\{f_n\}$ uniformly fuzzy converges to f. Again, $\{f_n\}$ is a family of continuous functions from (X, τ_N) to itself. For each $n \in \mathbb{N}$, f_n maps from C to the closed convex hull C_n of $\{x_n^i, i = 1, 2, 3, \cdots\}$. Since C is convex, then $C_n \subseteq C$. We constrict the restricted mapping $f_n : C_n \to C_n$, and it turns out that it maps the compact, convex subset of a finite-dimensional set C_n of $Y_n =$ the span of $\{x_n^i, i = 1, 2, \cdots, m(n)\}$ into itself. Thus, by the Browder fixed point theorem, $\exists\, x_n \in C_n \subseteq C$ such that $f_n(x_n) = x_n,\ \forall n \in \mathbb{N}$. Since C is compact w.r.t. τ_N, $\{x_n\}$ has a convergent subsequence, say $\{x_{n_k}\}$ w.r.t. fuzzy norm N and $\{x_{n_k}\} \to x_0$. Now, consider $t_0(> 0) \in \mathbb{R}$.

We have $N(f(x_0) - x_0, t_0)$

$$\geq N(f(x_0) - f(x_{n_k}), t_0/3) * N(f(x_{n_k}) - x_{n_k}, t_0/3) * N(x_{n_k} - x_0, t_0/3)$$

Taking $k \to \infty$ on both sides, we get $N(f(x_0) - x_0, t_0) = 1$. Again, $t_0 > 0$ is arbitrary.

$$\text{So, } N(f(x_0) - x_0, t) = 1,\ \forall t > 0$$
$$\Rightarrow f(x_0) = x_0.$$

□

Theorem 12. *Let $(X, N, *)$ be a fuzzy Banach space, C be a closed and convex subset of X, and $f : C \to C$ be a continuous mapping such that the image of C is contained ina compact set. Then, $\exists\, x_0 \in C$ such that $f(x_0) = x_0$.*

Proof. Let $B = f(C)$. Consider $K = \overline{Co(f(C))}$ (where $Co(f(C))$ is the convex combination of the element of $f(C)$). It is clear that K is a convex subset of X. We show that K is compact w.r.t. τ_N. We have $B(\subseteq C)$, a compact subset of X w.r.t. τ_N. Therefore, B is fuzzy totally bounded.
Let $\alpha_0 \in (0,1)$. Then, $\exists \beta_0 \in (0, \alpha_0/3)$ such that $(1-\beta_0) * (1-\beta_0) * (1-\beta_0) > (1-\alpha_0)$. Again, since $\beta_0 \in (0,1)$, $\exists \{x_1, x_2, \cdots, x_n\} \subseteq B$ such that:

$$B \subseteq \bigcup_{i=1}^{n} \{x_i\} + B(\theta, \beta_0, \beta_0) \tag{7}$$

Let $x \in Co(B)$. Thus, x is of the form $\sum_{j=1}^{m} \alpha_j y_j$, where $\sum_{j=1}^{m} \alpha_j = 1$. Again, each $y_j \in \bigcup_{i=1}^{n} \{x_i\} + B(\theta, \beta_0, \beta_0)$, $j = \{1, 2, \cdots, m(\leq) n\}$, Therefore, for each y_j, $\exists\, x_i$ for some $i \in \{1, 2, \cdots, n\}$ such that:

$$N(x_i - y_j, \beta_0) > 1 - \beta_0$$

Here:

$$\begin{aligned} &N(\sum_{j=1}^{m} \alpha_j x_j - \sum_{j=1}^{m} \alpha_j y_j, \beta_0) \\ &= N(x_j - y_j, \beta_0) > 1 - \beta_0 \\ &\therefore x \in \sum_{j=1}^{m} \alpha_j x_j + B(\theta, \beta_0, \beta_0) \end{aligned}$$

$\therefore x \in C_j + B(\theta, \beta_0, \beta_0)$, where $C_j = \sum_{j=1}^{m} \alpha_j x_j$. Since each fnls is a topological vector space:

$$\text{so, } \overline{Co(B)} = \bigcap_{\alpha \in (0,1)} Co(B) + B(\theta, \alpha, \alpha)$$

Thus:

$$\begin{aligned} \overline{Co(B)} &\subseteq Co(B) + B(\theta, \beta_0, \beta_0) \\ &\subseteq C_j + B(\theta, \beta_0, \beta_0) + B(\theta, \beta_0, \beta_0) \\ &\subseteq \overline{C_j} + B(\theta, \beta_0, \beta_0) + B(\theta, \beta_0, \beta_0) \end{aligned}$$

