

Article

# Feynman Integral and a Change of Scale Formula about the First Variation and a Fourier–Stieltjes Transform

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**Abstract:** We prove that the Wiener integral, the analytic Wiener integral and the analytic Feynman integral of the first variation of  $F(x) = \exp \left\{ \int_0^T \theta(t, x(t)) dt \right\}$  successfully exist under the certain condition, where  $\theta(t, u) = \int_{\mathbf{R}} \exp\{iuv\} d\sigma_t(v)$  is a Fourier–Stieltjes transform of a complex Borel measure  $\sigma_t \in \mathbf{M}(\mathbf{R})$  and  $\mathbf{M}(\mathbf{R})$  is a set of complex Borel measures defined on  $\mathbf{R}$ . We will find this condition. Moreover, we prove that the change of scale formula for Wiener integrals about the first variation of  $F(x)$  successfully holds on the Wiener space.

**Keywords:** Wiener space; Wiener integral; Feynman integral; Fourier–Stieltjes transform; first variation

## 1. Introduction

### 1.1. Motivation of the Wiener Integral from the Heat Equation

The solution of the heat (or diffusion) equation:

$$-\frac{\partial u}{\partial t} = -\frac{1}{2}\Delta u + V(\xi)u = Hu \quad (\xi \in \mathbf{R}^d, 0 \leq t), u(0, \cdot) = \psi(\cdot)$$

is of the form:

$$u(t, \xi) = (e^{-tH} \psi)(\xi) = E \left[ \exp \left\{ - \left( \int_0^t V(x(s) + \xi) ds \right) \right\} \psi(x(t) + \xi) \right], \quad (1)$$

where  $\psi \in L_2(\mathbf{R}^d)$ ,  $\xi \in \mathbf{R}^d$  and  $x(\cdot)$  is a  $\mathbf{R}^d$ -valued continuous function defined on  $[0, t]$  such that  $x(0) = 0$ ;  $E$  denotes the expectation with respect to the Wiener path starting at time  $t = 0$  ( $E$  is the Wiener integral);  $H = -\Delta + V$  is the energy operator (or Hamiltonian),  $\Delta$  is a Laplacian, and  $V : \mathbf{R}^d \rightarrow \mathbf{R}$  is a potential. (1) is called the Feynman–Kac formula. Applications of the Feynman–Kac formula (in various settings) have been given in the following areas: diffusion equations, the spectral theory of the Schrödinger operator, quantum mechanics, and statistical physics (see [1]).

### 1.2. Motivation of This Paper

Do the analytic Wiener integral and the analytic Feynman integral about the first variation of the function  $F(x) = \exp \left\{ \int_0^T \theta(t, x(t)) dt \right\}$  successfully exist?

Is the change of scale formula for the Wiener integral successfully satisfied about the first variation of  $F(x) = \exp \left\{ \int_0^T \theta(t, x(t)) dt \right\}$ ?

### 1.3. Research Flow About the Topic of a Change of Scale Formula from 1944

The concept of a transform of Wiener integrals was introduced in [2,3] (1944–1945). The behavior of a measure and a measurability under a change of scale on the Wiener space was expanded in [4] (1947). The first variation of a Wiener integral was defined in [5] (1951) and the translation pathology of Wiener integrals was proved in [6] (1954). The functional transform for Feynman integrals similar to Fourier transform was introduced in [7] (1972). Relationships among the Wiener integral and the analytic Feynman Integral was proved in [8] (1987). A change of scale formula for the Wiener integral was proved in [9]. The scale-Invariant Measurability on the Wiener space was proved in [10] (1979). Theorems about some Banach algebras of analytic Feynman integrable functionals was expanded in [11] (1980).

A change of scale formula for the Wiener integral of the cylinder function on the abstract Wiener space was proved in [12,13]. The relationship between the Wiener integral and the Fourier Feynman transform about the first variation was proved in [14–17]. The behavior of the scale factor for the Wiener integral of the unbounded function on the Wiener space was proved in [18].

### 1.4. Target of This Paper

The purpose of this paper is the following:

- (1) The first variation of a function  $F(x) = \exp \left\{ \int_0^T \theta(t, x(t)) dt \right\}$  exists under the certain condition. We find this condition.
- (2) The analytic Wiener integral and the analytic Feynman integral of the first variation of  $F(x)$  exist.
- (3) A change of scale formula for the Wiener integral holds for the first variation of  $F(x)$ .

## 2. Definitions and Preliminaries

Let  $C_0[0, T]$  denote the space of real-valued continuous functions  $x$  on  $[0, T]$  such that  $x(0) = 0$ . Let  $\mathcal{M}$  denote the class of all Wiener measurable subsets of  $C_0[0, T]$  and let  $m$  denote the Wiener measure and  $(C_0[0, T], \mathcal{M}, m)$  be a Wiener measure space, see [1].

The integral of a measurable function  $F$  defined on the Wiener measure space with respect to a Wiener measure  $m \in \mathcal{M}$  is called a Wiener integral of  $F$  and we denote it by  $\int_{C_0[0, T]} F(x) dm(x)$ .

A subset  $E$  of  $C_0[0, T]$  is said to be scale-invariant measurable if  $\rho E \in \mathcal{M}$  for each  $\rho > 0$ , and a scale-invariant measurable set  $N$  is said to be scale-invariant null if  $m(\rho N) = 0$  for each  $\rho > 0$ . A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (*s-a.e.*). If two functions  $F$  and  $G$  are equal *s-a.e.*, we write  $F \approx G$ . For more details, see [1].

