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The N-Dimensional Uncertainty Principle for the Free Metaplectic Transformation †

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Abstract: The free metaplectic transformation is an N-dimensional linear canonical transformation. This transformation operator is useful, especially for signal processing applications. In this paper, in order to characterize simultaneously local analysis of a function (or signal) and its free metaplectic transformation, we extend some different uncertainty principles (UP) from quantum mechanics including Classical Heisenberg's uncertainty principle, Nazarov's UP, Donoho and Stark's UP, Hardy's UP, Beurling's UP, Logarithmic UP, and Entropic UP, which have already been well studied in the Fourier transform domain.

Keywords: free metaplectic transformation; Uncertainty Principle; Nazarov's UP; Logarithmic's UP; Beurling's UP

100 MR Subject Classiflcation: 47B38; 42A38; 42B10; 70H15

1. Introduction

The free metaplectic transformation (FMT), which was first studied in Reference [1], is widely used in many fields such as filter design, pattern recognition, optics and analyzing the propagation of electromagnetic waves [2–5]. This useful transformation is also known as the N-dimensional nonseparable linear canonical transformation. In Table 1,

Table 1. Transition from the free metaplectic transformation (FMT) to various N-dimensional transformation.

Free Symplectic Matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$	Free Metaplectic Transformation
$A = (a_{kl})_{NN}$, $B = (b_{kl})_{NN}$, $C = (c_{kl})_{NN}$, $D = (d_{kl})_{NN}$	Nonseparable linear canonical transform
$A = D = \mathbf{0}$, $B = -C = I_N$	Fourier transform
$A = diag(a_{11}, \cdots, a_{NN})$, $B = diag(b_{11}, \cdots, b_{NN})$, $C = diag(c_{11}, \cdots, c_{NN})$, $D = diag(d_{11}, \cdots, d_{NN})$	Separable linear canonical transform
$A = D = diag(\cos\theta_1, \cdots, \cos\theta_N)$, $B = -C = diag(\sin\theta_1, \cdots, \sin\theta_N)$	Fractional Fourier transform
$A = D = diag(\cosh\phi_1, \cdots, \cosh\phi_N)$, $B = C = diag(\sinh\phi_1, \cdots, \sinh\phi_N)$	The Lorentz transform

Some specific transformations can be obtained by taking special values for symplectic matrices.

The uncertainty principle (UP) is usually understood as the relationship between the simultaneous expansion of functions and its Fourier transform. In essence, uncertainty deals with the problem of concentration. Therefore, it can process the interaction of data loss, sparsity and bandwidth limitation in signal recovery. Plenty of scholars have studied and popularized UP [1,6–9].

In Reference [10], Qingyue Zhang studies the Zak transform and uncertainty principles associated with the linear canonical transform. In Reference [11], Haiye Huo studies the Uncertainty Principles (UP) for the offset linear canonical transformation (OLCT) and extends the existing uncertainty theory from the Fourier transform domain to the OLCT domain. Until now, most of the existing results are considered in one-dimension. For N-dimensional FMT, Zhichao Zhang has done a number of studies, for example, Reference [12]. He extends the classical N-dimensional Heisenberg's uncertainty principle to the FMT. A lower bound for the complex signal's uncertainty product $\triangle u_{M_1}^2 \triangle u_{M_2}^2$ is given in Reference [12], where $M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ $M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$ are parameter matrices of the FMT. In Reference [13], he extends the classical N-dimensional Heisenberg's uncertainty principle to the fractional Fourier transform (FRFT). To the best of our knowledge, there are no other results published about UPs associated with the FMT. Hence, in this paper, we mainly extend the various forms of N-dimensional UPs from the Fourier transform to the FMT domain.

The function $\widehat{f}(u)$ denotes the N-dimensional Fourier transform (FT) of $f(x)$,

$$\widehat{f}(u) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i x u^T} dx.$$

and the inversion formula for the FT is shown to be

$$f(x) = \int_{\mathbb{R}^N} \hat{f}(u) e^{2\pi i x u^T} du$$

where $x = (x_1, x_2, \cdots, x_N)$, $u = (u_1, u_2, \cdots, u_N)$ and $xu^T = \sum_{k=1}^{N} x_k u_k$. On the basis of FT, the classical N-dimensional Heisenberg's uncertainty principle is given by the following:

If $f(x) \in L^2\left(\mathbb{R}^N\right)$, and $a, b \in \mathbb{R}^N$ are arbitrary, then

$$\int_{\mathbb{R}^N} \|x-a\|^2 |f(x)|^2 dx \int_{\mathbb{R}^N} \|u-b\|^2 \left|\widehat{f}(u)\right|^2 du \geq \frac{N^2}{16\pi^2} \|f\|_2^4 \tag{1}$$

where the norm of function f is L^2-norm, that is, $\|f\| = \left(\int_{\mathbb{R}^N} |f(x)|^2 dx\right)^{\frac{1}{2}}$.

Our paper is organized as follows—in the next section, we give some basic definitions and lemmas. In Section 3, we extend the corresponding results of the Classical Heisenberg's uncertainty principle, Nazarov's UP, Donoho and Stark's UP, Hardy's UP, Beurling's UP, Logarithmic UP and Entropic UP to the free metaplectic transformation domain, respectively.

2. Definitions and Preliminaries

This section give some useful definitions and lemmas about the free metaplectic transformation.

