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Article

Q-Extension of Starlike Functions Subordinated with a Trigonometric Sine Function

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Abstract: The main purpose of this article is to examine the q-analog of starlike functions connected with a trigonometric sine function. Further, we discuss some interesting geometric properties, such as the well-known problems of Fekete-Szegö, the necessary and sufficient condition, the growth and distortion bound, closure theorem, convolution results, radii of starlikeness, extreme point theorem and the problem with partial sums for this class.

Keywords: starlike functions; subordination; q-derivative operator; sine function

1. Introduction and Definitions

To understand all the concepts used in this article clearly we need to include and explain all the terms mentioned here. First, let $\mathcal{A}$ be the collection of functions which are holomorphic (or analytic) in $\mathcal{D} := \{z \in \mathbb{C} : |z| < 1\}$ and fulfill the subsequent Taylor series expansion:

$$f(z) = \sum_{k=1}^{\infty} a_k z^k,\ a_1 = 1,\ z \in \mathcal{D}. \tag{1}$$

In [1,2], Miller and Mocanu generalized the ideas that consist of differential inequalities for real to complex valued functions that laid the foundations for a new theory, known as "the method of differential subordination or admissible functions method". This technique is used in geometric function theory, as a tool that provides not only new results, but also solves complicated problems in a simple way. In complex valued function the characterization of a function can be obtained from a differential condition, for example, the Noshiro–Warschawski theorem [3]. Said theory is applicable in various fields, including ordinary differential equations, partial differential equations, harmonic functions, integral operators, Banach spaces and functions of several variables.

If f_1 and f_2 is in $\mathcal{A}$, then f_1 is subordinated by f_2 if a holomorphic function w can be find with the properties $w(0) = 0$ and $|w(z)| < |z|$ so that $f_1(z) = f_2(w(z))$ $(z \in \mathcal{D})$. In addition, if $f_1, f_2 \in \mathcal{D}$ are univalent, then:

$$f_1 \prec f_2 \quad \Leftrightarrow \quad f_1(\mathcal{D}) \subseteq f_2(\mathcal{D}) \quad \text{with} \quad f_1(0) = f_2(0).$$

Additionally, the Hadamard product (or convolution) between the functions $f_1, f_2 \in \mathcal{A}$ is described by

$$(f_1 * f_2)(z) = \sum_{k=1}^{\infty} a_k b_k \, z^k, \ (z \in \mathcal{D}),$$

where

$$f_1(z) = \sum_{k=1}^{\infty} a_k \, z^k \ \ and \ \ f_2(z) = \sum_{k=1}^{\infty} b_k \, z^k .$$

In 1994, Ma and Minda [4] introduced the following subset of holomorphic functions:

$$\mathcal{S}^*(h) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec h(z), \, z \in \mathcal{D} \right\}, \tag{2}$$

with the restriction that the image domain of h (h is a convex function with $Re h > 0$ in $\mathcal{D}$) is symmetric along the real axis and starlike about $h(0) = 1$ with $h'(0) > 0$. They investigated certain useful problems, including distortion, growth and covering theorems.

Now taking some particular functions instead of h in $\mathcal{S}^*(h)$, we achieve many sub-families of the collection $\mathcal{A}$ which have different geometric interpretations as for example:

(i) If $h(z) = \frac{1+Az}{1+Bz}$ with $-1 \leq B < A \leq 1$, then $\mathcal{S}^*[A,B] := \mathcal{S}^*\left(\frac{1+Az}{1+Bz}\right)$ is the set of Janowski starlike functions; see [5]. Some interesting problems such as convolution properties, coefficient inequalities, sufficient conditions, subordinates results and integral preserving were discussed recently in [6–10] for some of the generalized families associated with circular domains.

(ii) The class $\mathcal{S}^*_L := \mathcal{S}^*(\sqrt{1+z})$ was introduced by Sokól and Stankiewicz [11], consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$.

(iii) When we take $h(z) = e^z$, then we have $\mathcal{S}^*_e := \mathcal{S}^*(e^z)$ [12].