Throughout this paper, let  $\mathbf{R}^n$  denote the  $n$ -dimensional Euclidean space and let  $\mathbf{C}, \mathbf{C}_+$ , and  $\mathbf{C}_+^\sim$  denote the complex numbers, the complex numbers with positive real part, and the non-zero complex numbers with nonnegative real part, respectively.

**Definition 1** (see [1]). Let  $F$  be a complex-valued measurable function on  $C_0[0, T]$  such that the integral

$$J(F; \lambda) = \int_{C_0[0, T]} F(\lambda^{-\frac{1}{2}} x) dm(x) \tag{2}$$

exists for all real  $\lambda > 0$ . If there exists a function  $J^*(F; z)$  analytic on  $\mathbf{C}_+$  such that  $J^*(F; \lambda) = J(F; \lambda)$  for all real  $\lambda > 0$ , then we define  $J^*(F; z)$  to be the analytic Wiener integral of  $F$  over  $C_0[0, T]$  with parameter  $z$ , and for each  $z \in \mathbf{C}_+$ , we write

$$\int_{C_0[0, T]}^{anw_z} F(x) dm(x) = J^*(F; z). \tag{3}$$

Let  $q$  be a non-zero real number and let  $F$  be a function on  $C_0[0, T]$  whose analytic Wiener integral exists for each  $z$  in  $\mathbf{C}_+$ . If the following limit exists, then we call it the analytic Feynman integral of  $F$  over  $C_0[0, T]$  with a parameter  $q$  and we write

$$\int_{C_0[0,T]}^{anf_q} F(x)dm(x) = \lim_{z \rightarrow -iq} \int_{C_0[0,T]}^{anw_z} F(x), \tag{4}$$

where  $z$  approaches  $-iq$  through  $\mathbf{C}_+$  and  $i^2 = -1$ .

Now we introduce the Wiener Integration Formula.

Let  $C_0^{a,b}$  denote the space of real valued continuous functions  $x$  on  $[a, b]$  such that  $x(a) = 0$ . Let  $a = t_0 < t_1 < t_2 < \dots < t_n \leq b$ .

**Theorem 1** ([1], Theorem 3.3.5). *Let  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  be Lebesgue measurable. Then we have*

$$\begin{aligned} & \int_{C_0^{a,b}} f(x(t_1), x(t_2), \dots, x(t_n))dm(x) \\ &= \left[ \prod_{j=1}^n 2\pi(t_j - t_{j-1}) \right]^{-\frac{1}{2}} \int_{\mathbf{R}^n} f(\vec{u}) \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}} \right\} d\vec{u}, \end{aligned} \tag{5}$$

where  $u_0 = 0$  and where the equality is in the strong sense that if either side of (5) is defined (in the sense of the Lebesgue theory), whether finite or infinite, then so is the other side and they agree.

**Remark 1.** *If we let  $a = 0$  and  $b = T$  in (5), we have the following Wiener integration formula. Let  $C_0[0, T]$  be a Wiener space and let  $0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq T$ . Then*

$$\begin{aligned} & \int_{C_0[0,T]} f(x(t_1), x(t_2), \dots, x(t_n))dm(x) \\ &= \left[ \prod_{j=1}^n 2\pi(t_j - t_{j-1}) \right]^{-\frac{1}{2}} \int_{\mathbf{R}^n} f(\vec{u}) \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}} \right\} d\vec{u}, \end{aligned} \tag{6}$$

where  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is a Lebesgue measurable function and  $\vec{u} = (u_1, u_2, \dots, u_n)$  and  $u_0 = 0$  and  $d\vec{u} = du_1 du_2 \dots du_n$ .

**Definition 2** ([5]). *Let  $F$  be a measurable function on  $C_0[0, T]$ . Then for  $w \in C_0[0, T]$ ,*

$$\delta F(x|w) = \frac{\partial}{\partial h} F(x + hw)|_{h=0} \tag{7}$$

is called the first variation of  $F$  in the direction  $w \in C_0[0, T]$  (if it exists).

In the next section, we will use the following well-known integration formula:

$$\int_{\mathbf{R}} \exp\{-au^2 + ibu\} du = \sqrt{\frac{\pi}{a}} \exp\left\{-\frac{b^2}{4a}\right\}, \tag{8}$$

where  $a$  is a complex number with  $Re a > 0$  and  $b$  is a real number and  $i^2 = -1$ .

**Theorem 2** ([19], Morera’s Theorem). *Suppose  $f$  is a complex function in an open set  $\Omega$  such that*

$$\int_{\partial\Delta} f(z)dz = 0, \tag{9}$$

for every closed triangle  $\Delta \subset \Omega$ . Then  $f \in H(\Omega)$ , where  $H(\Omega)$  is the class of all analytic functions in an open set  $\Omega$  and  $\partial\Delta$  is a boundary of  $\Delta$ .

### 3. The Main Results

Define a function

$$F(x) = \exp\left\{\int_0^T \theta(t, x(t)) dt\right\}, \tag{10}$$

which has been used in the Quantum Mechanics, (see [1]).

**Definition 3** ([1,18]). Let  $\theta : [0, T] \times R \rightarrow \mathbf{C}$  be defined by

$$\theta(t, u) = \int_R \exp\{iuv\}d\sigma_t(v), \tag{11}$$

which is a Fourier–Stieltjes transform of a complex Borel measure  $\sigma_t \in \mathbf{M}(R)$ , where  $\mathbf{M}(R)$  is a set of complex Borel measures defined on  $R$ .