Definition 1. *(Zhang [12]) The metaplectic operator of a function $f(x) \in L^2(\mathbb{R}^N)$ with the free symplectic matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, (equivalently $(\det(B) \neq 0)$) is defined as*

$$\mathcal{L}_M[f](u) = \widehat{f_M}(u) = \frac{1}{\sqrt{|\det(B)|}} \int_{\mathbb{R}^N} f(x) e^{\pi i (uDB^{-1}u^T + xB^{-1}Ax^T) - 2\pi i x B^{-1} u^T} dx, \tag{2}$$

where $u = (u_1, \cdots, u_N)$, *and* $A = (a_{kl})$, $B = (b_{kl})$, $C = (c_{kl})$, $D = (d_{kl})$ *are all* $N \times N$ *real matrices satisfying the following constraints:*

$$AB^T = BA^T, CD^T = DC^T, AD^T - BC^T = I_N,$$

and where I_N *denotes an* $N \times N$ *identity matrix. The corresponding inverse formula is given by* $f(x) = \mathcal{L}_{M^{-1}}[\widehat{f_M}](x)$, *where* $M^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}$.

Here

$$\begin{aligned} MM^{-1} &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} \\ &= \begin{pmatrix} AD^T - BC^T & -AB^T + BA^T \\ CD^T - DC^T & -CB^T + DA^T \end{pmatrix} \\ &= \begin{pmatrix} I_N & 0 \\ 0 & I_N \end{pmatrix}. \end{aligned}$$

For the convenience of writing, $\mathcal{L}_{M^{-1}}(f)$ represents the inversion formula of the free metaplectic transformation about function f. The following definition is also a significant basic content for the study of free metaplectic transformation.

Definition 2. *(Zhang [12]) Let* $\hat{f}(u)$ *be the N-dimensional FT of* $f(x) \in L^2(\mathbb{R}^N)$, *and* $\widehat{f_M}(u)$ *be the free metaplectic transformation of* $f(x)$ *with the symplectic matrix* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. *Assume that* $xf(x), u\hat{f}(u) \in L^2(\mathbb{R}^N)$. *Then we can define*
(i) The covariance in the time domain:

$$\triangle x^2 = \int_{\mathbb{R}^N} \|x - x^0\|^2 |f(x)|^2 dx,$$

where the moment vector in the time domain is

$$x^0 = (x_1^0, \cdots, x_N^0),\ x_k^0 = \int_{\mathbb{R}^N} x_k |f(x)|^2 dx / \|f\|_2^2,\ k = 1, \cdots, N.$$

(ii) The covariance in the frequency domain:

$$\triangle u^2 = \int_{\mathbb{R}^N} \|u - u^0\|^2 \left|\widehat{f}(u)\right|^2 du,$$

where the moment vector in the frequency domain is

$$u^0 = (u_1^0, \cdots, u_N^0),\ u_k^0 = \int_{\mathbb{R}^N} u_k \left|\widehat{f}(u)\right|^2 du / \|f\|_2^2,\ k = 1, \cdots, N.$$

(iii) The covariance in the free metaplectic transformation domain:

$$\triangle u_M^2 = \int_{\mathbb{R}^N} \|u - u^{M,0}\|^2 \left|\widehat{f_M}(u)\right|^2 du,$$

where the moment vector in the free metaplectic transformation domain is

$$u^{M,0} = (u_1^{M,0}, \cdots, u_N^{M,0}),\ u_k^{M,0} = \int_{\mathbb{R}^N} u_k \left|\widehat{f_M}(u)\right|^2 du / \|f\|_2^2,\ k = 1, \cdots, N.$$

We first give the definition of infinitesimal of the same order.

Definition 3. *(Gordon, Kusraev and Kutateladze [14]) Let $f(x)$ and $g(x)$ be two infinitely small functions as $x \to a$. We call $f(x)$ an infinitesimal of the same order of $g(x)$, record as $f(x) = \mathcal{O}(g(x))$, if*

$$0 < \lim_{x \to a} \frac{f(x)}{g(x)} < \infty.$$

The following is the definition of the Schwartz class. It is useful in Section 3.

Definition 4. *(Grochenig [9]) The Schwartz class consists of all C^∞-functions f on $\mathbb{R}^N$, such that*

$$\sup |D^\alpha X^\beta f(x)| < \infty$$

for all $\alpha, \beta \in \mathbb{Z}_+^N$, where $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}}$ for the partial derivative and $X^\alpha f(x) = x^\alpha f(x)$ for the multiplication operator. $\mathcal{S}(\mathbb{R}^N)$ denotes the Schwartz class.

In addition, Gamma function will also be used in this paper.

Definition 5. *In real number field, Gamma function's formula is that*

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt.$$

As an extension of factorial, Gamma function is defined in the complex range of meromorphic function, which is usually written as $\Gamma(x)$. When the variable of a function is a positive integer, then $\Gamma(n+1) = n!$

For a set $T \subseteq \mathbb{R}^N$, the characteristic function of T is

$$\chi_T(x) = \begin{cases} 1, & x \in T \\ 0, & x \notin T \end{cases}.$$

Let $P : E \to F$ be a bounded linear operator. Then the operator norm of P is

$$\|P\| = \sup_{|x| \le 1} \|Px\| = \sup_{|x|=1} \|Px\| = \sup_{x \in E, x \neq 0} \frac{\|Px\|}{\|x\|}.$$

more theories on operator norm can be seen in Reference [15].

The key players are the transition from free metaplectic transform to Fourier transform and the research on norm of free metaplectic transformation. The corresponding results are given by the following lemmas.