(iv) The family $\mathcal{S}^*_R := \mathcal{S}^*(h(z))$ with $h(z) = 1 + \frac{z}{k}\frac{k+z}{k-z}$, $k = \sqrt{2}+1$ is studied in [13].

(v) By setting $h(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, the family $\mathcal{S}^*(h)$ reduces to $\mathcal{S}^*_{car}$ introduced by Sharma and his coauthors [14], consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the cardioid given by

$$(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0;$$

see also [15,16]. For more special cases of the set $\mathcal{S}^*(h)$, see [17–19].

Recently in 2019, Cho and his coauthors [20] established the following class $\mathcal{S}^*_{\sin}$ by selecting the function $1 + \sin z$ instead of the function h as:

$$\mathcal{S}^*_{\sin} = \left\{ f \in \mathcal{A} : \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \sin z \right\}. \tag{3}$$

Geometrically, the ratio $\frac{zf'(z)}{f(z)}$ lies in an eight-shaped region in the right half plane. They investigated the inverse inclusion relations of this family with the already known subfamilies of analytic functions. Later on for this family, the third Hankel determinants were studied by the authors in [21].

The classical calculus with no limit is known as quantum calculus or just q-calculus. This exceptional theory emerged via Jackson [22,23]. The readers were influenced by the q-calculus learning owing to its contemporary usage of numerous arguments as for example; in quantum theory, special functions theory, differential equations, number theory, operator theory, combinatorics, numerical analysis and certain other similar theories; see [24,25]. The early work of q-calculus in the field of geometric function theory (GFT) was done by Ismail et al. (see [26]) by generalizing the set of starlike functions into a q-analogue, known as the set of q-starlike functions. Another important development in this direction was the work of Anastassiu and Gal [27,28], who gave the q-generalizations of certain complex operators (particularly Picard and Gauss–Weierstrass singular integral operators). Following the same idea, Srivastava [29] presented some strong footing by giving some applications of q-calculus in this field by using q-analogues of hypergeometric functions. In this direction, some good valuable contributions were made by researchers, including Srivastava [30], Agrawal [31], Seoudy and Aouf [32], Agrawal and Sahoo [33], Arif and Ahmad [34], Kanas and Răducanu [35], Arif, Srivastava and Umar [36] and Haq et al. [37]. See also the articles [38–43].

For $q \in]0,1[$ and $z \in \mathcal{D}$, the q-analog derivative of f is defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}. \tag{4}$$

If we take $f(z) = \sum_{n=1}^{\infty} a_n z^n$, then for $n \in \mathbb{N}$ (natural number set) and $z \in \mathcal{D}$

$$\partial_q f(z) := \partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n]_q \, a_n z^{n-1}, \tag{5}$$

with

$$[n]_q := \frac{1-q^n}{1-q} = 1 + \sum_{l=1}^{n-1} q^l \;\; for \;\; n \neq 0 \text{ and } [0]_q = 0.$$

Using the above mentioned concepts, we now define the following family $\mathcal{S}^*_{\sin}(q)$ of starlike functions by:

$$\mathcal{S}^*_{\sin}(q) = \left\{ f \in \mathcal{A} : \left(\frac{z\partial_q f(z)}{f(z)} - 1 \right) \prec \sin z, \; z \in \mathcal{D} \right\}. \tag{6}$$

We note that $\lim_{q \to 1^-} \mathcal{S}^*_{\sin}(q) \equiv \mathcal{S}^*_{\sin}$, the class given by Equation (3). In this paper, we study some essential properties, such as the inequality of Fekete-Szegö, convolution problems, necessary and sufficient conditions, coefficient inequality, growth and distortion bounds, closure theorem, extreme point theorem and the partial sums problem.

The following two lemmas are used in the paper. However, before the statements of lemmas we define the class P of functions with a positive real part.