Here, $\overline{C_j}$ is a closed bounded subset of $Y_j = Span\{x_j\}_{j=1}^{m}$. Therefore, $\overline{C_j}$ is compact w.r.t. τ_N. Thus, $\exists\, \{p_k\}_{k=1}^{r} \in \overline{C_j} \subseteq \overline{Co(B)}$ such that:

$$\overline{C_j} \subseteq \bigcup_{k=1}^{r} \{p_k\} + B(\theta, \beta_0, \beta_0)$$

$$\begin{aligned} \therefore \overline{Co(B)} &\subseteq \bigcup_{k=1}^{r} \{p_k\} + B(\theta, \beta_0, \beta_0) + B(\theta, \beta_0, \beta_0) + B(\theta, \beta_0, \beta_0) \\ &\subseteq \bigcup_{k=1}^{r} \{p_k\} + B(\theta, \alpha, \alpha) \end{aligned}$$

Thus, we get that $K = \overline{Co(B)}$ is totally bounded and complete, i.e., compact w.r.t. τ_N. Again, $f(K) \subseteq f(C) \subseteq K$. By theorem 11, $\exists x_0 \in K$ such that $f(x_0) = x_0$. □

4. Darbo's Generalization of the Schauder-Type Fixed Point Theorem Using the Concept of the Measure of Non-Compactness

In this section, we first consider two types of fuzzy bounded subsets of a KM-type fuzzy metric space (i.e., M is a left-continuous function w.r.t. t, and $*$ is left-continuous at $(1,1)$). We renamed them as strongly and weakly and studied the relation between them. After that, the measure of the non-compactness of a strongly fuzzy bounded subset of the fuzzy metric space is defined. Using this concept, a family of ψ-set contraction mapping is specified, and Darbo's generalization of the Schauder-type fixed point theorem is established for these types of contraction mappings.

Definition 15. *(Strongly fuzzy boundedness) Let $(X, M, *)$ be a fuzzy metric space. A subset Q of X is said to be strongly fuzzy bounded if $\exists\, t > 0$ such that for each $\alpha \in (0,1)$:*

$$x \in B_M(y, \alpha, t) \;\forall\, x, y \in Q$$

i.e., fuzzy diameter of Q ($f-\delta(Q)$) less than ∞ where
$f-\delta(Q) = \bigvee_{\alpha\in(0,1)} \bigvee_{x,y\in Q} \wedge\{t > 0 : M(x,y,t) > 1-\alpha\}$ *(defined by Bag and Samanta in the paper [28].)*

An example is presented to understand the strongly fuzzy boundedness more clearly.

Example 2. *Let $X = \mathbb{R}^2$ (the set of all ordered pairs of the elements of the set of all real numbers) and $\|x\|' = |x_1| + |x_2|$, $\|x\|'' = (|x_1|^2 + |x_2|^2)^{1/2}$, where $x = (x_1, x_2)$ are two norms on X. Clearly, $\|x\|' \geq \|x\|''$, $\forall x \in X$. Define a function $M : X \times X \times \mathbb{R} \to [0,1]$ by:*

$$M(x,y,t) = \begin{cases} 1, \; t \geq \|x-y\|' \\ 1/2, \; \|x-y\|'' \leq t < \|x-y\|' \\ 0, \; t < \|x-y\|'' \end{cases}$$

Then, M is a fuzzy metric on X w.r.t. the min t-norm. Clearly,

$$\wedge\{t > 0 : M(x,y,t) > 1-\alpha\} = \begin{cases} \|x-y\|', \; 0 < \alpha < 1/2 \\ \|x-y\|'', \; 1/2 \leq \alpha < 1 \end{cases}$$

Consider $A = B(0,1)$. Now:

$$\begin{aligned} f-\delta(A) &= \bigvee_{\alpha\in(0,1)} \bigvee_{x,y\in A} \wedge\{t > 0 : M(x,y,t) > 1-\alpha\} \\ &= diam A \; w.r.t. \| \;\|' \\ &= 1 < \infty \end{aligned}$$

$\therefore$ A is strongly fuzzy bounded.

The fuzzy boundedness defined in Definition 4 is renamed as the weakly fuzzy bounded subset of a fuzzy metric space $(X, M, *)$. From the two definitions, it is clear that strongly fuzzy bounded implies the weakly fuzzy boundedness, but the converse may not be. This can be justified by the following example.

Example 3. *Consider the fuzzy metric:*

$$M(x,y,t) = \begin{cases} \frac{t}{t+|x-y|}, t > |x-y| \\ 0, t \leq |x-y| \end{cases}$$

Now, $\wedge\{t > 0 : M(x,y,t) > 1-\alpha\} = \frac{\alpha}{1-\alpha}|x-y|$. *Let* $A = [0,1]$. *Since* $|x-y| \le 1 < 2$, *so* A *is weakly fuzzy bounded. However,* $\bigvee_{\alpha\in(0,1)} \frac{\alpha}{1-\alpha} = \infty$. *Thus,* A *is not strongly fuzzy bounded.*

The weakly fuzzy bounded subset of a fuzzy metric space can also be defined as if $\beta(A) = 1$, then A is weakly fuzzy bounded where $\beta(A) = \underset{t>0}{Sup}\ \underset{x,y\in A}{inf}\ M(x,y,t)$. The equivalence between these two definitions was already proved in the paper [29].