Lemma 1. *Let $f \in L^2(\mathbb{R}^N)$ and $g(x) = f(x) e^{\pi i x B^{-1} A x^T}$, then the relationship between $\widehat{g}(u)$ and $\widehat{f_M}(u)$ is*

$$|\widehat{g}(u)| = \sqrt{|\det(B)|} \left| \widehat{f_M}(uB^T) \right|. \tag{3}$$

Proof. From Definition 1, for all $u \in \mathbb{R}^N$, the free metaplectic transformation of uB^T

$$\begin{aligned}\widehat{f_M}(uB^T) &= \frac{1}{\sqrt{|\det(B)|}}\int_{\mathbb{R}^N} f(x)e^{\pi i(uB^TDB^{-1}Bu^T+xB^{-1}Ax^T)-2\pi ixB^{-1}Bu^T}dx \\ &= \frac{1}{\sqrt{|\det(B)|}}\int_{\mathbb{R}^N} f(x)e^{\pi i(uB^TDu^T+xB^{-1}Ax^T)-2\pi ixu^T}dx \\ &= \frac{1}{\sqrt{|\det(B)|}}e^{\pi iuB^TDu^T}\int_{\mathbb{R}^N} f(x)e^{\pi ixB^{-1}Ax^T}e^{-2\pi ixu^T}dx.\end{aligned}$$

Let $g(x) = f(x)e^{\pi ixB^{-1}Ax^T}$, then

$$|g(x)| = |f(x)|. \tag{4}$$

Furthermore,

$$\widehat{f_M}(uB^T) = \frac{1}{\sqrt{|\det(B)|}}e^{\pi iuB^TDu^T}\widehat{g}(u).$$

Taking modulus on both sides of the equation, then

$$\left|\widehat{f_M}(uB^T)\right| = \frac{1}{\sqrt{|\det(B)|}}|\widehat{g}(u)|,$$

that is,

$$|\widehat{g}(u)| = \sqrt{|\det(B)|}\left|\widehat{f_M}(uB^T)\right|. \tag{5}$$

□

Next, we provide the fundamental result for the free metaplectic transformation. The Parseval formula is generalized.

Lemma 2. *If* $f \in L^1\left(\mathbb{R}^N\right) \cap L^2\left(\mathbb{R}^N\right)$, *then*

$$\left\|\widehat{f_M}\right\|_2 = \|f\|_2. \tag{6}$$

Proof. By Definition 1,

$$\widehat{f_M}(u) = \frac{1}{\sqrt{|\det(B)|}}e^{\pi iuDB^{-1}u^T}\widehat{g}\left(u\left(B^{-1}\right)^T\right).$$

Equation (6) follows from the calculation

$$\begin{aligned}\left\|\widehat{f_M}\right\|_2^2 &= \int_{\mathbb{R}}\left|\widehat{f_M}(u)\right|^2du = \int_{\mathbb{R}}\frac{1}{|\det(B)|}\left|\hat{g}\left(u(B^{-1})^T\right)\right|^2du \\ &= \int_{\mathbb{R}}|\hat{g}(w)|^2dw = \int_{\mathbb{R}}|g(x)|^2dx = \int_{\mathbb{R}}|f(x)|^2dx = \|f\|_2^2.\end{aligned}$$

□

3. Uncertainty Principles for the Free Metaplectic Transformation Domain

Apparently, there are many different forms of UP's in the Fourier transform domain. Since the free metaplectic transformation is a generalized version of the Fourier transform and the linear canonical transform, it is natural and interesting to study this different forms of UP's in the free metaplectic transformation domain. it can not give the exact location in both f and its free metaplectic transformation.

3.1. Classical Heisenberg's Uncertainty Principle

In Reference [12], Zhang gives the measurement estimate about the lower bound of the product of two free metaplectic transformations under the premise of classical uncertainty. In this subsection, the lower bound of uncertainty corresponding to the free metaplectic transformation is given by the following theorem.

Theorem 1. *Let $f(x) \in L^2(\mathbb{R}^N)$. If $\triangle x^2$ and $\triangle u_M^2$ are the variances $f(x)$ and the free metaplectic transformation of $f(x)$, then*

$$\triangle x^2 \triangle u_M^2 \geq \frac{N^2 \|f\|_2^4}{16\pi^2} \sigma_{min}^2(B).$$

where $\sigma_{min}(B)$ denotes the minimum singular value of matrix B.

Proof. Applying N-dimensional Heisenberg's uncertainty principle to $g(x)$, we have

$$\int_{\mathbb{R}^N} \|x-a\|^2 |g(x)|^2 dx \int_{\mathbb{R}^N} \|u-b\|^2 \left|\widehat{g}(u)\right|^2 du \geq \frac{N^2}{16\pi^2} \|g\|_2^4. \tag{7}$$

Equality holds if and only if $g(x)$ is an N-dimensional Gaussian function. The combination of (4), (5) and (7) yields

$$|\det(B)| \int_{\mathbb{R}^N} \|x-a\|^2 |f(x)|^2 dx \int_{\mathbb{R}^N} \|u-b\|^2 \left|\widehat{f_M}(uB^T)\right|^2 du \geq \frac{N^2}{16\pi^2} \|f\|_2^4.$$

Let $w = uB^T$, then $u = w\left(B^T\right)^{-1}$. Therefore,

$$\int_{\mathbb{R}^N} \|x-a\|^2 |f(x)|^2 dx \int_{\mathbb{R}^N} \left\|w\left(B^T\right)^{-1} - b\right\|^2 \left|\widehat{f_M}(w)\right|^2 dw \geq \frac{N^2}{16\pi^2} \|f\|_2^4.$$

Furthermore,

$$\begin{aligned}
\left\|w\left(B^T\right)^{-1} - b\right\|^2 &= \left\|(w - bB^T)\left(B^T\right)^{-1}\right\|^2 \\
&= \left\langle (w - bB^T)\left(B^T\right)^{-1}, (w - bB^T)(B^T)^{-1} \right\rangle \\
&= (w - bB^T)\left(B^T\right)^{-1} B^{-1}(w - bB^T)^T \\
&= (w - bB^T)\left(BB^T\right)^{-1}(w - bB^T)^T.
\end{aligned}$$