#### Notation 1.

(1) Let  $\Delta_n$  be defined by

$$\Delta_n(T) \equiv \{(t_1, t_2, \dots, t_n) | 0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq T\}, t_0 = 0 (1 \leq n).$$

(2) Let  $M_n'' = M_n''(\Delta_n(T) \times R^n)$  be the set of a countably additive complex Borel measure defined on the product space  $\Delta_n(T) \times R^n$  ( $1 \leq n$ ), see [1].

Throughout this paper, we suppose that  $\vec{v} = (v_1, v_2, \dots, v_n)$  and  $w \in C_0[0, T]$  with  $|w| < \infty$  and  $\sum_{n=1}^{\infty} \|\mu_n\| < \infty$ , where  $\|\cdot\|$  is a total variation of a countably additive complex Borel measure, and we suppose that  $\sum_{j=1}^n |v_j| < \infty$  and

$$\sum_{n=1}^{\infty} \int_{\Delta_n(T) \times R^n} \left[ \sum_{j=1}^n |v_j| \right] d|\mu_n|(\vec{t}, \vec{v}) < \infty. \tag{12}$$

**Lemma 1** ([20]). Let  $F : C_0[0, T] \rightarrow \mathbf{C}$  be defined by (10) and (11). Then we have that

$$F(x) = \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times R^n} \exp\left\{i \sum_{j=1}^n v_j x(t_j)\right\} d\mu_n(\vec{t}, \vec{v}), \tag{13}$$

where  $\mu_n$  is a countably additive Borel measure defined on  $\Delta_n(T) \times R^n$  for each  $n = 1, 2, \dots$ , and  $d\mu_n(\vec{t}, \vec{v}) = \left[ \prod_{j=1}^n d\sigma_{t_j}(v_j) dt_j \right]$ . (For more details, see [20]).

First we calculate the first variation of  $F(x)$  and we find a condition of the existence of the first variation of  $F(x)$ .

**Lemma 2.** The first variation of  $F(x)$  in (10) exists under the condition (12) and is of the form :

$$\delta F(x|w) = \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times R^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp\left\{i \sum_{j=1}^n v_j x(t_j)\right\} d\mu_n(\vec{t}, \vec{v}). \tag{14}$$

**Proof.** By the definition of the first variation in [5] and Dominated Convergence Theorem, we have that

$$\begin{aligned}
 \delta F(x|w) &\equiv \frac{\partial}{\partial h} F(x + hw)|_{h=0} \\
 &= \lim_{h \rightarrow 0} \left[ \lim_{\Delta h \rightarrow 0} \frac{1}{\Delta h} \left[ F(x + (h + \Delta h)w) - F(x + hw) \right] \right] \\
 &= \lim_{h \rightarrow 0} \left[ \lim_{\Delta h \rightarrow 0} \frac{1}{\Delta h} \left[ \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times R^n} \exp \left\{ i \sum_{j=1}^n v_j (x(t_j) + (h + \Delta h) w(t_j)) \right\} \right. \right. \\
 &\quad \left. \left. - \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times R^n} \exp \left\{ i \sum_{j=1}^n v_j (x(t_j) + h w(t_j)) \right\} d\mu_n(\vec{t}, \vec{v}) \right] \right] \\
 &= \lim_{h \rightarrow 0} \left[ \lim_{\Delta h \rightarrow 0} \frac{1}{\Delta h} \left[ \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times R^n} \exp \left\{ i \sum_{j=1}^n v_j (x(t_j) + h w(t_j)) \right\} \right. \right. \\
 &\quad \left. \left. \times \left[ \exp \left\{ i \Delta h \sum_{j=1}^n v_j w(t_j) \right\} - 1 \right] d\mu_n(\vec{t}, \vec{v}) \right] \right] \\
 &= \lim_{h \rightarrow 0} \left[ \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times R^n} \exp \left\{ i \sum_{j=1}^n v_j (x(t_j) + h w(t_j)) \right\} \right. \\
 &\quad \left. \times \left[ \lim_{\Delta h \rightarrow 0} \frac{\exp \left\{ i \Delta h \sum_{j=1}^n v_j w(t_j) \right\} - 1}{\Delta h} \right] d\mu_n(\vec{t}, \vec{v}) \right] \\
 &= \lim_{h \rightarrow 0} \left[ \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times R^n} \exp \left\{ i \sum_{j=1}^n v_j (x(t_j) + h w(t_j)) \right\} \right. \\
 &\quad \left. \times \left[ \lim_{\Delta h \rightarrow 0} \frac{\left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ i \Delta h \sum_{j=1}^n v_j w(t_j) \right\}}{1} \right] d\mu_n(\vec{t}, \vec{v}) \right] \\
 &= \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times R^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ i \sum_{j=1}^n v_j x(t_j) \right\} d\mu_n(\vec{t}, \vec{v}), \tag{15}
 \end{aligned}$$

where we use L'Hospital's Rule in the fifth equality. For  $w \in C_0[0, T]$  with  $|w| < \infty$ ,

$$\begin{aligned}
 |\delta F(x|w)| &\leq \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times R^n} \left| \sum_{j=1}^n v_j w(t_j) \right| d|\mu_n|(\vec{t}, \vec{v}) \\
 &\leq \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times R^n} |w| \sum_{j=1}^n |v_j| d|\mu_n|(\vec{t}, \vec{v}) \\
 &= |w| \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times R^n} \left[ \sum_{j=1}^n |v_j| \right] d|\mu_n|(\vec{t}, \vec{v}). \tag{16}
 \end{aligned}$$

Therefore, the first variation  $\delta F(x|w)$  of  $F(x)$  exists under the condition (12).  $\square$

Now, we prove the existence of the analytic Wiener integral of the first variation of  $F(x)$  in (10).

