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Distance and Similarity Measures for Octahedron Sets and Their Application to MCGDM Problems

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Abstract: In this paper, in order to apply the concept of octahedron sets to multi-criteria group decision-making problems, we define several similarity and distance measures for octahedron sets. We present a multi-criteria group decision-making method with linguistic variables in octahedron set environment. We give a numerical example for multi-criteria group decision-making problems.

Keywords: octahedron set; distance measure; similarity measure

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1. Introduction

In real world, we frequently encounter with decision-making problems with uncertainty and vagueness that can be difficult to solve with the classical methods. A number of techniques have been developed to solve uncertinities; similarity measures are one of tools solving decision-making problems. Chen and Hsiao [1] studied some similarity measures for fuzzy sets introduced by Zadeh [2]. Pramanik and Mondal [3] defined the concept of weighted fuzzy similarity measure (called a tangent similarity measure) and applied it to medical diagnosis. Hwang and Yang [4] made a new similarity measure for intuitionistic fuzzy sets proposed by Atanassov [5]. Pramanik and Mondal [6] proposed intuitionistic fuzzy similarity measure based on tangent function and applied it to multi–attribute decision. Ren and Wang [7] introduced the notion of similarity measures for interval-valued intuitionstic fuzzy sets proposed by Atanassov and Gargov [8]. Baroumi and Smarandache [9] dealt with several similarity measures between neutrosophic sets and applied them to decision-making problems introduced by Smarandache [10]. Ye [11] defined a similarity measure for interval neutrosophic sets and applied it to decision-making method. Sahin and Liu [12] introduced various distance and similarity measures between single-valued neutrosophic hesitant fuzzy sets (see Reference [13,14]) and discussed MADM problems based on the single-valued neutrosophic hesitant fuzzy information. Kaur and Garg [15,16] applied it to decision-making and studied cubic intuitionistic fuzzy aggregation operators (see Reference [17,18]). Pramanik et al. [19] proposed a similarity measure for cubic neutrosophic sets and applied it a multi-criteria group decision-making (MCGDM) method.

Recently, Kim et al. [20] defined an octahedron set composed of an interval-valued fuzzy set, an intuitionistic set and a fuzzy set that will provide more information about uncertainty and vagueness. The purpose of this paper is to review recent research into the octahedron set applications with MCGDM method. This paper proposes a new methodology for the theory of octahedron set with defining several distance and similarity measures between octahedron sets. Moreover, we prove that each is distance and similarity measure. This new application offers some important insights into octahedron sets with MCGDM method based on a similarity measure for octahedron sets. The findings reported here shed new light on an application area of octahedron sets. This approach will prove useful in expanding our understanding of how the application of octahedron sets can be done. The findings of this study could have a number of important implications for future practice with analyzing our numerical example for MCGDM problems that is given in this paper. In order to apply the concept of octahedron sets to MCGDM problems, this paper is presented as follows: In Section 2, we list some basic notions that are needed in next section. In Section 3, we propose several distance and similarity measures between octahedron sets and prove that each is distance and similarity measure. In Section 4, we present a MCGDM method based on a similarity measure for octahedron sets environments. In addition, we give a numerical example for MCGDM problems to demonstrate the usefulness and applicability of our proposed method. There is a relatively small body of literature that is concerned with the application of soft octahedron sets. In recent years, there has been an increasing amount of literature on the application of set theories. One of the aim of this research was to improve the works on the application of octahedron sets by proposing suitable examples in this paper. All these findings will provide a base to researchers who want to work in the field of the application of octahedron sets and will help to strengthen the foundations of the other MCGDM problems in octahedron set environment, such as economic policy, foreign policy between countries, trade policy, financial policy, etc., by using big data.

2. Preliminaries

Let $I \oplus I = \{\bar{a} = (a^{\in}, a^{\notin}) \in I \times I : a^{\in} + a^{\notin} \leq 1\}$, where $I = [0, 1]$. Then, each member $\bar{a}$ of $I \oplus I$ is called an intuitionistic point or intuitionistic number. In particular, we denote $(0, 1)$ and $(1, 0)$ as $\bar{0}$ and $\bar{1}$, respectively. Refer to Reference [21] for the definitions of the order $(\leq)$ and the equality $(=)$ of two intuitionistic numbers, and the infimum and the supremum of any intuitionistic numbers.