Since $(BB^T)^{-1}$ is real and symmetric matrix by definition, there exits unitary matrix P such that

$$P^T(BB^T)^{-1}P = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdot & \\ & & & \cdot \\ & & & \lambda_n \end{pmatrix}, \tag{8}$$

where $\lambda_1, \cdots, \lambda_n$ are eigenvalues of $(BB^T)^{-1}$. Now suppose that $\lambda_{min}(BB^T)$ represents the minimum eigenvalue of matrix BB^T, $\lambda_{max}(BB^T)$ represents the maximum eigenvalue of matrix BB^T. We have

$$\begin{aligned} \|w(B^T)^{-1} - b\|^2 &= (w - bB^T)\left(BB^T\right)^{-1}(w - bB^T)^T \\ &\leq \frac{1}{\lambda_{min}(BB^T)}(w - bB^T)PEP^T(w - bB^T)^T \\ &= \frac{1}{\lambda_{min}(BB^T)}\|w - bB^T\|^2. \end{aligned}$$

Then the result can be written as follows,

$$\int_{\mathbb{R}^N} \|x - a\|^2 |f(x)|^2 dx \int_{\mathbb{R}^N} \left\|w - bB^T\right\|^2 \left|\widehat{f_M}(w)\right|^2 dw \geq \frac{N^2}{16\pi^2}\|f\|_2^4 \lambda_{min}\left(BB^T\right).$$

Since the relationship $\lambda_{min}(BB^T) = \sigma^2_{min}(B)$, the right of the inequality can be simplified as $\frac{N^2}{16\pi^2}\|f\|_2^4 \sigma^2_{min}(B)$. The equality holds if and only if

$$\lambda_{min}(BB^T) = \lambda_{max}(BB^T) = \sigma^2(B).$$

It is easy to see that if $\int_{\mathbb{R}^N} \|x - a\|^2 |f(x)|^2 dx$ is finite, then it is minimized at x^0, that is $\triangle x^2$; similarly, $\int_{\mathbb{R}^N} \|w - bB^T\|^2 \left|\widehat{f_M}(w)\right|^2 dw$ is minimized at $bB^T = u^{M,0}$, that is $\triangle u_M^2$. Therefore,

$$\triangle x^2 \triangle u_M^2 \geq \frac{N^2}{16\pi^2}\|f\|_2^4 \sigma^2_{min}(B).$$

From the above discussion, if $f(x)$ is an N-dimensional Gaussian function and

$$\lambda_{min}(BB^T) = \lambda_{max}(BB^T) = \sigma^2(B),$$

then the equality holds. □

In particular, the result of Theorem 1 is consistent with Reference [16] for the N-dimentional FRFT if $A = D = diag(\cos\theta_1, \cdots, \cos\theta_N)$, $B = -C = diag(\sin\theta_1, \cdots, \sin\theta_N)$.

3.2. Nazarov's UP

Measured by smallness of support, Nazarov's UP was first proposed by F.L. Nazarov in 1993 [17]. It argues what happens if a nonzero function and its Fourier transform are only small outside a compact set. We first introduce Nazarov's UP for the Fourier transform.

Proposition 1. *(Jaming [18]) There exists a constant C, such that for finite Lebesgue measurable sets S, $E \subset \mathbb{R}^N$ and for every $f \in L^2\left(\mathbb{R}^N\right)$,*

$$Ce^{Cmin\left(|S||E|,\ |S|^{1/N}w(E),\ w(S)|E|^{1/N}\right)}\left(\int_{\mathbb{R}^N\setminus S}|f(x)|^2dx + \int_{\mathbb{R}^N\setminus E}\left|\widehat{f}(u)\right|^2 du\right) \geq \|f\|_2^2$$

where $w(E)$ is the mean width of E. In geometry, the mean width is a measure of the "size" of a body.

Motivated by Proposition 1, we extend the Nazarov's UP to the free metaplectic transformation domain.

Theorem 2. *There exists a constant C, such that for finite Lebesgue measurable sets S, $E \subset \mathbb{R}^N$ and for every $f \in L^2\left(\mathbb{R}^N\right)$,*

$$Ce^{Cmin\left(|S||E|,|S|^{1/N}w(E),w(S)|E|^{1/N}\right)}\left(\int_{\mathbb{R}^N\setminus S}|f(x)|^2dx+\int_{\mathbb{R}^N\setminus(EB^T)}\left|\widehat{f}_M(u)\right|^2du\right)\geq\|f\|_2^2$$

where $w(E)$ is the mean width of E.

Proof. Let $w = uB^T$ and $C' = Ce^{Cmin\left(|S||E|,|S|^{1/N}w(E),w(S)|E|^{1/N}\right)}$. For $g(x)$, by Lemma 1, we have

$$\int_{\mathbb{R}^N}|g(x)|^2dx\leq C'\left(\int_{\mathbb{R}^N\setminus S}|g(x)|^2dx+\int_{\mathbb{R}^N\setminus E}|\widehat{g}(u)|^2du\right). \tag{9}$$

Substituting (4), (5) into (9), we obtain

$$\int_{\mathbb{R}^N}|f(x)|^2dx\leq C'\left(\int_{\mathbb{R}^N\setminus S}|f(x)|^2dx+|\det(B)|\int_{\mathbb{R}^N\setminus E}\left|\widehat{f}_M(uB^T)\right|^2du\right).$$

Therefore,

$$\int_{\mathbb{R}^N}|f(x)|^2dx\leq C'\left(\int_{\mathbb{R}^N\setminus S}|f(x)|^2dx+\int_{\mathbb{R}^N\setminus\left(EB^T\right)}\left|\widehat{f}_M(w)\right|^2dw\right),$$

which completes the proof. □

3.3. Donoho and Stark's UP

The classical uncertainty principle is based on the interpretation of the standard deviation $\triangle_f x$ as the size of the "essential support" of f. By considering other notions of the support, it lead to different versions of the uncertainty principle. Considering the concentration degree of energy in a certain area, the beautiful uncertainty principle in this case has been received by Donoho and Stark in Reference [7].