Definition 16. *(Kuratowski's measure of non-compactness) Let* $(X, M, *)$ *be a fuzzy metric space and* Q *be a strongly fuzzy bounded subset of* X*. Then, Kuratowski's measure of the non-compactness of* Q *denoted by* $\psi(Q)$ *is defined as:*

$$\psi(Q) = inf\{\epsilon > 0;\ Q \subseteq \bigcup_{i=1}^{n} S_i,\ S_i \subseteq X,\ f-\delta(S_i) < \epsilon\ \forall i \in \{1,2,\cdots,n\}\}$$

From the definition, it is clear that $\psi(Q) < f-\delta(Q)$, for each strongly fuzzy bounded subset Q of X.

Definition 17. *(α-level Kuratowski measure of non-compactness) Let* $(X, M, *)$ *be a fuzzy metric space and* Q *be a weakly fuzzy bounded subset (or strongly fuzzy bounded subset) of* X*. Then, for each* $\alpha \in (0,1)$*, the* α*-level Kuratowski measure of the non-compactness of* Q *denoted by* $\psi_\alpha(Q)$ *is defined as:*

$$\psi_\alpha(Q) = inf\{\epsilon > 0;\ Q \subseteq \bigcup_{i=1}^{n} S_i,\ S_i \subseteq X,\ \alpha-\delta(S_i) < \epsilon\ \forall i \in \{1,2,\cdots,n\}\}$$

where $\alpha-\delta(S_i) = \bigvee_{x,y\in S_i} \wedge\{t > 0 : M(x,y,t) > 1-\alpha\}$*, defined by Bag and Samanta [28].*

From the definition of $\psi(Q)$ and $\psi_\alpha(Q)$, it is clear that if Q is a strongly fuzzy bounded subset, then $\psi_\alpha(Q) \le \psi(Q)\ \forall\alpha \in (0,1)$, i.e., $\bigvee_{\alpha\in(0,1)} \psi_\alpha(Q) \le \psi(Q)$.

Lemma 8. *Let* $Q,\ Q_1,\ Q_2$ *be strong fuzzy bounded subsets of a complete fuzzy metric space* $(X, M, *)$*. Then:*

(i) $\psi(Q) = 0 \iff \bar{Q}$ *is compact w.r.t.* τ_M.
(ii) $\psi(Q) = \psi(\bar{Q})$
(iii) $Q_1 \subseteq Q_2 \implies \psi(Q_1) \le \psi(Q_2)$
(iv) $\psi(Q_1 \cup Q_2) = max\{\psi(Q_1), \psi(Q_2)\}$

Again, if $(X, N, *)$ *is an fnls, then the followings properties also hold.*

(v) $\psi(Q_1 + Q_2) \le \psi(Q_1) + \psi(Q_2)$
(vi) $\psi(Q + x_0) = \psi(Q)$
(vii) $\psi(rQ_1) = |r|\psi(Q_1)$
(viii) $\psi(ConvQ) = \psi(Q)$

Proof. (i) First, we suppose that $\psi(Q) = 0$. Then, for each $\epsilon > 0\ \exists\ \{S_i\}_{i=1}^n$ with $f-\delta(S_i) < \epsilon$ such that $Q \subseteq \bigcup_{i=1}^{n} S_i$. Now, if Q is totally bounded, then $\bar{Q}$ is also, and we get the required result. Let $\alpha_0 \in (0,1)$ and $\epsilon_0 > 0$. Consider a fixed $x_i \in S_i$ for each $i = \{1,2,\cdots,n\}$. Then, it is clear that $S_i \subseteq B_M(x_i, \alpha_0, \epsilon_0)$.

$$\therefore Q \subseteq \bigcup_{i=1}^{n} S_i \subseteq \bigcup_{i=1}^{n} B_M(x_i, \alpha_0, \epsilon_0)$$

Thus, Q is totally bounded. Conversely, suppose that $\bar{Q}$ is compact w.r.t. τ_M. Then, Q is totally bounded. Let $\epsilon > 0$ be given. Then, for any $\alpha \in (0,1)$, and for, $\epsilon > 0$, $\exists\ \{x_1, x_2, \cdots, x_n\}$ such that

$Q \subseteq \bigcup_{i=1}^{n} B_M(x_i, \alpha, \epsilon/2)$. Consider:

$$S_i = \bigvee_{\alpha \in (0,1)} \{y \in X, \wedge\{t > 0 : M(x_i, y, t) > 1 - \alpha < \epsilon/2\}, \forall\, i = \{1, 2, \cdots, n\}.$$

Then, $Q \subseteq \bigcup_{i=1}^{n} S_i$, where $f - \delta(S_i) < \epsilon$ for each $i = \{1, 2, \cdots, n\}$. Since $\epsilon > 0$ is arbitrary, thus $\psi(Q) = 0$.