Let P denote the family of all functions p_1 that are analytic in $\mathcal{D}$ with positive real parts and have the following series representation:

$$p_1(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \;\; z \in \mathcal{D}. \tag{7}$$

Lemma 1. *[4] If $f \in \mathcal{P}$ has the expansion form given in Equation (7), then for $\vartheta \in \mathbb{C}$,*

$$\left| c_2 - \vartheta c_1^2 \right| \leq 2 \, \max\{1; |2\vartheta - 1|\}. \tag{8}$$

Lemma 2. *[4] If $f \in \mathcal{P}$ and is represented by Equation (7), then*

$$\left| c_2 - v c_1^2 \right| \leq \begin{cases} -4v + 2 & for \quad v \leq 0 \\ 2 & for \quad 0 \leq v \leq 1 \\ 4v - 2 & for \quad v \geq 1. \end{cases}$$

2. Major Contributions

Theorem 1. *Let $f \in \mathcal{S}^*_{\sin}(q)$ have the representation given in Equation (1). Then for $\vartheta \in \mathbb{C}$*

$$\left| a_3 - \vartheta a_2^2 \right| \leq \frac{1}{q[2]_q} \max\left\{ 1, \left| \frac{\vartheta [2]_q - 1}{q} \right| \right\}.$$

Proof. Let $f \in \mathcal{S}^{*}_{\sin}(q)$. Then one can conveniently write Equation (6) in terms of the Schwarz function w as

$$\frac{z\partial_q f(z)}{f(z)} = 1 + \sin(w(z)). \tag{9}$$

Additionally, if $p_1 \in \mathcal{P}$, then

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{10}$$

Alternatively

$$w(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{c_1 z + c_2 z^2 + \cdots}{2 + c_1 z + c_2 z^2 + \cdots}.$$

From Equations (1) and (5), we easily have

$$\frac{z\partial_q f(z)}{f(z)} = 1 + q a_2 z + q\left([2]_q a_3 - a_2^2\right) z^2 + \cdots, \tag{11}$$

and on the other hand

$$1 + \sin\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{1}{2}c_1 + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right) z^2 + \cdots. \tag{12}$$

From the last two equations, we get

$$a_2 = \frac{1}{2q}c_1, \tag{13}$$

$$a_3 = \frac{1}{2q[2]_q}c_2 - \frac{1}{4q[2]_q}\left(\frac{q-1}{q}\right)c_1^2. \tag{14}$$

Now using Equations (13) and (14), we obtain

$$\left|a_3 - \vartheta a_2^2\right| = \frac{1}{2q[2]_q}\left|c_2 - \frac{\left(\left([2]_q\right)^2 - 3[2]_q + 2\right) + \vartheta q [2]_q}{2q^2} c_1^2\right|. \tag{15}$$

By applying Lemma 1 to Equation (15) we get

$$\left|a_3 - \vartheta a_2^2\right| \le \frac{1}{q[2]_q} \max\left\{1, \left|\frac{\vartheta [2]_q - 1}{q}\right|\right\};$$

hence, proof is complete. □

If we put $\lambda = 1$, in Theorem 1, we deduce the result below.

Corollary 1. *Let $f \in \mathcal{S}^{*}_{\sin}(q)$. Then*

$$\left|a_3 - a_2^2\right| \le \frac{1}{q[2]_q}.$$

By making $q \to 1^{-}$ in Theorem 1, we achieve:

Corollary 2. *[21] Let $f \in \mathcal{S}^{*}_{\sin}$. Then for $\vartheta \in \mathbb{C}$*

$$\left|a_3 - \vartheta a_2^2\right| \le \frac{1}{2}\max\{1, |2\vartheta - 1|\}.$$

Theorem 2. *Let $f \in \mathcal{S}^*_{\sin}(q)$ and is of the form given by Equation (1). Then*

$$\left|a_3 - va_2^2\right| \leq \begin{cases} \frac{1-v[2]_q}{q^2[2]_q}, & \text{for} \quad v \leq \frac{2-[2]_q}{[2]_q}, \\ \frac{1}{q[2]_q}, & \text{for} \quad \frac{2-[2]_q}{[2]_q} \leq v \leq 1, \\ \frac{v[2]_q-1}{q^2[2]_q}, & \text{for} \quad v \geq 1. \end{cases}$$

Proof. Using Equations (13) and (14), we have

$$\left|a_3 - va_2^2\right| = \frac{1}{2q\,[2]_q}\left|c_2 - \frac{\left(\left([2]_q\right)^2 - 3\,[2]_q + 2\right) + vq\,[2]_q}{2q^2}c_1^2\right|. \tag{16}$$