Definition 6. *(Donoho and Stark [7]) A function $f \in L^2\left(\mathbb{R}^N\right)$ is ϵ-concentrated on a measurable set $T \subseteq \mathbb{R}^N$, if*

$$\left(\int_{T^c}|f(x)|^2dx\right)^{1/2}\leq\epsilon\|f\|_2.$$

Here T^c is the complement of set T on $\mathbb{R}^N$. The following proposition is the Donoho and Stark's UP in fourier domain.

Proposition 2. *(Grochenig [9]) Suppose that $f \in L^2\left(\mathbb{R}^N\right)$ and $f \neq 0$ is ϵ_T-concentrated on $T \subseteq \mathbb{R}^N$ and $\hat{f}$ is ϵ_Ω-concentrated on $\Omega \subseteq \mathbb{R}^N$. Then*

$$|T||\Omega|\geq(1-\epsilon_T-\epsilon_\Omega)^2.$$

Similarly, it can be extended to the free metaplectic transformation domain.

Theorem 3. *Suppose that $f \in L^2\left(\mathbb{R}^N\right)$ and $f \neq 0$ is ϵ_T-concentrated on $T \subseteq \mathbb{R}^N$ and $\widehat{f}_M$ is ϵ_Ω-concentrated on $\Omega \subseteq \mathbb{R}^N$. Then*

$$|T||\Omega|\geq|\det(B)|\left(1-\epsilon_T-\epsilon_\Omega\right)^2.$$

Proof. Without loss of generality, we assume that T and Ω have finite measure. In this proof we introduce two operators that occur frequently in problems of band-limited functions. Let

$$P_Tf=\chi_T\cdot f$$

and

$$Q_\Omega f(x) = \mathcal{L}_{M^{-1}}\left(\chi_\Omega \cdot \widehat{f_M}\right)(x).$$

With this notation f is ϵ_T-concentrated on T if and only if

$$\|f - P_T f\|_2 \leq \epsilon_T \|f\|_2,$$

and $\widehat{f}_M$ is ϵ_Ω-concentrated on $\Omega \subseteq \mathbb{R}^N$ if and only if

$$\|f - Q_\Omega f\|_2 = \left\|\mathcal{L}_{M^{-1}}\left(\chi_{\Omega^c} \cdot \widehat{f}_M\right)\right\|_2 = \left\|\chi_{\Omega^c} \cdot \widehat{f}_M\right\|_2 \leq \epsilon_\Omega \|f\|_2.$$

Since $\|Q_\Omega\|_{op} \leq 1$, we obtain that

$$\|f - Q_\Omega P_T f\|_2 \leq \|f - Q_\Omega f\|_2 + \|Q_\Omega(f - P_T f)\|_2 \leq (\epsilon_\Omega + \epsilon_T)\|f\|_2.$$

Consequently

$$\|Q_\Omega P_T f\|_2 \geq \|f\|_2 - \|f - Q_\Omega P_T f\|_2 \geq (1 - \epsilon_T - \epsilon_\Omega)\,\|f\|_2. \tag{10}$$

Next we compute the integral kernel and the Hilbert-Schmidt norm of $Q_\Omega P_T$.

Let $h(w) = \chi_\Omega(w) \cdot e^{\pi i w[DB^{-1} - (DB^{-1})^T]w^T}$, then

$$\begin{aligned}
& Q_\Omega P_T f(x) \\
= \;& \mathcal{L}_{M^{-1}}\left(\chi_\Omega \cdot (\widehat{P_T f})_M\right)(x) \\
= \;& \int_\Omega \int_T \frac{1}{|\det(B)|} f(t) e^{\pi i t B^{-1} A t^T} e^{-\pi i x (B^{-1}A)^T x^T} \\
& e^{\pi i w[DB^{-1} - (DB^{-1})^T]w^T} e^{-2\pi i w (B^T)^{-1}(t-x)^T} dt dw. \\
= \;& \int_{\mathbb{R}^N} \int_T \frac{1}{|\det(B)|} h(w) f(t) e^{\pi i t B^{-1} A t^T} e^{-\pi i x (B^{-1}A)^T x^T} e^{-2\pi i w (B^T)^{-1}(t-x)^T} dt dw.
\end{aligned}$$

It follows from that the inverse formula of the free metaplectic transformation. Clearly, $f \in L^2(T) \subseteq L^1(T)$. Since both T and Ω have finite measure, the double integral converges absolutely. By Fubini's theorem, the order of integration can be exchanged, that is,

$$Q_\Omega P_T f(x) = \int_{\mathbb{R}^N} f(t) K(x,t) dt,$$

where the kernel

$$K(x,t) = \frac{\chi_T(t)}{|\det(B)|} e^{\pi i t B^{-1} A t^T - \pi i x (B^{-1}A)^T x^T} \widehat{h}\left((t-x)B^{-1}\right).$$

The Hilbert-Schmidt norm (Appendix A) of $Q_\Omega P_T$ is given by

$$\|Q_\Omega P_T\|_{HS}^2 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |K(x,t)|^2 dx dt.$$