**Theorem 3.** For  $z \in \mathbf{C}^+$ , the analytic Wiener integral of the first variation of  $F(x)$  in (10) exists and is given by

$$\begin{aligned}
 &\int_{C_0[0,T]}^{anw_z} \delta F(x|w) dm(x) \\
 &= \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times R^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n v_k^2 \right\} d\mu_n(\vec{t}, \vec{v}). \tag{17}
 \end{aligned}$$

**Proof.** By Fubini’s Theorem, Wiener integration formula (6) and Lemma 2, we have that for real  $\lambda > 0$  and for  $w \in C_0[0, T]$ ,

$$\begin{aligned}
 g(\lambda) &= \int_{C_0[0,T]} \delta F(\lambda^{-\frac{1}{2}} x|w) dm(x) \\
 &= \int_{C_0[0,T]} \left[ \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ i \lambda^{-\frac{1}{2}} \sum_{j=1}^n v_j x(t_j) \right\} d\mu_n(\vec{t}, \vec{v}) \right] dm(x) \\
 &= \prod_{j=1}^n \left( \frac{\lambda}{2\pi(t_j - t_{j-1})} \right)^{\frac{1}{2}} \int_{\mathbf{R}^n} \left[ \sum_{n=1}^{\infty} \int_{\Delta(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \right. \\
 &\quad \left. \times \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n \frac{[u_j - u_{j-1}]^2}{t_j - t_{j-1}} + i \sum_{j=1}^n u_j v_j \right\} d\mu_n(\vec{t}, \vec{v}) \right] d\vec{u} \\
 &= \sum_{n=1}^{\infty} \int_{\Delta(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \\
 &\quad \times \left[ \int_{\mathbf{R}^n} \left[ \prod_{j=1}^n \left( \frac{\lambda}{2\pi(t_j - t_{j-1})} \right)^{\frac{1}{2}} \right] \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n \frac{[u_j - u_{j-1}]^2}{t_j - t_{j-1}} + i \sum_{j=1}^n u_j v_j \right\} d\vec{u} \right] d\mu_n(\vec{t}, \vec{v}) \\
 &= \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ -\frac{1}{2\lambda} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n v_k^2 \right\} d\mu_n(\vec{t}, \vec{v}), \tag{18}
 \end{aligned}$$

where for  $\lambda \in \mathbf{C}^+$ ,

$$\begin{aligned}
 &\int_{C_0[0,T]} \exp \left\{ i \lambda^{-\frac{1}{2}} \sum_{j=1}^n v_j x(t_j) \right\} dm(x) \\
 &= \prod_{j=1}^n \left( \frac{\lambda}{2\pi(t_j - t_{j-1})} \right)^{\frac{1}{2}} \int_{\mathbf{R}^n} \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n \frac{[u_j - u_{j-1}]^2}{t_j - t_{j-1}} + i \sum_{j=1}^n u_j v_j \right\} d\vec{u} \\
 &= \exp \left\{ -\frac{1}{2\lambda} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n v_k^2 \right\}. \tag{19}
 \end{aligned}$$

We use the following function in the second equality of (18), when we applied the Wiener integration formula (6):

$$f(x(t_1), x(t_2), \dots, x(t_n)) = \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ i \lambda^{-\frac{1}{2}} \sum_{j=1}^n v_j x(t_j) \right\} d\mu_n(\vec{t}, \vec{v}).$$

Now we prove the existence of the analytic Wiener integral of the first variation  $F(x)$  in (10). For  $z \in \mathbf{C}^+$ , let

$$g^*(z) = \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n v_k^2 \right\} d\mu_n(\vec{t}, \vec{v}). \tag{20}$$

Then  $g^*(\lambda) = g(\lambda)$  for all real  $\lambda > 0$ . For  $z \in \mathbf{C}^+$ ,

$$\begin{aligned}
 |g^*(z)| &\leq \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ |w| \sum_{j=1}^n |v_j| \right] \left| \exp \left\{ -\frac{\bar{z}}{2z\bar{z}} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n v_k^2 \right\} \right| d|\mu_n|(\vec{t}, \vec{v}) \\
 &\leq \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ |w| \sum_{j=1}^n |v_j| \right] d|\mu_n|(\vec{t}, \vec{v}) \\
 &= |w| \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ \sum_{j=1}^n |v_j| \right] d|\mu_n|(\vec{t}, \vec{v}) < \infty, \tag{21}
 \end{aligned}$$

because  $Re(z) = Re(\bar{z}) > 0$ . By Dominated Convergence Theorem,  $g^*(z)$  is a continuous function of  $z \in \mathbf{C}^+$ .

Since the function :  $\exp \left\{ -\frac{1}{2z} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n v_k^2 \right\}$  is an analytic function of  $z \in \mathbf{C}^+$  for each  $\vec{v} = (v_1, v_2, \dots, v_n) \in \mathbf{R}^n$ , we have that  $\int_{\Gamma} \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n v_k^2 \right\} dz = 0$  for all rectifiable simple closed curves  $\Gamma$  in  $\mathbf{C}^+$  by Cauchy Integral Theorem.