(ii) We first prove that $f - \delta(Q) = f - \delta(\bar{Q})$. Then, the required result follows immediately. Obviously, $f - \delta(Q) \leq f - \delta(\bar{Q})$. For the reverse part, let $x, y \in \bar{Q}$. Then, $\exists\, \{x_n\}$ and $\{y_n\}$ in Q such that:

$$\lim_{n\to\infty} M(x_n, x, t) = 1\ \forall t > 0$$
$$\lim_{n\to\infty} M(y_n, y, t) = 1\ \forall t > 0$$

Here:

$$\begin{aligned}
&\wedge\{t > 0 : M(x, y, t) > 1 - \alpha\} \\
&\leq \wedge\{t > 0 : M(x, x_n, t) > 1 - \beta\} + \wedge\{t > 0 : M(x_n, y_n, t) > 1 - \beta\} + \wedge\{t > 0 : M(y_n, y, t) > 1 - \beta\} \\
&(\beta \leq \alpha) \\
&\implies \wedge\{t > 0 : M(x, y, t) > 1 - \alpha\} \leq \liminf_{n\to\infty} \wedge\{t > 0 : M(x_n, y_n, t) > 1 - \beta\} \leq f - \delta(Q) \\
&\therefore f - \delta(\bar{Q}) \leq f - \delta(Q)
\end{aligned}$$

Thus, we arrive at the required conclusion.

(iii) For $Q_1 \subseteq Q_2$, the set $\{\epsilon > 0;\ Q_2 \subseteq \bigcup_{i=1}^{n} S_i,\ S_i \subseteq X,\ f - \delta(S_i) < \epsilon$ $\forall i \in \{1, 2, \cdots, n\}\} \subseteq \{\epsilon > 0;\ Q_1 \subseteq \bigcup_{i=1}^{n} S_i,\ S_i \subseteq X,\ f - \delta(S_i) < \epsilon$ $\forall i \in \{1, 2, \cdots, n\}\}. \therefore \psi(Q_1) \leq \psi(Q_2)$.

(iv) From (iii), $\psi(Q_1 \cup Q_2) \leq max\{\psi(Q_1), \psi(Q_2)\}$ follows. The reverse part is similar to a crisp set. For the references, please see [30].

For (v), (vi), (vii), and (viii), we first prove that in an fnls, the following properties hold.

(1) $f - \delta(Q_1 + Q_2) \leq f - \delta(Q_1) + f - \delta(Q_2)$.
(2) $f - \delta(Q + x_0) = f - \delta(Q)$
(3) $f - \delta(rQ) = |r| f - \delta(Q)$
(4) $f - \delta(ConvQ) = f - \delta(Q)$

Then rest of the proof of (v), (vi), (vii), and (viii) is similar to the classical version of this theorem.

(1) Now:

$$\begin{aligned}
f - \delta(Q_1 + Q_2) &= \bigvee_{\alpha \in (0,1)} \bigvee_{x, y \in Q_1 + Q_2} \wedge\{t > 0 : N(x - y, t) > 1 - \alpha\} \\
&= \bigvee_{\alpha \in (0,1)} \bigvee_{x_1, x_2 \in Q_1, y_1, y_2 \in Q_2} \wedge\{t > 0 : N(x_1 + y_1 - x_2 - y_2, t) > 1 - \alpha\} \\
&\leq \bigvee_{\alpha \in (0,1)} \bigvee_{x_1, x_2 \in Q_1} \wedge\{t > 0 : N(x_1 - x_2, t) > 1 - \alpha\} \\
&+ \bigvee_{\alpha \in (0,1)} \bigvee_{y_1, y_2 \in Q_2} \wedge\{t > 0 : N(y_1 - y_2, t) > 1 - \alpha\} \\
&= f - \delta(Q_1) + f - \delta(Q_2)
\end{aligned}$$

(2) Again:

$$
\begin{aligned}
f-\delta(Q) &= \bigvee_{\alpha\in(0,1)} \bigvee_{x,y\in Q} \wedge\{t>0 : N(x-y,t)>1-\alpha\}\\
&= \bigvee_{\alpha\in(0,1)} \bigvee_{x,y\in Q} \wedge\{t>0 : N(x+x_0-y-x_0,t)>1-\alpha\}\\
&= f-\delta(Q+x_0)
\end{aligned}
$$

(3)

$$
\begin{aligned}
f-\delta(rQ) &= \bigvee_{\alpha\in(0,1)} \bigvee_{x,y\in rQ} \wedge\{t>0 : N(x-y,t)>1-\alpha\}\\
&= \bigvee_{\alpha\in(0,1)} \bigvee_{x_1,y_1\in Q} \wedge\{t>0 : N(rx_1-ry_1,t)>1-\alpha\}\\
&= \bigvee_{\alpha\in(0,1)} \bigvee_{x_1,y_1\in Q} \wedge\{|r|t/|r|>0 : N(x_1-y_1,t/|r|)>1-\alpha\}\\
&= |r| \bigvee_{\alpha\in(0,1)} \bigvee_{x_1,y_1\in Q} \wedge\{t>0 : N(x_1-y_1,t)>1-\alpha\}\\
&= |r| f-\delta(Q)
\end{aligned}
$$