Using Lemma 2 to Equation (16), we obtain the required result. □

Theorem 3. *Let $f \in \mathcal{A}$. Then $f \in \mathcal{S}^*_{\sin}(q)$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z - \mathcal{H}qz^2}{(1-z)(1-qz)}\right] \neq 0, \tag{17}$$

for all $\mathcal{H} = \mathcal{H}_\theta = \frac{1+\sin\left(e^{i\theta}\right)}{\sin\left(e^{i\theta}\right)}$, and also for $\mathcal{H} = 1$.

Proof. If $f \in \mathcal{S}^*_{\sin}(q)$, then $f(z) \neq 0$, $\forall\, z \in \mathcal{D}^* = \mathcal{D}\setminus\{0\}$ and so $\frac{1}{z}f(z) \neq 0$, for $z \in \mathcal{D}$. Thus we achieve an equivalent result to Equation (17) for $\mathcal{H} = 1$. Now, from Equation (6), there occurs a Schwarz function w such that

$$\frac{z\partial_q f(z)}{f(z)} = 1 + \sin(w(z)),$$

and if we take $w(z) = e^{i\theta}$, $\theta \in [0, 2\pi]$, we get

$$\frac{z\partial_q f(z)}{f(z)} \neq 1 + \sin\left(e^{i\theta}\right), \tag{18}$$

or

$$z\partial_q f(z) - f(z)\left(1 + \sin\left(e^{i\theta}\right)\right) \neq 0. \tag{19}$$

Using the relations

$$z\partial_q f(z) = f(z) * \left[\frac{z}{(1-z)(1-qz)}\right] \text{ and } f(z) = f(z) * \frac{z}{1-z},$$

and Equation (19), becomes

$$\frac{1}{z}\left[f(z) * \frac{z - \mathcal{H}qz^2}{(1-z)(1-qz)}\right] \neq 0,$$

where $\mathcal{H}$ is given above and the direct part of the proof is completed.

Conversely, suppose the Equation (17) hold true. Additionally, let $\Phi(z) = \frac{z\partial_q f(z)}{f(z)}$ be holomorphic in $\mathcal{D}$ with $\Phi(0) = 1$. Further, suppose that $\Psi(z) = 1 + \sin z$, $z \in \mathcal{D}$. It is clear from Equation (18) that $\Psi(\partial\mathcal{D}) \cap \Phi(\mathcal{D}) = \phi$. Hence, the simply connected domain $\Phi(\mathcal{D})$ is contained in connected component of $\mathbb{C} \setminus \Psi(\partial\mathcal{D})$. The univalence of "$\Psi$", together with fact $\Phi(0) = \Psi(0) = 1$, shows that $\Phi \prec \Psi$ and this gives $f \in \mathcal{S}^*_{\sin}(q)$. □

Theorem 4. *A necessary and sufficient criteria for a holomorphic function $f \in \mathcal{S}^{*}_{\sin}(q)$ is*

$$\sum_{n=2}^{\infty}\left(\frac{[n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}{\sin\left(e^{i\theta}\right)}\right)a_n z^{n-1}\neq 1. \tag{20}$$

Proof. In the light of above Theorem 3, we have $f \in \mathcal{S}^{*}_{\sin}(q)$ if and only if

$$\begin{aligned}
0 &\neq \frac{1}{z}\left[f(z)*\frac{z-\mathcal{H}qz^2}{(qz-1)(z-1)}\right]\\
&= \frac{1}{z}\left[f(z)*\frac{z}{(qz-1)(z-1)}-f(z)*\frac{\mathcal{H}qz^2}{(qz-1)(z-1)}\right]\\
&= \left[\partial_q f(z)+\mathcal{H}\left(\frac{f(z)}{z}-\partial_q f(z)\right)\right].
\end{aligned}$$