As for all  $z \in \mathbf{C}^+$ ,

$$\left| \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n v_k^2 \right\} \right| \leq |w| \sum_{j=1}^n |v_j| < \infty, \tag{22}$$

we can apply Fubini’s Theorem to the integral  $\int_{\Gamma} g^*(z) dz$  and we have  $\int_{\Gamma} g^*(z) dz = 0$ .

By Morera’s Theorem in [19],  $g^*(z)$  is an analytic function of  $z$  in  $\mathbf{C}^+$ . Therefore the analytic Wiener integral of the first variation of  $F(x)$  successfully exists and is given by (17) for  $z \in \mathbf{C}^+$  and for  $w \in C_0[0, T]$ .  $\square$

**Remark 2.** In (19), we use the formula : For real  $\lambda > 0$  and for each  $j = 1, 2, \dots, n$ ,

$$\begin{aligned} & \left( \frac{\lambda}{2\pi(t_j - t_{j-1})} \right)^{\frac{1}{2}} \int_{\mathbf{R}} \exp \left\{ -\frac{\lambda[u_j - u_{j-1}]^2}{2(t_j - t_{j-1})} + iu_j w \right\} du_j \\ &= \exp \left\{ iu_{j-1} w - \frac{(t_j - t_{j-1})w^2}{2\lambda} \right\}, \end{aligned} \tag{23}$$

which is proved by the transformation  $z = \left[ \frac{\lambda}{(t_j - t_{j-1})} \right]^{\frac{1}{2}} (u_j - u_{j-1})$  and by (8).

Now, we obtain the analytic Feynman integral of the first variation  $\delta F(x|w)$  of functions  $F : C_0[0, T] \rightarrow \mathbf{C}$  in (10) and prove the existence of the analytic Feynman integral.

**Theorem 4.** The analytic Feynman integral of the first variation of  $F(x)$  in (10) exists and

$$\begin{aligned} & \int_{C_0[0, T]}^{anf_q} \delta F(x|w) dm(x) \\ &= \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ -\frac{i}{2q} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n v_k^2 \right\} d\mu_n(\vec{t}, \vec{v}). \end{aligned} \tag{24}$$

**Proof.** For  $m = 1, 2, 3 \dots$  and for  $z_m \in \mathbf{C}^+$ , let

$$f_m(\vec{v}) = \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ -\frac{1}{2z_m} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n u_k^2 \right\} \tag{25}$$

and let  $z_m \rightarrow -iq$  whenever  $m \rightarrow \infty$ . Then  $f_m(\vec{v}) \rightarrow f(\vec{v})$ , where

$$f(\vec{v}) = \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ -\frac{i}{2q} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n u_k^2 \right\} \tag{26}$$

and  $|f_m(\vec{v})| \leq |w| \sum_{j=1}^n |v_j| < \infty$  for all  $m = 1, 2, \dots$ .

By Dominated Convergence Theorem and Theorem 3, the analytic Feynman integral of the first variation of  $F(x)$  in (10) exists and

$$\begin{aligned}
 & \int_{C_0[0,T]}^{anf_q} \delta F(x|w) dm(x) \\
 = & \lim_{z_m \rightarrow -iq} \int_{C_0[0,T]}^{anw_{z_m}} \delta F(x|w) dm(x) \\
 = & \lim_{z_m \rightarrow -iq} \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ -\frac{1}{2z_m} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n u_k^2 \right\} d\mu_n(\vec{t}, \vec{v}) \\
 = & \lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} f_m(\vec{v}) d\mu_n(\vec{t}, \vec{v}) \\
 = & \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} f(\vec{v}) d\mu_n(\vec{t}, \vec{v}) \\
 = & \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ -\frac{i}{2q} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n u_k^2 \right\} d\mu_n(\vec{t}, \vec{v}), \tag{27}
 \end{aligned}$$

□

To prove harmonious relationships among the Wiener integral, the analytic Wiener integral and the analytic Feynman integral of the first variation and to prove the change of scale formula for the Wiener integral of the first variation of  $F(x)$  in (10), we first have to prove the following property:

**Theorem 5.** For  $z \in \mathbf{C}^+$ , the function

$$\exp \left\{ \frac{1-z}{2} \sum_{j=1}^n \frac{[x(t_j) - x(t_{j-1})]^2}{t_j - t_{j-1}} \right\} \delta F(x|w) \tag{28}$$

is a Wiener integrable function of  $x \in C_0[0, T]$ .



**Proof.** By Fubini Theorem, Wiener integration formula (6), (12), (14) and (19), we have that for  $z \in \mathbf{C}^+$ ,