(4) $f-\delta(Q) \le f-\delta(ConvQ)$ is obvious as $Q \subseteq ConvQ$. We only show that for a fixed $\alpha_0 \in (0,1)$ and for a fixed $x_0(\neq \theta) \in X$,
$\bigvee_{y\in Conv(Q)} \wedge\{t>0 : N(x_0-y,t)>1-\alpha_0\} = \bigvee_{y\in Q} \wedge\{t>0 : N(x_0-y,t)>1-\alpha_0\}$.

Since α_0 and x_0 are arbitrary, thus
$f-\delta(ConvQ) \le f-\delta(Q)$. Consider $y \in Conv(Q)$. Thus, $y = \sum_{i=1}^{n} \lambda_i x_i,\ x_i \in Q,\ \sum_{i=1}^{n} \lambda_i = 1$.

$$
\begin{aligned}
\therefore\ & \bigvee_{y\in Conv(Q)} \wedge\{t>0 : N(x_0-y,t)>1-\alpha_0\}\\
&= \bigvee_{x_i\in Q} \wedge\{t>0 : N(\sum_{i=1}^{n} \lambda_i x_0 - \sum_{i=1}^{n} \lambda_i x_i, t)>1-\alpha\}\\
&= \bigvee_{x_i\in Q} \wedge\{t>0 : N(x_0-x_i, t/\sum_{i=1}^{n} \lambda_i)>1-\alpha\}\\
&= \bigvee_{x_i\in Q} \wedge\{t>0 : N(x_0-x_i,t)>1-\alpha\}\\
&\le \bigvee_{y\in Q} \wedge\{t>0 : N(x_0-y,t)>1-\alpha\}
\end{aligned}
$$

$\therefore$ we arrive at the required conclusion. □

Definition 18. *(Axiomatic approach) Let $(X, M, *)$ be a complete fuzzy metric space and $\mathcal{B}$ the family of strongly fuzzy bounded subsets of X. A map $\psi : \mathcal{B} \to [0,\infty)$ is called a measure of non-compactness if it satisfies the following properties:*

(1) $\psi(B) = 0 \iff$ *B is fuzzy totally bounded,* $\forall B \in \mathcal{B}$.
(2) $\psi(B) = \psi(\bar{B}), \forall B \in \mathcal{B}$.
(3) $\psi(B_1 \cup B_2) = max\{\psi(B_1), \psi(B_2)\}, \forall B_1,\ B_2 \in \mathcal{B}$

Using this axiomatic approach, we give some examples of the measure of the non-compactness in a fuzzy metric space.

Example 4. *Let $X = \mathbb{R}^2$ (the set of all ordered pairs of the elements of the set of all real numbers) and $\|x\|' = |x_1| + |x_2|, \|x\|'' = (|x_1|^2 + |x_2|^2)^{1/2}$, where $x = (x_1, x_2)$ are two norms on X. Clearly,*

$\|x\|' \geq \|x\|''$, $\forall x \in X$. *Define a function* $M : X \times X \times \mathbb{R} \to [0,1]$ *by:*

$$M(x,y,t) = \begin{cases} 1, \ t \geq \|x-y\|' \\ 1/2, \ \|x-y\|'' \leq t < \|x-y\|' \\ 0, \ t < \|x-y\|'' \end{cases}$$

Then, M is a fuzzy metric on X w.r.t. the min t-norm. Clearly,

$$\wedge\{t > 0 : M(x,y,t) > 1-\alpha\} = \begin{cases} \|x-y\|', \ 0 < \alpha < 1/2 \\ \|x-y\|'', \ 1/2 \leq \alpha < 1 \end{cases}$$

Define functions ψ_1 *and* ψ_2 *from the set of all strongly fuzzy bounded subsets of X to* $[0,\infty)$ *by:*

$$\psi_1(B) = \begin{cases} 0, \ if B \text{ is totally fuzzy bounded} \\ 1, otherwise \end{cases}$$

and $\psi_2(B) = f - \delta(B)$.

Both ψ_1 *and* ψ_2 *satisfy all the conditions of Definition 18. Therefore, both are the measure of the non-compactness of fuzzy metric space* (X, M, min).

Theorem 13. *Let* $(X, M, *)$ *be a complete fuzzy metric space. If* $\{F_n\}$ *is a decreasing sequence of non-empty closed, strongly fuzzy bounded subsets of X such that* $\lim_{n\to\infty} \psi(F_n) = 0$, *then the intersection* $F_\infty = \bigcap_{n=1}^{\infty} F_n$ *is a non-empty compact subset of X w.r.t.* τ_M.