So we have

$$\begin{aligned}\int_{\mathbb{R}^N} |K(x,t)|^2 dx &= \frac{\chi_T(t)}{\det^2(B)} \int_{\mathbb{R}^N} \left|\widehat{h}\left((t-x)B^{-1}\right)\right|^2 dx \\ &= \frac{\chi_T(t)}{\det^2(B)} \int_{\mathbb{R}^N} \left|\widehat{h}(uB^{-1})\right|^2 du \\ &= \frac{\chi_T(t)}{|\det(B)|} \int_{\mathbb{R}^N} \left|\widehat{h}(w)\right|^2 dw \\ &= \frac{\chi_T(t)}{|\det(B)|} |\Omega|,\end{aligned}$$

therefore

$$\|Q_\Omega P_T\|_{HS}^2 = \frac{|\Omega|}{|\det(B)|} \int_{\mathbb{R}^N} \chi_T(t) dt = \frac{1}{|\det(B)|} |T||\Omega|. \tag{11}$$

Finally, by (10), (11) and the operator norm $\|Q_\Omega P_T\|_{op}$ is dominated by the Hilbert Schmidt norm, we have

$$\begin{aligned}(1-\epsilon_T-\epsilon_\Omega)^2 \|f\|_2^2 &\leq \|Q_\Omega P_T f\|_2^2 \\ &\leq \|Q_\Omega P_T\|_{op}^2 \|f\|_2^2 \\ &\leq \|Q_\Omega P_T\|_{HS}^2 \|f\|_2^2 \\ &= \frac{1}{|\det(B)|} |T||\Omega| \|f\|_2^2.\end{aligned}$$

Consequently,

$$|T||\Omega| \geq |\det(B)| (1-\epsilon_T-\epsilon_\Omega)^2.$$

□

3.4. Hardy's UP

Hardy's UP was first introduced by G.H. Hardy in 1933 [19]. Its localization is measured by fast decrease of a function and its Fourier transform. Hardy's UP in the Fourier transform domain [20] was given as follows.

Proposition 3. *(Escauriaza, Kenig, Ponce and Vega [20]) If* $f(x) = \mathcal{O}(e^{-|x|^2/\beta^2})$, $\hat{f}(\omega) = \mathcal{O}((2\pi)^{N/2} e^{-16\pi^2|\omega|^2/\alpha^2})$ *and* $1/\alpha\beta > 1/4$, *then* $f \equiv 0$. *Also, if* $1/\alpha\beta = 1/4$, *then*

$$f = Ce^{-|x|^2/\beta^2}.$$

where C is a constant, $C \in \mathbb{C}$.

Based on Proposition 2, we derive the corresponding Hardy's UP for the free metaplectic transformation.

Theorem 4. *If* $f(x) = \mathcal{O}(e^{-|x|^2/\beta^2})$, $\widehat{f_M}(u) = \mathcal{O}((2\pi)^{N/2} e^{-16\pi^2|u(B^T)^{-1}|^2/\alpha^2})$ *and* $1/\alpha\beta > 1/4$, *then* $f \equiv 0$. *Also, if* $1/\alpha\beta = 1/4$, *then*

$$f = Ce^{-|x|^2/\beta^2 - \pi i x B^{-1} A x^T},$$

where C is a constant, $C \in \mathbb{C}$.

Proof. Let $g(x) = f(x)e^{\pi i x B^{-1} A x^T}$. By (4) and (5), we have

$$|g(x)| = |f(x)| = \mathcal{O}(e^{-|x|^2/\beta^2}),$$

and

$$|\widehat{g}(u)| = \sqrt{|\det(B)|}\left|\widehat{f_M}(uB^T)\right| = \mathcal{O}((2\pi)^{N/2}e^{-16\pi^2|u|^2/\alpha^2}).$$

Since $g(x)$ satisfies Proposition 2 and $1/\alpha\beta > 1/4$, we have $g \equiv 0$. The equation $f \equiv 0$ is immediately obtained. Also, if $1/\alpha\beta = 1/4$, then $g(x)$ is a constant multiple of $e^{-|x|^2/\beta^2}$. Further,

$$f(x) = Ce^{-|x|^2/\beta^2 - \pi i x B^{-1} A x^T}.$$

This completes the proof. □

Clearly, the Proposition 2 can be received if $A = \mathbf{0}$ and $B = E$ in Theorem 4.

3.5. Beurling's UP

Beurling's UP is a variant of Hardy's UP. It implies the weak form of Hardy's UP immediately. Beurling's UP in the Fourier transform domain is as follows.

Proposition 4. *(Bonami, Demange and Jaming [21]) Let $f \in L^2(\mathbb{R}^N)$ and $n \geq 0$. Then*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)||\hat{f}(y)|}{(1+\|x\|+\|y\|)^n} e^{2\pi|\langle x,y\rangle|}dxdy < +\infty.$$

if and only if f may be written as

$$f(x) = P(x)e^{-\pi\langle Ax,x\rangle},$$

where A is a real positive definite symmetric matrix and P is a polynomial of degree$< \frac{n-N}{2}$.

Here a polynomial of degree means that the degree of the highest degree term of a polynomial.

In particular, for $n \leq N$, the function f is identically 0. Beurling-Hörmander's original theorem is the above theorem for $N = 1$ and $n = 0$. An extension to $N \geq 1$ but still $n = 0$ first has been given by S.C. Bagchi and S. K. Ray in Reference [22] in a weaker form. Then S. K. Ray and E. Naranayan give in the present form. Next, we formulate the Beurling's UP for the free metaplectic transformation domain.

Theorem 5. *Let $f \in L^2(\mathbb{R}^N)$ and $n \geq 0$. Then*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)|\left|\widehat{f_M}(\omega)\right|}{\sqrt{\det(B)}(1+\|x\|+\|\omega\tilde{B}\|)^n} e^{2\pi|\langle x,\omega\tilde{B}\rangle|}dxd\omega < +\infty.$$

if and only if f may be written as

$$f(x) = P(x)e^{-\pi i x B^{-1} A x^T - \pi\langle Ax,x\rangle},$$

where A is a real positive definite symmetric matrix and P is a polynomial of degree$< \frac{n-N}{2}$.