Using series form of f and $z\partial_q f$, we have

$$\begin{aligned}
0 &\neq \frac{1}{z}\left[z+\sum_{n=2}^{\infty}[n]_q a_n z^n-\mathcal{H}\sum_{n=2}^{\infty}\left([n]_q-1\right)a_n z^n\right]\\
&= 1-\sum_{n=2}^{\infty}\left((\mathcal{H}-1)[n]_q-\mathcal{H}\right)a_n z^{n-1}\\
&= 1-\sum_{n=2}^{\infty}\left(\frac{[n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}{\sin\left(e^{i\theta}\right)}\right)a_n z^{n-1};
\end{aligned}$$

hence the relation (20) is proved. □

Theorem 5. *Let $f \in \mathcal{A}$ be of the type of Equation (1) and satisfy*

$$\sum_{n=2}^{\infty}\left|\frac{[n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}{\sin\left(e^{i\theta}\right)}\right||a_n|<1. \tag{21}$$

Then $f \in \mathcal{S}^{*}_{\sin}(q)$.

Proof. In order to establish this theorem, we use relation (20). We have

$$\begin{aligned}
\left|1-\sum_{n=2}^{\infty}\left(\frac{[n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}{\sin\left(e^{i\theta}\right)}\right)a_n z^{n-1}\right| &> 1-\sum_{n=2}^{\infty}\left|\left(\frac{[n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}{\sin\left(e^{i\theta}\right)}\right)a_n z^{n-1}\right||a_n|\left|z^{n-1}\right|\\
&= 1-\sum_{n=2}^{\infty}\left|\frac{[n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}{\sin\left(e^{i\theta}\right)}\right||a_n|\left|z^{n-1}\right|\\
&> 1-\sum_{n=2}^{\infty}\left|\frac{[n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}{\sin\left(e^{i\theta}\right)}\right||a_n|>0,
\end{aligned}$$

and hence by virtue of Theorem 4, the proof is completed. □

Theorem 6. *Let $f \in \mathcal{S}^{*}_{\sin}(q)$, and $|z|=r$. Then*

$$r-\left|\frac{\sin\left(e^{i\theta}\right)}{[2]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}\right|r^2\leq|f(z)|\leq r+\left|\frac{\sin\left(e^{i\theta}\right)}{[2]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}\right|r^2.$$

Proof. Consider

$$\begin{aligned}
\left|f(z)\right| &\leq r+\sum_{n=2}^{\infty}\left|a_n\right|r^n \\
&\leq r+r^2\sum_{n=2}^{\infty}\left|a_n\right| \\
&\leq r+\left|\frac{\sin\left(e^{i\theta}\right)}{[2]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}\right|r^2.
\end{aligned}$$

On other hand,

$$\begin{aligned}
\left|f(z)\right| &\geq r-\sum_{n=2}^{\infty}\left|a_n\right|r^n \\
&\geq r-r^2\sum_{n=2}^{\infty}\left|a_n\right| \\
&\geq r-\left|\frac{\sin\left(e^{i\theta}\right)}{[2]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}\right|r^2.
\end{aligned}$$

□

Theorem 7. *Let $f\in\mathcal{S}^*_{\sin}(q)$, and $|z|=r$. Then*

$$r-\left|\frac{[2]_q\sin\left(e^{i\theta}\right)}{[2]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}\right|r^2\leq\left|\partial_q f(z)\right|\leq r+\left|\frac{[2]_q\sin\left(e^{i\theta}\right)}{[2]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}\right|r^2.$$

Proof. The proof is similar to that of Theorem 6 and it is omitted. □

Theorem 8. *Let $f_k\in\mathcal{S}^*_{\sin}(q)$, $k=1,2,\cdots,l$, such that*

$$f_k(z)=z+\sum_{n=2}^{\infty}a_{n,k}z^n.$$

Then $h(z)=\sum_{k=1}^{\infty}\delta_k f_k(z)$, where $\sum_{k=1}^{\infty}\delta_k=1$ is in the class $\mathcal{S}^*_{\sin}(q)$.