$$\begin{aligned}
 & \left| \int_{C_0[0,T]} \exp \left\{ \frac{1-z}{2} \sum_{j=1}^n \frac{[x(t_j) - x(t_{j-1})]^2}{t_j - t_{j-1}} \right\} \delta F(x|w) dm(x) \right| \\
 = & \left| \int_{C_0[0,T]} \exp \left\{ \frac{1-z}{2} \sum_{j=1}^n \frac{[x(t_j) - x(t_{j-1})]^2}{t_j - t_{j-1}} \right\} \right. \\
 & \quad \times \left[ \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ i \sum_{j=1}^n v_j x(t_j) \right\} d\mu_n(\vec{t}, \vec{v}) \right] dm(x) \Big| \\
 = & \left| \prod_{j=1}^n [2\pi(t_j - t_{j-1})]^{-\frac{1}{2}} \int_{\mathbf{R}^n} \exp \left\{ \frac{1-z}{2} \sum_{j=1}^n \frac{[u_j - u_{j-1}]^2}{t_j - t_{j-1}} \right\} \right. \\
 & \quad \times \left[ \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ i \sum_{j=1}^n u_j v_j \right\} d\mu_n(\vec{t}, \vec{v}) \right] \\
 & \quad \times \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \frac{[u_j - u_{j-1}]^2}{t_j - t_{j-1}} \right\} d\vec{u} \Big| \\
 = & \left| \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \left[ \prod_{j=1}^n [2\pi(t_j - t_{j-1})]^{-\frac{1}{2}} \right. \right. \\
 & \quad \times \left. \int_{\mathbf{R}^n} \exp \left\{ -\frac{z}{2} \sum_{j=1}^n \frac{[u_j - u_{j-1}]^2}{t_j - t_{j-1}} + i \sum_{j=1}^n u_j v_j \right\} d\vec{u} \right] d\mu_n(\vec{t}, \vec{v}) \Big| \\
 = & \left| \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] z^{-\frac{n}{2}} \right. \\
 & \quad \times \left[ \prod_{j=1}^n \left( \frac{z}{2\pi(t_j - t_{j-1})} \right)^{\frac{n}{2}} \int_{\mathbf{R}^n} \exp \left\{ -\frac{z}{2} \sum_{j=1}^n \frac{[u_j - u_{j-1}]^2}{t_j - t_{j-1}} + i \sum_{j=1}^n u_j v_j \right\} d\vec{u} \right] d\mu_n(\vec{t}, \vec{v}) \Big| \\
 = & \left| z^{-\frac{n}{2}} \sum_{j=1}^n \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n v_k^2 \right\} d\mu_n(\vec{t}, \vec{v}) \right| \\
 = & \left| z^{-\frac{n}{2}} \sum_{j=1}^n \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ -\frac{\bar{z}}{2z\bar{z}} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n v_k^2 \right\} d\mu_n(\vec{t}, \vec{v}) \right| \\
 \leq & |z|^{-\frac{n}{2}} \left( \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} (|w| \sum_{j=1}^n |v_j|) \times 1 d|\mu_n|(\vec{t}, \vec{v}) \right) \\
 = & |z|^{-\frac{n}{2}} |w| \left( \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \sum_{j=1}^n |v_j| d|\mu_n|(\vec{t}, \vec{v}) \right) \\
 < & \infty, \tag{29}
 \end{aligned}$$

because for  $z \in \mathbf{C}^+$ ,

$$\left| \exp \left\{ -\frac{\bar{z}}{2z\bar{z}} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n v_k^2 \right\} \right| \leq 1.$$

We use the following function in the second equality of (29), when we applied the Wiener integration formula (6):

$$\begin{aligned}
 & f(x(t_1), x(t_2), \dots, x(t_n)) \\
 &= \exp \left\{ \frac{1-z}{2} \sum_{j=1}^n \frac{[x(t_j) - x(t_{j-1})]^2}{t_j - t_{j-1}} \right\} \\
 & \quad \times \left[ \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbb{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ i \sum_{j=1}^n v_j x(t_j) \right\} d\mu_n(\vec{t}, \vec{v}) \right]. \tag{30}
 \end{aligned}$$

Therefore we have the desired result.  $\square$

**Remark 3.** (1) In [14], Y.S. Kim proved the relationship between the Fourier-Feynman transform and the Wiener integral about the first variation of

$$F(x) = \int_H \exp \left\{ i(h, x)^\sim \right\} d\mu(h)$$

in the Fresnel class under the condition that

$$\int_H |h| d|\mu|(h) < \infty$$

on the abstract Wiener space.

(2) In [17], Y.S. Kim proved the relationship between the Fourier-Feynman transform and the convolution about the first variation of

$$F(x) = \hat{\mu}((h_1, x)^\sim, \dots, (h_n, x)^\sim) = \int_{\mathbb{R}^n} \exp \left\{ i \sum_{j=1}^n (h_j, x)^\sim v_j \right\} \mu(d\vec{v})$$

under the condition that

$$\int_{\mathbb{R}^n} \left[ \sum_{j=1}^n v_j^2 \right]^{\frac{1}{2}} |\mu|(d\vec{v}) < \infty$$

on the abstract Wiener space.

(3) In this paper, we find a condition :

$$\sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbb{R}^n} \left[ \sum_{j=1}^n |v_j| \right] d|\mu_n|(\vec{t}, \vec{v}) < \infty$$

that the first variation of  $F(x)$  in (10) exists, where

$$F(x) = \exp \left\{ \int_0^T \theta(t, x(t)) dt \right\} = \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbb{R}^n} \exp \left\{ i \sum_{j=1}^n v_j x(t_j) \right\} d\mu_n(\vec{t}, \vec{v})$$

on the Wiener space.

Now we establish the harmonious relationship between the analytic Wiener integral and the Wiener integral of the first variation.

That is, we prove that the analytic Wiener integral of the first variation can be successfully expressed by the sequence of Wiener integrals of the first variation.