Definition 1 (Reference [5]). *For a nonempty set X, a mapping $A : X \to I \oplus I$ is called an intuitionistic fuzzy set (briefly, IF set) in X, where, for each $x \in X$, $A(x) = (A^{\in}(x), A^{\notin}(x))$, and $A^{\in}(x)$ and $A^{\notin}(x)$ represent the degree of membership and the degree of nonmembership of an element x to A, respectively. Let $(I \oplus I)^X$ denote the set of all IF sets in X and for each $A \in (I \oplus I)^X$, we write $A = (A^{\in}, A^{\notin})$. In particular, $\bar{\mathbf{0}}$ and $\bar{\mathbf{1}}$ denote the IF empty set and the IF whole set in X defined by, respectively:*

For each $x \in X$,

$$\bar{\mathbf{0}}(x) = \bar{0} \text{ and } \bar{\mathbf{1}}(x) = \bar{1}.$$

Refer to Reference [5] for the definitions of the inclusion, the equality, the intersection, and the union of intuitionistic fuzzy set and operations ^{c}, $[\,]$, $\diamond$ on $(I \oplus I)^X$.

The set of all closed subintervals of I is denoted by $[I]$, and members of $[I]$ are called interval numbers and are denoted by $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$,, etc., where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$. In particular, if $a^- = a^+$, then we write as $\widetilde{a} = \mathbf{a}$. Refer to Reference [20] for the definitions of the order and the equality of two interval numbers, as well as the infimum and the supremum of any interval numbers.

Definition 2 (Reference [22,23]). *For a nonempty set X, a mapping $A : X \to [I]$ is called an interval-valued fuzzy set (briefly, an IVF set) in X. Let $[I]^X$ denote the set of all IVF sets in X. For each $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the degree of membership of an element x to A, where $A^-, A^+ \in I^X$ are called a lower fuzzy set and an upper fuzzy set in X, respectively. For each $A \in [I]^X$, we write $A = [A^-, A^+]$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy empty whole set in X defined by, respectively: for each $x \in X$,*

$$\widetilde{0}(x) = \mathbf{0} \text{ and } \widetilde{1}(x) = \mathbf{1}.$$

Refer to Reference [22,23] for the definitions of the inclusion, the equality, the intersection, and the union of intuitionistic fuzzy set and operation ^{c} on $[I]^X$.

Now, members of $[I] \times (I \oplus I) \times I$ are written by $\widetilde{\widetilde{a}} =< \widetilde{a}, \bar{a}, a >=< [a^-, a^-], (a^{\in}, a^{\notin}), a >$, $\widetilde{\widetilde{b}} =< \widetilde{b}, \bar{b}, b >=< [b^-, b^-], (b^{\in}, b^{\notin}), b >$, etc., and are called octahedron numbers. Furthermore, we define the following order relations between $\widetilde{\widetilde{a}}$ and $\widetilde{\widetilde{b}}$ (see Reference [20]):

(i) (Equality) $\widetilde{\widetilde{a}} = \widetilde{\widetilde{b}} \Leftrightarrow \widetilde{a} = \widetilde{b},\ \bar{a} = \bar{b},\ a = b$,
(ii) (Type 1-order) $\widetilde{\widetilde{a}} \leq_1 \widetilde{\widetilde{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \leq b^+,\ a^{\in} \leq b^{\in},\ a^{\notin} \geq b^{\notin},\ a \leq b$,
(iii) (Type 2-order) $\widetilde{\widetilde{a}} \leq_2 \widetilde{\widetilde{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \leq b^+,\ a^{\in} \leq b^{\in},\ a^{\notin} \geq b^{\notin},\ a \geq b$,
(iv) (Type 3-order) $\widetilde{\widetilde{a}} \leq_3 \widetilde{\widetilde{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \geq b^+,\ a^{\in} \geq b^{\in},\ a^{\notin} \leq b^{\notin},\ a \leq b$,
(v) (Type 4-order) $\widetilde{\widetilde{a}} \leq_4 \widetilde{\widetilde{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \leq b^+,\ a^{\in} \geq b^{\in},\ a^{\notin} \leq b^{\notin},\ a \geq b$.