Proof. We have

$$h(z)=\sum_{k=1}^{\infty}\delta_k f_k(z)=z+\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\delta_k a_{n,k}z^n.$$

Consider

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left|\frac{[n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}{\sin\left(e^{i\theta}\right)}\right|\left(\sum_{k=1}^{\infty}\delta_k\left|a_{n,k}\right|\right) \\
=\;&\sum_{k=1}^{\infty}\delta_k\left(\sum_{n=2}^{\infty}\left|\frac{[n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}{\sin\left(e^{i\theta}\right)}\right|\left|a_{n,k}\right|\right) \\
<\;&\sum_{k=1}^{\infty}\delta_k=1;
\end{aligned}$$

hence $h\in\mathcal{S}^*_{\sin}(q)$. □

Theorem 9. *The class $\mathcal{S}^*_{\sin}(q)$ is a convex set.*

Proof. Let $f, g \in \mathcal{S}^*_{\sin}(q)$ with

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \;\; g(z) = \sum_{n=1}^{\infty} b_n z^n, \; a_1 = b_1 = 1.$$

We prove that the function $h(z) = \eta f(z) + (1-\eta) g(z)$, with $0 \leq \eta \leq 1$, is in the class $\mathcal{S}^*_{\sin}(q)$. We have

$$h(z) = z + \sum_{n=2}^{\infty} [\eta a_n + (1-\eta) b_n] z^n.$$

Thus to prove $h \in \mathcal{S}^*_{\sin}(q)$, we show that

$$\sum_{n=2}^{\infty} \left| \frac{[n]_q - (1 + \sin(e^{i\theta}))}{\sin(e^{i\theta})} \right| |\eta a_n + (1-\eta) b_n| < 1.$$

We have, using Theorem 5,

$$\begin{aligned} &\sum_{n=2}^{\infty} |\eta a_n + (1-\eta) b_n| \left| \frac{[n]_q - (1 + \sin(e^{i\theta}))}{\sin(e^{i\theta})} \right| \\ &\leq \eta \sum_{n=2}^{\infty} \left| \frac{[n]_q - (1 + \sin(e^{i\theta}))}{\sin(e^{i\theta})} \right| |a_n| + \\ &\qquad (1-\eta) \sum_{n=2}^{\infty} \left| \frac{[n]_q - (1 + \sin(e^{i\theta}))}{\sin(e^{i\theta})} \right| |b_n| \\ &< \eta(1) + (1-\eta)(1) = 1; \end{aligned}$$

hence $h \in \mathcal{S}^*_{\sin}(q)$. □

Theorem 10. *Let $f \in \mathcal{S}^*_{\sin}(q)$. Then, for $|z| < r^*$*

$$Re\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \;\; (0 \leq \alpha < 1), \tag{22}$$

where

$$r^* = \inf \left\{ \left| \frac{(1-\alpha)\left([n]_q - (1 + \sin(e^{i\theta}))\right)}{(n-\alpha)\sin(e^{i\theta})} \right| \right\}^n, \; \textit{for } n \in \mathbb{N}\backslash\{1\}.$$

Proof. To establish inequality (22), it is enough to derive that

$$\left| \frac{f(z) - zf'(z)}{(2\alpha - 1) f(z) - zf'(z)} \right| < 1.$$

We have

$$\begin{aligned} \left| \frac{f(z) - zf'(z)}{(2\alpha - 1) f(z) - zf'(z)} \right| &= \left| \frac{\sum_{n=2}^{\infty} (n-1) a_n z^n}{(1-\alpha) z + \sum_{n=2}^{\infty} (n+1-2\alpha) a_n z^n} \right| \\ &\leq \sum_{n=2}^{\infty} \left(\frac{n-\alpha}{1-\alpha} \right) |a_n| |z|^n. \end{aligned} \tag{23}$$

From Equation (21), we have

$$\sum_{n=2}^{\infty}\left|\frac{[n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}{\sin\left(e^{i\theta}\right)}\right||a_n|<1.$$