Proof. Let $g(x) = f(x)e^{\pi i x B^{-1} A x^T}$. By (4), (5) and the Proposition 4, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g(x)||\hat{g}(u)|}{(1+\|x\|+\|u\|)^n} e^{2\pi|\langle x,u\rangle|}dxdu \\
= &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)||\sqrt{|\det(B)|}\left|\widehat{f_M}(uB^T)\right|}{(1+\|x\|+\|u\|)^n} e^{2\pi|\langle x,u\rangle|}dxdu \\
= &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)||\widehat{f_M}(\omega)|}{\sqrt{\det(B)}(1+\|x\|+\|\omega(B^T)^{-1}\|)^n} e^{2\pi|\langle x,\omega\tilde{B}\rangle|}dxd\omega < +\infty
\end{aligned}$$

if and only if $g(x) = P(x)e^{-\pi\langle Ax,x\rangle}$. Furthermore, $f(x) = P(x)e^{-\pi ixB^{-1}Ax^T-\pi\langle Ax,x\rangle}$. □

From Theorem 5, we know that it is not possible for a nonzero function f and its free metaplectic transformation to decrease very rapidly simultaneously.

3.6. Logarithmic's UP

In Reference [23], a simple argument based on a sharp form of Pitt's inequality is used to obtain a logarithmic estimate of uncertainty.

Proposition 5. *(Beckner [23]) For $f \in \mathcal{S}(\mathbb{R}^N)$ and $0 \leq \alpha < N$*

$$\int_{\mathbb{R}^N} |\xi|^{-\alpha}|\hat{f}(\xi)|^2 d\xi \leq C_\alpha \int_{\mathbb{R}^N} |x|^\alpha |f(x)|^2 dx,$$

where

$$C_\alpha = \pi^\alpha [\Gamma(\frac{N-\alpha}{4})/\Gamma(\frac{N+\alpha}{4})]^2. \tag{12}$$

Here $\mathcal{S}(\mathbb{R}^N)$ denotes the Schwartz class.

Therefore, the Pitt's inequality can be extended to the free metaplectic transformation domain. In the following, we will denote $(B^T)^{-1}$ by $\tilde{B}$.

Theorem 6. *For $f \in \mathcal{S}(\mathbb{R}^N)$ and $0 \leq \alpha < N$*

$$\int_{\mathbb{R}^N} |\omega\tilde{B}|^{-\alpha}|\widehat{f}_M(\omega)|^2 d\omega \leq C_\alpha \int_{\mathbb{R}^N} |x|^\alpha |f(x)|^2 dx$$

where C_α is given by (12).

Proof. Let $g(x) = f(x)e^{\pi ixB^{-1}Ax^T}$, then $g(x)$ satisfies the condition of Proposition 5, then

$$\int_{\mathbb{R}^N} |u|^{-\alpha}|\hat{g}(u)|^2 du \leq C_\alpha \int_{\mathbb{R}^N} |x|^\alpha |g(x)|^2 dx. \tag{13}$$

Put (4), (5) into (13), we have

$$|\det(B)| \int_{\mathbb{R}^N} |u|^{-\alpha}|\widehat{f_M}(uB^T)|^2 du \leq C_\alpha \int_{\mathbb{R}^N} |x|^\alpha |f(x)|^2 dx.$$

Let $\omega = uB^T$, then

$$\int_{\mathbb{R}^N} |\omega\tilde{B}|^{-\alpha}|\widehat{f_M}(\omega)|^2 d\omega \leq C_\alpha \int_{\mathbb{R}^N} |x|^\alpha |f(x)|^2 dx.$$

□

Based on the generalized Pitt's inequality for the free metaplectic transformation proposed in Theorem 6, we investigate the logarithmic UP associated with the free metaplectic transformation.

Theorem 7. *Let $f \in \mathcal{S}(\mathbb{R}^N)$ and $\|f\| = 1$, then*

$$\int_{\mathbb{R}^N} \ln|\omega\tilde{B}||\widehat{f_M}(\omega)|^2 d\omega + \int_{\mathbb{R}^N} \ln|x||f(x)|^2 dx \geq \frac{\Gamma'(N/4)}{\Gamma(N/4)} - \ln\pi. \tag{14}$$

Proof. Let $H(\alpha) \int_{\mathbb{R}^N} |\omega\tilde{B}|^{-\alpha} \left|\widehat{f_M}(\omega)\right|^2 d\omega - C_\alpha \int_{\mathbb{R}^N} |x|^\alpha |f(x)|^2 dx$, where C_α is given by (12). Taking the derivative of $H(\alpha)$ about the variable α, we obtain

$$H'(\alpha) = -\int_{\mathbb{R}^N} \ln|\omega\tilde{B}| \, |\omega\tilde{B}|^{-\alpha} \left|\widehat{f_M}(\omega)\right|^2 d\omega - C_\alpha \int_{\mathbb{R}^N} |x|^\alpha \ln|x| |f(x)|^2 dx - C'_\alpha \int_{\mathbb{R}^N} |x|^\alpha |f(x)|^2 dx.$$

where

$$C'_\alpha = \pi^\alpha \ln\pi [\frac{\Gamma(\frac{N-\alpha}{4})}{\Gamma(\frac{N+\alpha}{4})}]^2 - \frac{\pi^\alpha}{2} \frac{\Gamma(\frac{N-\alpha}{4})\Gamma'(\frac{N-\alpha}{4}) + \Gamma^2(\frac{N-\alpha}{4})\Gamma'(\frac{N+\alpha}{4})/\Gamma(\frac{N+\alpha}{4})}{\Gamma^2(\frac{N+\alpha}{4})}.$$