From the above orders, we can define the inf and the sup of octahedron numbers as follows.

Definition 3 (Reference [20])**.** *Let* $\widetilde{\widetilde{a}},\ \widetilde{\widetilde{b}} \in [I] \times (I \oplus I) \times I$. *Then,*

(i) $\widetilde{\widetilde{a}} \wedge^1 \widetilde{\widetilde{b}} = \left\langle [a^- \wedge b^-, a^+ \wedge b^+], (a^{\in} \wedge b^{\in}, a^{\notin} \vee b^{\notin}), a \wedge b \right\rangle,$

$\widetilde{\widetilde{a}} \wedge^2 \widetilde{\widetilde{b}} = \left\langle [a^- \wedge b^-, a^+ \wedge b^+], (a^{\in} \wedge b^{\in}, a^{\notin} \vee b^{\notin}), a \vee b \right\rangle,$

$\widetilde{\widetilde{a}} \wedge^3 \widetilde{\widetilde{b}} = \left\langle [a^- \wedge b^-, a^+ \wedge b^+], (a^{\in} \vee b^{\in}, a^{\notin} \wedge b^{\notin}), a \wedge b \right\rangle,$

$\widetilde{\widetilde{a}} \wedge^4 \widetilde{\widetilde{b}} = \left\langle [a^- \wedge b^-, a^+ \wedge b^+], (a^{\in} \vee b^{\in}, a^{\notin} \wedge b^{\notin}), a \vee b \right\rangle,$

(ii) $\widetilde{\widetilde{a}} \vee^1 \widetilde{\widetilde{b}} = \left\langle [a^- \vee b^-, a^+ \vee b^+], (a^{\in} \vee b^{\in}, a^{\notin} \wedge b^{\notin}), a \vee b \right\rangle,$

$\widetilde{\widetilde{a}} \vee^2 \widetilde{\widetilde{b}} = \left\langle [a^- \vee b^-, a^+ \vee b^+], (a^{\in} \vee b^{\in}, a^{\notin} \wedge b^{\notin}), a \wedge b \right\rangle,$

$\widetilde{\widetilde{a}} \vee^3 \widetilde{\widetilde{b}} = \left\langle [a^- \vee b^-, a^+ \vee b^+], (a^{\in} \wedge b^{\in}, a^{\notin} \vee b^{\notin}), a \vee b \right\rangle,$

$\widetilde{\widetilde{a}} \vee^4 \widetilde{\widetilde{b}} = \left\langle [a^- \vee b^-, a^+ \vee b^+], (a^{\in} \wedge b^{\in}, a^{\notin} \vee b^{\notin}), a \wedge b \right\rangle.$

Definition 4 (Reference [20])**.** *Let X be a nonempty set, and let $\mathbf{A} = [A^-, A^+] \in [I]^X$, $A = (A^{\in}, A^{\notin}) \in (I \oplus I)^X$, $\lambda \in I^X$. Then, the triple $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$ is called an octahedron set in X. In fact, $\mathcal{A} : X \to [I] \times (I \oplus I) \times I$ is a mapping. In particular, the octahedron empty (resp. whole) set in X, denoted by $\ddot{0}$ (resp. $\ddot{1}$), is an octahedron set in X defined by:*

$$\ddot{0} = \left\langle \widetilde{0}, \bar{\mathbf{0}}, 0 \right\rangle,\ \ddot{1} = \left\langle \widetilde{1}, \bar{\mathbf{1}}, 1 \right\rangle.$$

It is obvious that, for each $A \in 2^X$, $\chi_{\mathcal{A}} = \left\langle [\chi_A, \chi_A], (\chi_A, \chi_{A^c}), \chi_A \right\rangle \in \mathcal{O}(X)$ and then $2^X \subset \mathcal{O}(X)$, where 2^X denotes the set of all subsets of X and χ_A denotes the characteristic function of A. Furthermore, we can easily see that, for each $\mathbf{A} = \langle A, \lambda \rangle \in \mathcal{C}(X)$, $\mathbf{A} = \langle A, (A^-, A^+), \lambda \rangle$, $\mathbf{A} = \langle A, (\lambda, \lambda^c), \lambda \rangle \in \mathcal{O}(X)$ and then $\mathcal{C}(X) \subset \mathcal{O}(X)$. In this case, we denote $\langle A, (A^-, A^+), \lambda \rangle$ and $\langle A, (\lambda, \lambda^c), \lambda \rangle$ as $\mathcal{A}_A$ and $\mathcal{A}_\lambda$, respectively. In fact, we can consider octahedron sets as a generalization of cubic sets.