The relation (23) is bounded by 1 if

$$\left(\frac{n-\alpha}{1-\alpha}\right)|z|^n<\left|\frac{[n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)}{\sin\left(e^{i\theta}\right)}\right|,$$

it implies that

$$|z|<\left(\left|\frac{(1-\alpha)\left([n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)\right)}{(n-\alpha)\sin\left(e^{i\theta}\right)}\right|\right)^n=r^*,$$

and hence the proof is completed. □

Theorem 11. *Let us choose the function*

$$f_n(z)=z+\left|\frac{\sin\left(e^{i\theta}\right)}{\left([n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)\right)}\right|z^n,\ \textit{for } n\in\mathbb{N}\backslash\{1\}. \tag{24}$$

with $f_1(z)=z$. Then $f\in\mathcal{S}^*_{\sin}(q)$ if and only if f can be written as

$$f(z)=\sum_{n=1}^{\infty}\delta_n f_n(z),\ \textit{for } \delta_n\geq 0, \tag{25}$$

with $\sum_{n=1}^{\infty}\delta_n=1$.

Proof. Let the relation (25) hold true. Then

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left|\frac{\left([n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)\right)}{\sin\left(e^{i\theta}\right)}\right||a_n| \\
&= \sum_{n=2}^{\infty}\left|\frac{\left([n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)\right)}{\sin\left(e^{i\theta}\right)}\right|\left|\frac{\sin\left(e^{i\theta}\right)}{\left([n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)\right)}\right|\delta_n \\
&< \sum_{n=2}^{\infty}\delta_n=(1-\delta_1)<1;
\end{aligned}$$

hence by Theorem 5, $f\in\mathcal{S}^*_{\sin}(q)$. Conversely, if $f\in\mathcal{S}^*_{\sin}(q)$, then we have

$$\delta_n=\left|\frac{\left([n]_q-\left(1+\sin\left(e^{i\theta}\right)\right)\right)}{\sin\left(e^{i\theta}\right)}\right||a_n|,\ \text{for } n\geq 2,$$

and $\delta_1=1-\sum_{n=2}^{\infty}\delta_n$. Then function is of the form given by Equation (24) and this complete the proof. □

3. Partial Sum Problems

In this section, we examine the partial sum problems of certain analytic functions contained in the family $\mathcal{S}^*_{\sin}(q)$. We produce some new findings that have a connection between the analytical functions and their partial sum sequences. If a function $f \in \mathcal{A}$ has the series form given in Equation (1), then the partial sum f_m of f is described by

$$f_m(z) = z + \sum_{n=2}^{m} a_n z^n \text{ with } f_1(z) = z.$$

In 1928, Szegö [44] proved an interesting result which states that if $f \in \mathcal{S}^*$, then

$$\begin{cases} f_m \in \mathcal{S}^* & for \quad |z| < \frac{1}{4} \\ f_m \in \mathcal{C} & for \quad |z| < \frac{1}{8}. \end{cases}$$

This result motivated researchers to study the problem of partial sums for sub-families of analytic, univalent and multivalent functions. In [45], Silverman determined sharp lower bounds on the real parts of the quotients between the normalized convex or star-like functions and their consequences of partial sums. Additionally, Singh [46], Shiel-Small [47], Robertson [48], Ruscheweyh [49], Ponnusamy et al. [50], Srivastava et al. [25] and Owa et al. [24], have derived some beautiful results involving the partial sums.

Theorem 12. *Let $f \in \mathcal{S}^*_{\sin}(q)$ be given by Equation (1) and satisfy condition (21). Then*

$$Re\left\{\frac{f(z)}{f_m(z)}\right\} \geq 1 - \frac{1}{d_{m+1}}, \tag{26}$$

and

$$Re\left\{\frac{f_m(z)}{f(z)}\right\} \geq \frac{d_{m+1}}{d_{m+1}+1}, \tag{27}$$

where

$$d_n = \left|\frac{[n]_q - \left(1 + \sin\left(e^{i\theta}\right)\right)}{\sin\left(e^{i\theta}\right)}\right|$$

and

$$d_n \geq \begin{cases} 1 & for \quad n = 2, 3, \ldots, m \\ d_{m+1} & for \quad n = m+1, \ldots. \end{cases} \tag{28}$$

The above given results are the best ones.