Proof. Here, $\psi(F_\infty) \leq \lim_{n\to\infty} \psi(F_n) = 0$. Thus, by Lemma 8, F_∞ is compact w.r.t. τ_M, as F_∞ is closed. Now, we will show that F_∞ is non-empty. Since $\lim_{n\to\infty} \psi(F_n) = 0$, so $\lim_{n\to\infty} \psi_\alpha(F_n) = 0$, $\forall \alpha \in (0,1)$. Let $\{x_n\} \subseteq X$ and $x_n \in F_n$, i.e., $\{x_n\} \in F_1$; $\{x_n\}_{n=2}^{\infty} \in F_2$, and so on. Consider $\alpha_0 \in (0,1)$. Thus, $\lim_{n\to\infty} \psi_{\alpha_0}(F_n) = 0$.

By Definition 17, for every $n \in \mathbb{N}$, $F_n \subseteq \bigcup_{i=1}^{k_n} F_i^n$ such that $\alpha_0 - \delta(F_i^n) < \psi_{\alpha_0}(F_n) + 1/n$. Since $\{x_n\} \subseteq F_1$ $\exists \{x_n^1\} \subseteq F_i^1 \cap F_2 \subseteq F_1$ for some $i = \{1, 2, \cdots, k_n\}$, so $\alpha_0 - \delta(x_n^1) < \psi_{\alpha_0}(F_1) + 1$. Consider a subsequence $\{x_n^2\}$ of $\{x_n^1\}$ with $\{x_n^1\} \subseteq F_i^2 \cap F_2$, for some $i = \{1, 2, \cdots, k_n\}$. Thus, $\alpha_0 - \delta(x_n^2) < \psi_{\alpha_0}(F_2) + 1/2$.

Similarly, we get a subsequence of $\{x_n^j\}$ of $\{x_n^{j-1}\}$ with $\alpha_0 - \delta(x_n^j) < \psi_{\alpha_0}(F_j) + 1/j$, i.e., $\lim_{j\to\infty} \alpha_0 - \delta(x_n^j) = 0$. This is true for any $\alpha_0 \in (0,1)$.

Thus, $\lim_{j\to\infty} \alpha - \delta(x_n^j) = 0$, $\forall \alpha \in (0,1)$.

For any $\alpha \in (0,1)$ and $\epsilon > 0$ $\forall j_1, j_2 \in \mathbb{N}$, $\exists N_0(\alpha, \epsilon) \in \mathbb{N}$ such that:

$$\begin{aligned} &\wedge \{t > 0 : M(x_n^{j_1}, x_n^{j_2}, t) > 1-\alpha\} < \epsilon \ \forall j_1, j_2 > N_0 \\ &\implies M(x_n^{j_1}, x_n^{j_2}, \epsilon) > 1-\alpha \ \forall j_1, j_2 > N_0 \\ &\implies \lim_{j_1, j_2 \to\infty} M(x_n^{j_1}, x_n^{j_2}, t) = 1 \ \forall t > 0. \end{aligned}$$

$\therefore\ \{x_n^j\}_j$ is a Cauchy sequence w.r.t. $(X, M, *)$, i.e., converges to $x \in F_n\ \forall n \in \mathbb{N}$.
i.e. $x \in \bigcap_{n=1}^{\infty} F_n = F_\infty$. Thus, F_∞ is non-empty. $\square$

Definition 19. *Let $(X, M, *)$ be a complete fuzzy metric space and $f : X \to X$ be a fuzzy continuous mapping. Then, f is called a ψ-set contraction if there exists $k \in [0, 1)$ such that for all strongly fuzzy bounded subsets C of X, the following relation holds, $\psi(f(C)) \le k\psi(C)$, where ψ is the measure of the noncompactness of C.*

This definition is inspired by the α-set contraction in the classical set theory. For the references, please see the book [9].

Theorem 14. *(Darbo's generalization of the Schauder-type fixed point theorem) Let $(X, N, *)$ be a fuzzy Banach space and C be a closed, strongly fuzzy bounded, and convex subset of X. If $f : C \to C$ is a ψ-set contraction, then f has a fixed point in C.*

Proof. For each $n \in \mathbb{N}$, consider $C_n = \overline{Convf(C_{n-1})}$. Clearly, $C_{n+1} \subseteq C_n\ \forall n$. Now, $C_\infty = \bigcap_{n=1}^{\infty} C_n$, which is a closed and convex set, and $\psi(C) \le \lim_{n\to\infty} \psi(C_n)$. Again, $\lim_{n\to\infty} \psi(C_n) = 0$ ($\because\ \psi(f(C_n)) \le k^n\psi(C)$).