**Theorem 6.** For  $z \in \mathbf{C}^+$ ,

$$\begin{aligned} & \int_{C_0[0,T]} \delta F(x|w) dm(x) \\ &= z^{\frac{n}{2}} \int_{C_0[0,T]} \exp \left\{ \frac{1-z}{2} \sum_{j=1}^n \frac{[x(t_j) - x(t_{j-1})]^2}{t_j - t_{j-1}} \right\} \delta F(x|w) dm(x). \end{aligned} \tag{31}$$

**Proof.** By Wiener integration formula (6), Lemma 2, Theorem 3 and Theorem 5, we have that for  $z \in \mathbf{C}^+$  and for  $w \in C_0[0, T]$ ,

$$\begin{aligned} & \int_{C_0[0,T]} \exp \left\{ \frac{1-z}{2} \sum_{j=1}^n \frac{[x(t_j) - x(t_{j-1})]^2}{t_j - t_{j-1}} \right\} \delta F(x|w) dm(x) \\ &= \int_{C_0[0,T]} \exp \left\{ \frac{1-z}{2} \sum_{j=1}^n \frac{[x(t_j) - x(t_{j-1})]^2}{t_j - t_{j-1}} \right\} \\ & \quad \times \left[ \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ i \sum_{j=1}^n v_j x(t_j) \right\} d\mu_n(\vec{t}, \vec{v}) \right] dm(x) \\ &= \prod_{j=1}^n \left[ 2\pi(t_j - t_{j-1}) \right]^{-\frac{1}{2}} \int_{\mathbf{R}^n} \exp \left\{ \frac{1-z}{2} \sum_{j=1}^n \frac{[u_j - u_{j-1}]^2}{t_j - t_{j-1}} \right\} \\ & \quad \times \left[ \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ i \sum_{j=1}^n v_j u_j \right\} \right] \\ & \quad \times \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \frac{[u_j - u_{j-1}]^2}{t_j - t_{j-1}} \right\} d\mu_n(\vec{t}, \vec{v}) d\vec{u} \\ &= z^{-\frac{n}{2}} \left( \prod_{j=1}^n \left[ \frac{z}{2\pi(t_j - t_{j-1})} \right]^{\frac{1}{2}} \int_{\mathbf{R}^n} \exp \left\{ -\frac{z}{2} \sum_{j=1}^n \frac{[u_j - u_{j-1}]^2}{t_j - t_{j-1}} \right\} \right. \\ & \quad \times \left. \left[ \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ i \sum_{j=1}^n v_j u_j \right\} d\mu_n(\vec{t}, \vec{v}) \right] d\vec{u} \right) \\ &= z^{-\frac{n}{2}} \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbf{R}^n} \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n u_k^2 \right\} d\mu_n(\vec{t}, \vec{v}) \\ &= z^{-\frac{n}{2}} \int_{C_0[0,T]} \delta F(x|w) dm(x). \end{aligned} \tag{32}$$

□

Now, we prove that the first variation of  $F(x)$  satisfies successfully the change of scale formula for the Wiener integral on the Wiener space.

**Theorem 7.** For positive real  $\rho > 0$ ,

$$\begin{aligned} & \int_{C_0[0,T]} \delta F(\rho x |w) dm(x) \\ &= \rho^{-n} \int_{C_0[0,T]} \exp \left\{ \frac{\rho^2 - 1}{2\rho^2} \sum_{j=1}^n \frac{[x(t_j) - x(t_{j-1})]^2}{t_j - t_{j-1}} \right\} \delta F(x|w) dm(x). \end{aligned} \tag{33}$$

**Proof.** By Theorem 6, we have that for real  $z > 0$ ,

$$\begin{aligned} & \int_{C_0[0,T]} \delta F(z^{-\frac{1}{2}}x|w) dm(x) \\ &= z^{\frac{n}{2}} \int_{C_0[0,T]} \exp \left\{ \frac{1-z}{2} \sum_{j=1}^n \frac{[x(t_j) - x(t_{j-1})]^2}{t_j - t_{j-1}} \right\} \delta F(x|w) dm(x). \end{aligned} \tag{34}$$

If we let  $z = \rho^{-2}$  in the above equation, we can have the desired result.  $\square$

Now, we establish the harmonious relationship between the analytic Feynman integral and the Wiener integral of the first variation of  $F(x)$ .

That is, we prove that the analytic Feynman integral of the first variation can be successfully expressed as the limit of the sequence of Wiener integrals of the first variation on the Wiener space.

**Theorem 8.**

$$\begin{aligned} & \int_{C_0[0,T]}^{anf_q} \delta F(x|w) dm(x) \\ &= \lim_{k \rightarrow \infty} z_k^{\frac{n}{2}} \int_{C_0[0,T]} \exp \left\{ \frac{1-z_k}{2} \sum_{j=1}^n \frac{[x(t_j) - x(t_{j-1})]^2}{t_j - t_{j-1}} \right\} \delta F(x|w) dm(x), \end{aligned} \tag{35}$$

whenever  $\{z_k\} \rightarrow -iq$  through  $\mathbf{C}_+$ .

**Proof.** By Definition 1, Lemma 2 and Theorem 6, we have that

$$\begin{aligned} & \int_{C_0[0,T]}^{anf_q} \delta F(x|w) dm(x) \\ &= \lim_{k \rightarrow \infty} \int_{C_0[0,T]}^{anw_{z_k}} F(x) dm(x) \\ &= \lim_{k \rightarrow \infty} z_k^{\frac{n}{2}} \int_{C_0[0,T]} \exp \left\{ \frac{1-z_k}{2} \sum_{j=1}^n \frac{[x(t_j) - x(t_{j-1})]^2}{t_j - t_{j-1}} \right\} \delta F(x|w) dm(x), \end{aligned} \tag{36}$$

whenever  $\{z_k\} \rightarrow -iq$  through  $\mathbf{C}_+$ .  $\square$

#### 4. Conclusions and Discussion

The first variation of  $F(x) = \exp \left\{ \int_0^T \theta(t, x(t)) dt \right\}$  successfully exists under the condition (12). The Wiener integral, the analytic Wiener integral and the analytic Feynman integral of the first variation of  $F(x)$  successfully exist. Furthermore, the first variation of  $F(x)$  successfully satisfies the change of scale formula for the Wiener integral.