Proof. To prove relation (26), let us write

$$\begin{aligned} d_{m+1}\left\{\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right)\right\} &= 1 + \frac{d_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{m} a_n z^{n-1}} \\ : &= \frac{1 + w(z)}{1 - w(z)}, \end{aligned} \tag{29}$$

where

$$w(z) = \frac{d_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}{2 + 2\sum_{n=2}^{m} a_n z^{n-1} + d_{m+1}\sum_{n=m+1}^{\infty} a_n z^{n-1}}.$$

Now

$$|w(z)| = \frac{d_{m+1}\sum_{n=m+1}^{\infty} |a_n|}{2 - 2\sum_{n=2}^{m} |a_n| - d_{m+1}\sum_{n=m+1}^{\infty} |a_n|} \leq 1$$

if and only if

$$\sum_{n=2}^{m} |a_n| + d_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq 1. \tag{30}$$

Finally, to show relation (30), it is sufficient to establish that the left-side of relation (30) is bounded above by $\sum_{n=2}^{\infty} d_n |a_n|$ and it is equal to

$$\sum_{n=2}^{m} (d_n - 1) |a_n| + \sum_{n=m+1}^{\infty} (d_n - d_{m+1}) |a_n| \geq 0.$$

The last inequality is true because of relation (28). To show that the inequality (26) is sharp, let us consider the function

$$f(z) = z + \frac{1}{d_{m+1}} z^{m+1}.$$

Then for $z = re^{i\frac{\pi}{m}}$, we have

$$\begin{aligned} \frac{f(z)}{f_m(z)} &= 1 + \frac{1}{d_{m+1}} z^m \to 1 - \frac{1}{d_{m+1}} r^m \\ &= \frac{d_{m+1} - 1}{d_{m+1}}. \end{aligned}$$

To derive inequality (27), let us write

$$\begin{aligned} (1 + d_{m+1}) \left\{ \frac{f_m(z)}{f(z)} - \left(\frac{d_{m+1}}{d_{m+1} + 1} \right) \right\} &= 1 - \frac{(1 + d_{m+1}) \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{m} a_n z^{n-1}} \\ &= \frac{1 + w(z)}{1 - w(z)}, \end{aligned}$$

where

$$w(z) = \frac{-(1 + d_{m+1}) \sum_{n=m+1}^{\infty} a_n z^{n-1}}{2 + 2\sum_{n=2}^{m} a_n z^{n-1} - (1 + d_{m+1}) \sum_{n=m+1}^{\infty} a_n z^{n-1}}.$$

Now

$$|w(z)| = \frac{(1 + d_{m+1}) \sum_{n=m+1}^{\infty} |a_n|}{2 - 2\sum_{n=2}^{m} |a_n| - (1 + d_{m+1}) \sum_{n=m+1}^{\infty} |a_n|} \leq 1$$

if the following inequality holds

$$\sum_{n=2}^{m} |a_n| + (1 + d_{m+1}) \sum_{n=m+1}^{\infty} |a_n| \leq 1. \tag{31}$$

Finally, to obtain inequality (31), it is enough to show that the left side of inequality (31) is bounded by $\sum_{n=2}^{\infty} d_n |a_n|$ and is equivalent to

$$\sum_{n=2}^{m} (d_n - 1) |a_n| + \sum_{n=m+1}^{\infty} (d_n - d_{m+1} - 1) |a_n| \geq 0,$$

which is true due to relation (28). □

4. Conclusions

Utilizing the principle of subordinations, we have defined the family of q-starlike functions connected with a particular trigonometric function such as sine functions. The new class generalizes the class of starlike functions subordinated with sine function which was introduced by Cho et al. [20] in which the radii problems were investigated. For the newly defined class, we have first investigated the familiar Fekete-Szegö type problems. After that, we have proved some convolution results which were used in proving the necessary and sufficient condition for the defined class. The problem of partial sums has been established with the help of sufficiency criteria for this newly defined class. Some other problems, such as radii of starlikeness, closure theorem, growth and distortion bounds and extreme point theorem have also been studied here for this class. Moreover, the present idea can be extended to prove some other problems, such as the Hankel determinant, the sufficiency criterion and convolution conditions for this class. Furthermore, these results can also be obtained for starlike functions associated with cosine functions. This class was recently studied in [51].

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