Furthermore, $f(C_\infty) = f(\bigcap_{n=1}^{\infty} C_n) \subseteq \bigcap_{n=1}^{\infty} f(C_n) \subseteq \bigcap_{n=1}^{\infty} C_{n+1} = C_\infty$. By Theorem 13, C_∞ is compact and non-empty. Thus, $f : C_\infty \to C_\infty$ is a continuous mapping from a compact, convex set to itself. Thus, by Theorem 11, $\exists\, x_0 \in C_\infty$ such that $f(x_0) = x_0$. This completes the proof. $\square$

Example 5. *Let $X = C[0, 1]$ (the set of all continuous functions over [0,1]) and $\|x\|' = \underset{0\le t\le 1}{Sup} |x(t)|$, $\|x\|'' = \int_0^1 |x(t)|dt$ be two norms on X. Clearly, $\|x\|' \ge \|x\|'', \forall x \in X$.*

Define a function $N : X \times \mathbb{R} \to [0, 1]$ by:

$$N(x,t) = \begin{cases} 1,\ t \ge \|x\|' \\ 1/2,\ \|x\|'' \le t < \|x\|' \\ 0,\ t < \|x\|'' \end{cases}$$

Then, (X, N, min) is fuzzy Banach space. Clearly,

$$\wedge\{t > 0 : N(x,t) > 1-\alpha\} = \begin{cases} \|x\|',\ 0 < \alpha < 1/2 \\ \|x\|'',\ 1/2 \le \alpha < 1 \end{cases}$$

Define a function $f : \overline{B(\theta, 1/2, 1)} \to \overline{B(\theta, 1/2, 1)}$ with $\|f(x) - f(y)\|' \le k\|x - y\|''$, $\forall x, y \in \overline{B(\theta, 1/2, 1)}$ and $\psi(C) = f - \delta(C) = \bigvee_{x,y\in C} \|x - y\|'$, where C is a strongly fuzzy bounded subset of $\overline{B(\theta, 1/2, 1)}$. Clearly, $\psi(f(C)) = \bigvee_{x,y\in C} \|f(x) - f(y)\|' \le k \bigvee_{x,y\in C} \|x - y\|'' \le k \bigvee_{x,y\in C} \|x - y\|' = k\psi(C)$.
$\therefore f$ is a ψ-set contraction mapping. By Theorem 14, f has a fixed point in C.

Remark 3. *In Example 5, $\overline{B(\theta, 1/2, 1)} = \{y \in X, \|x - y\|'' \le 1\}$. It is a closed, convex, bounded subset in $(X, \|\ \|'')$, where $\|x\|'' = \int_0^1 |x(t)|dt$. However, $(X, \|\ \|'')$ is not a Banach space. Therefore, the classical version of Darbo's generalization of the Schauder-type fixed point theorem will not be able to give the existence result of a fixed point of f, which is defined in Example 5. In this scene, our theorem is more general than its classical form.*

5. Conclusions

Schauder's fixed point theorem and its generalizations play a pivotal role in this context of nonlinear functional analysis. The aim of this paper is to study different types of Schauder's fixed point theorems in the context of fuzzy settings. For this reason, two types of fuzzy convergence are defined for a sequence of linear operators whose domain and range space are the fnlss. Moreover, the notion of two types of fuzzy bounded subsets of a fuzzy metric space is formulated, and the relation between them is studied. Further, the concept of Kuratowski's measure of non-compactness in a fuzzy metric and an fnls are introduced for both fuzzy bounded subsets. This concept is used as a tool to prove Darbo's generalization of the Schauder-type fixed point theorem. This is the first instance of studying the measure of the non-compactness in fuzzy settings. There is a huge scope of further research in this area, and many fixed point theorems can be developed by using these types of measures of non-compactness. Schauder's fixed point theorem has various applications in the theory of differential equations such as Peano's existence theorem for the first-order differential equations, the existence of the positive solution to the second-order singular differential equations, the existence of periodic orbits of rapidly symmetric systems, and so on. The theorems developed in this manuscript will promote future studies on the fuzzified area of the above-mentioned differential equations, as well as in the fuzzy neural networks.