The Wiener integral, the analytic Wiener integral and the analytic Feynman integral of the first variation of  $F(x)$  exist under the condition that  $\sum_{n=1}^{\infty} \|\mu_n\| < \infty$  and  $\sum_{n=1}^{\infty} \int_{\Delta_n(T) \times R^n} \left[ \sum_{j=1}^n |v_j| \right] d|\mu_n|(\vec{t}, \vec{v}) < \infty$ .

But the Wiener integral, the analytic Wiener integral and the analytic Feynman integral of  $F(x)$  exist and all properties of  $F(x)$  hold under one condition that  $\sum_{n=1}^{\infty} \|\mu_n\| < \infty$  :

$$\begin{aligned}
 (1). \int_{C_0[0,T]} F(x) dm(x) &= \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbb{R}^n} \exp \left\{ -\frac{1}{2} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n u_k^2 \right\} d\mu_n(\vec{t}, \vec{v}) \\
 (2). \int_{C_0[0,T]}^{anw_z} F(x) dm(x) &= \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbb{R}^n} \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n u_k^2 \right\} d\mu_n(\vec{t}, \vec{v}) \\
 (3). \int_{C_0[0,T]}^{anf_q} F(x) dm(x) &= \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbb{R}^n} \exp \left\{ -\frac{i}{2q} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n u_k^2 \right\} d\mu_n(\vec{t}, \vec{v}) \\
 (4). \int_{C_0[0,T]}^{anw_z} F(x) dm(x) &= z^{\frac{n}{2}} \int_{C_0[0,T]} \exp \left\{ \frac{1-z}{2} \sum_{j=1}^n \frac{[x(t_j) - x(t_{j-1})]^2}{t_j - t_{j-1}} \right\} F(x) dm(x) \\
 (5). \int_{C_0[0,T]} F(\rho x) dm(x) &= \rho^{-n} \int_{C_0[0,T]} \exp \left\{ \frac{\rho^2 - 1}{2\rho^2} \sum_{j=1}^n \frac{[x(t_j) - x(t_{j-1})]^2}{t_j - t_{j-1}} \right\} F(x) dm(x) \\
 (6). \int_{C_0[0,T]}^{anf_q} F(x) dm(x) &= \lim_{k \rightarrow \infty} z_k^{\frac{n}{2}} \int_{C_0[0,T]} \exp \left\{ \frac{1 - z_k}{2} \sum_{j=1}^n \frac{[x(t_j) - x(t_{j-1})]^2}{t_j - t_{j-1}} \right\} F(x) dm(x), \tag{37}
 \end{aligned}$$

because  $|\int_{C_0[0,T]} F(x) dm(x)| \leq \sum_{n=1}^{\infty} \int_{\Delta_n(T) \times \mathbb{R}^n} 1 d|\mu_n|(\vec{t}, \vec{v}) = \sum_{n=1}^{\infty} \|\mu_n\| < \infty$ . That is,

$$\begin{aligned}
 (1). \left| \int_{C_0[0,T]} F(x) dm(x) \right| &\leq \sum_{n=1}^{\infty} \|\mu_n\| < \infty \\
 (2). \left| \int_{C_0[0,T]}^{anw_z} F(x) dm(x) \right| &\leq \sum_{n=1}^{\infty} \|\mu_n\| < \infty \\
 (3). \left| \int_{C_0[0,T]}^{anf_q} F(x) dm(x) \right| &\leq \sum_{n=1}^{\infty} \|\mu_n\| < \infty. \tag{38}
 \end{aligned}$$

**Example 1.** Let  $F : C_0[0, T] \rightarrow \mathbf{C}$  be defined by  $F(x) = \exp \left\{ i \sum_{j=1}^n v_j x(t_j) \right\}$  with  $\sum_{j=1}^n |v_j| < \infty$  and let  $\Delta_n(T)$  be defined as in Notation 1. Then for  $w \in C_0[0, T]$  with  $|w| < \infty$ ,

$$\delta F(x|w) = \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ i \sum_{j=1}^n v_j x(t_j) \right\}$$

and  $|\delta F(x|w)| \leq |w| \left[ \sum_{j=1}^n |v_j| \right] < \infty$ , using the proof of Lemma 2. The analytic Wiener integral and the analytic Feynman integral exist and for  $z \in \mathbf{C}^+$ ,

$$\int_{C_0[0,T]}^{anw_z} \delta F(x/w) dm(x) = \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ -\frac{1}{2z} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n v_k^2 \right\}$$

and

$$\int_{C_0[0,T]}^{anf_q} \delta F(x/w) dm(x) = \left[ i \sum_{j=1}^n v_j w(t_j) \right] \exp \left\{ -\frac{i}{2q} \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=j}^n v_k^2 \right\}$$

whenever  $z \rightarrow -iq$  through  $\mathbf{C}^+$ , using the proof of Theorems 3 and 4. Furthermore, we can prove Theorems 5–8 about the first variation of  $F(x)$  exploiting proofs of those Theorems.

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