By Theorem 6, we know $H(\alpha) \leq 0$ for $0 \leq \alpha < N$. From assumptions and Lemma 2,

$$H(0) = \int_{\mathbb{R}^N} \left|\widehat{f_M}(\omega)\right|^2 d\omega - \int_{\mathbb{R}^N} |f(x)|^2 dx = 0.$$

We get

$$H'(0+) \leq 0.$$

Consequently,

$$\int_{\mathbb{R}^N} \ln|\omega\tilde{B}| \left|\widehat{f_M}(\omega)\right|^2 + \int_{\mathbb{R}^N} \ln|x| |f(x)|^2 dx \geq \frac{\Gamma'(N/4)}{\Gamma(N/4)} - \ln\pi.$$

□

3.7. Entropic UP

Entropic UP is a fundamental tool in information theory, quantum physical and harmonic analysis. Its localization is measured by Shannon entropy. Let ρ be a probability density function on $\mathbb{R}^N$. The Shannon entropy of ρ is denoted by

$$E(\rho) = -\int_{\mathbb{R}^N} \rho(t) \ln\rho(t) dt. \tag{15}$$

We revisit entropic UP associated with the Fourier transform in following proposition.

Proposition 6. *(Folland, Sitaram [24]) If $f \in L^2(\mathbb{R}^N)$ and $\|f\|_2 = 1$, we have*

$$E(|f|^2) + E(|\hat{f}|^2) \geq N(1 - \ln 2),$$

whenever the left side is well defined.

Next, we propose the entropic UP in the free metaplectic transformation domain.

Theorem 8. *If $f \in L^2(\mathbb{R}^N)$ and $\|f\|_2 = 1$, we have*

$$E(|f|^2) + E(|\widehat{f_M}|^2) \geq N(1 - \ln 2) + \ln|\det(B)|.$$

Proof. Let $g(x) = f(x)e^{\pi i x B^{-1} A x^T}$, then $g(x) \in L^2(\mathbb{R}^N)$. Thus

$$-\int_{\mathbb{R}^N} |g(x)|^2 \ln|g(x)|^2 dx - \int_{\mathbb{R}^N} |\hat{g}(u)|^2 \ln|\hat{g}(u)|^2 du \geq N(1 - \ln 2) \tag{16}$$

by Proposition 6. Substitute (4), (5) into (16),

$$-\int_{\mathbb{R}^N} |f(x)|^2 \ln|f(x)|^2 dx - \int_{\mathbb{R}^N} |\det(B)| \left|\widehat{f_M}(uB^T)\right|^2 \ln\left(|\det(B)| \left|\widehat{f_M}(uB^T)\right|^2\right) du \geq N(1 - \ln 2).$$

Let $\omega = uB^T$, then

$$-\int_{\mathbb{R}^N} |f(x)|^2 \ln |f(x)|^2 dx - \int_{\mathbb{R}^N} \left|\widehat{f_M}(\omega)\right|^2 \ln\left(|\det(B)| \left|\widehat{f_M}(\omega)\right|^2\right) d\omega \geq N(1-\ln 2).$$

That is,

$$\begin{aligned} &-\int_{\mathbb{R}^N} |f(x)|^2 \ln |f(x)|^2 dx - \int_{\mathbb{R}^N} |\widehat{f_M}(\omega)|^2 \ln |\widehat{f_M}(\omega)|^2 d\omega \\ &\geq N(1-\ln 2) + \ln|\det(B)| \int_{\mathbb{R}^N} |\widehat{f_M}(\omega)|^2 d\omega. \end{aligned} \tag{17}$$

By Lemma 2,

$$\int_{\mathbb{R}^N} \left|\widehat{f_M}(\omega)\right|^2 d\omega = \int_{\mathbb{R}^N} |f(x)|^2 dx = 1. \tag{18}$$

Put (18) into (17), we have

$$-\int_{\mathbb{R}^N} |f(x)|^2 \ln |f(x)|^2 dx - \int_{\mathbb{R}^N} \left|\widehat{f_M}(\omega)\right|^2 \ln \left|\widehat{f_M}(\omega)\right|^2 d\omega \geq N(1-\ln 2) + \ln|\det(B)|.$$

□

4. Concluding Remarks

Our paper mainly studies the UP for the free metaplectic transformation of $f \in L^2(\mathbb{R}^N)$. We first give the relationship between the Fourier transformation and the free metaplectic transformation by Lemmas 1 and 2, then we study the UP for the free metaplectic transformation with seven different forms. In Section 3, we first show that it can not give the exact location in both f and its free metaplectic transformation by Theorem 1 . Then we give the Nazarov's UP, Donoho and Stark's UP, Hardy's UP, Beurling's UP, Logarithmic UP, and Entropic UP in the free metaplectic transformation domain by Theorems 2–8.

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Appendix A

Hilbert-Schmidt Operators. (Grochenig [9])A bounded operator $A : \mathcal{H} \to \mathcal{H}$ is called a Hilbert-Schmidt operator if $\sum_{n=1}^{\infty} \|Ae_n\|_{\mathcal{H}}^2 < \infty$ for some orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of $\mathcal{H}$. The Hilbert-Schmidt norm of A is given by

$$\|A\|_{H.S.} = \left(\sum_{n=1}^{\infty} \|Ae_n\|_{\mathcal{H}}^2\right)^{1/2},$$

and this quantity is independent of the choice of the orthonormal basis $\{e_n\}$. If A is an integral operator with kernel k, that is, $Af(x) = \int_{\mathbb{R}^d} k(x,y) f(y) dy$, then A is Hilbert-Schmidt if and only if $k \in L^2(\mathbb{R}^{2d})$, and in this case $\|A\|_{H.S.} = \|k\|_2$.

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