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Reference

1. Chu, J.; Torres, P.J. Applications of Schauder's fixed point theorem to singular differential equations. *Bull. Lond. Math. Soc.* **2007**, *39*, 653–660. [CrossRef]
2. Dizaji, A.F.; Sepiani, H.A.; Ebrahimi, F.; Allahverdizadeh, A.; Sepiani, H.A. Schauder fixed point theorem based existence of periodic solution for the response of Duffing's oscillator. *J. Mech. Sci. Technol.* **2009**, *23*, 2299–2307. [CrossRef]
3. Li, S.; Tang, X.; Luo, H.; Applications of Schauder's fixed point theorem to singular radially symmetric systems. *J. Fixed Point Theory Appl.* **2019**, *21*, 46. [CrossRef]
4. Zhu, H.; Feng, C. Existence and global uniform asymptotic stability of pseudo almost periodic solutions for Cohen-Grossberg neural networks with discrete and distributed delays. *Math. Probl. Eng.* **2014**. [CrossRef]
5. Liang, X.; Wang, L.; Wang, R. Random Attractor of Reaction-Diffusion Hopfield Neural Networks Driven by Wiener Processes. *Math. Probl. Eng.* **2018**. [CrossRef]
6. Li, Y.; Qin, J.; Li, B. Periodic solutions for quaternion-valued fuzzy cellular neural networks with time-varying delays, *Adv. Differ. Equations* **2019**, *2019*, 63.
7. Tychonoff, A.N. Ein Fixpunktsatz. *Math. Ann.* **1935**, *111*, 767–776. [CrossRef]
8. Hukuhara, M. Sur i'existence des points invariants d'une transformation dans I'espace fonctionnel. *Jpn. J. Math.* **1950**, *20*, 1–4. [CrossRef]
9. Istratescu, V.I. *Fixed Point Theory: An Introduction*; Springer: Berlin/Heidelberg, Germany, 1981.
10. Fan, Ky. A Generalization of Tychonoff's Fixed Point Theorem. *Math. Ann.* **1961**, *142*, 305–310. [CrossRef]
11. Pouso, R.L. Schauder's fixed-point theorem: New applications and a new version for discontinuous operators. *Bound. Value Probl.* **2012**, *2012*, 92.
12. Agarwal, R.P.; Arshad, S.; O'Regan, D.; Lupulescu, V. A Schauder fixed point theorem in semilinear spaces and applications. *Fixed Point Theory Appl.* **2013**, *2013*, 306. [CrossRef]

13. Du, W.-S. A Short Note on a Simple Proof of Schauder's Fixed Point Theorem and its Generalization without Continuity and Compactness Assumptions. *Int. J. Math. Anal.* **2016**, *10*, 933–938. [CrossRef]
14. Xiao, J.-Z.; Zhu, X.-H. Fixed point theorems in generating spaces of quasi-norm family and applications. *Fixed Point Theory Appl.* **2006**, *1*, 61623. [CrossRef]
15. Zhang, S.S; Guo, J.L. Probability integrals, Gateaux differentials and the Schauder principle in probabilistic normed linear spaces. *J. Sichuan Univ.* **1989**, *26*, 127–135.
16. Bag, T.; Samanta, S.K. Some fixed point theorems in fuzzy normed linear spaces. *Inf. Sci.* **2007**, *177*, 3271–3289. [CrossRef]
17. Bag, T.; Samanta, S.K. Finite dimensional fuzzy normed linear spaces. *Ann. Fuzzy Math. Inform.* **2013**, *6*, 271–283.
18. Chatterjee, S.; Bag, T.; Samanta, S.K. Uniform boundedness principle in generalized fuzzy normed linear spaces. *Afrika Matematika* **2020**. [CrossRef]
19. Wu, H.-C. Hausdorff Topology Induced by the Fuzzy Metric and the Fixed Point Theorems in Fuzzy Metric Spaces, *J. Korean Math. Soc.* **2015**, *52*, 1287–1303. [CrossRef]
20. Gregori, V.; Romaguera, S. Some properties of fuzzy metric spaces. *Fuzzy Sets Syst.* **2000**, *115*, 485–489. [CrossRef]
21. Bag, T.; Samanta, S.K. Finite dimensional fuzzy normed linear spaces. *J. Fuzzy Math.* **2003**, *11*, 687–705.
22. Chatterjee, S.; Bag, T.; Samanta, S.K. Fuzzy compact linear operator. *Adv. Fuzzy Math.* **2017**, *12*, 215–228.
23. Adhya, S.; Ray, A. D. On Lebesgue Property for Fuzzy Metric Spaces. *arXiv* **2018**, arXiv:1812.03093.
24. Bînzar, T.; Pater, F.; Nuaduaban, S. A study of boundedness in fuzzy normed linear spaces. *Symmetry* **2019**, *11*, 923. [CrossRef]
25. Sadeqi, I. Some fixed point theorems in fuzzy reflexive Banach spaces. *Chaos Solitons Fractals* **2009**, *41*, 2606–2612. [CrossRef]
26. Sadeqi, I.; Kia, F.S. Fuzzy normed linear space and its topological structure. *Chaos Solitons Fractals* **2009**, *40*, 2576–2589. [CrossRef]
27. Bag, T.; Samanta, S.K. Fuzzy bounded linear operators. *Fuzzy Sets Syst.* **2005**, *151*, 513–547. [CrossRef]
28. Bag, T.; Samanta, S.K. Fixed point theorems on fuzzy normed linear spaces. *Inf. Sci.* **2006**, *176*, 2910–2931. [CrossRef]
29. Xiao, J.-Z.; Zhu, X.-H.; Zhou, H. On the topological structure of KM fuzzy metric spaces and normed spaces. *IEEE Trans. Fuzzy Syst.* **2019**, *28*, 1575–1584. [CrossRef]
30. Malkowsky, E.; Rakocević, V. An introduction into the theory of sequence spaces and measures of noncompactness. *Zb. Rad.* **2000**, *17*, 143–234.