Definition 5 (Reference [20])**.** *Let X be a nonempty set, and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in \mathcal{O}(X)$. Then, we can define following order relations between $\mathcal{A}$ and $\mathcal{B}$:*

(i) (Equality) $\mathcal{A} = \mathcal{B} \Leftrightarrow \mathbf{A} = \mathbf{B},\ A = B,\ \lambda = \mu$,
(ii) (Type 1-order) $\mathcal{A} \subset_1 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \subset B, \lambda \leq \mu$,
(iii) (Type 2-order) $\mathcal{A} \subset_2 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \subset B, \lambda \geq \mu$,
(iv) (Type 3-order) $\mathcal{A} \subset_3 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \supset B, \lambda \leq \mu$,
(v) (Type 4-order) $\mathcal{A} \subset_4 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \supset B, \lambda \geq \mu$.

Definition 6 (Reference [20]). *Let X be a nonempty set, and let* $(\mathcal{A}_j)_{j\in J} = (\langle \mathbf{A}_j, A_j, \lambda_j\rangle)_{j\in J}$ *be a family of octahedron sets in X. Then, the Type i-union* $\cup^i$ *and Type i-intersection* $\cap^i$ *of* $(\mathcal{A}_j)_{j\in J}$, $(i = 1, 2, ,3, 4)$, *are defined as follows, respectively:*

(i) (Type i-union) $\bigcup^1_{j\in J} \mathcal{A}_j = \left\langle \bigcup_{j\in J} \mathbf{A}_j, \bigcup_{j\in J} A_j, \bigcup_{j\in J} \lambda_j \right\rangle,$

$$\bigcup^2_{j\in J} \mathcal{A}_j = \left\langle \bigcup_{j\in J} \mathbf{A}_j, \bigcup_{j\in J} A_j, \bigcap_{j\in J} \lambda_j \right\rangle,$$

$$\bigcup^3_{j\in J} \mathcal{A}_j = \left\langle \bigcup_{j\in J} \mathbf{A}_j, \bigcap_{j\in J} A_j, \bigcup_{j\in J} \lambda_j \right\rangle,$$

$$\bigcup^4_{j\in J} \mathcal{A}_j = \left\langle \bigcup_{j\in J} \mathbf{A}_j, \bigcap_{j\in J} A_j, \bigcap_{j\in J} \lambda_j \right\rangle,$$

(ii) (Type i-intersection) $\bigcap^1_{j\in J} \mathcal{A}_j = \left\langle \bigcap_{j\in J} \mathbf{A}_j, \bigcap_{j\in J} A_j, \bigcap_{j\in J} \lambda_j \right\rangle,$

$$\bigcap^2_{j\in J} \mathcal{A}_j = \left\langle \bigcap_{j\in J} \mathbf{A}_j, \bigcap_{j\in J} A_j, \bigcup_{j\in J} \lambda_j \right\rangle,$$

$$\bigcap^3_{j\in J} \mathcal{A}_j = \left\langle \bigcap_{j\in J} \mathbf{A}_j, \bigcup_{j\in J} A_j, \bigcap_{j\in J} \lambda_j \right\rangle,$$

$$\bigcap^4_{j\in J} \mathcal{A}_j = \left\langle \bigcap_{j\in J} \mathbf{A}_j, \bigcup_{j\in J} A_j, \bigcup_{j\in J} \lambda_j \right\rangle.$$

Let $\mathbb{R}$ be the real space. Then, for any intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ of $\mathbb{R}$, the Hausdorff distance $d_H(A, B)$ between A and B is defined by:

$$d_H(A, B) = \max(\mid a_1 - b_1 \mid, \mid a_2 - b_2 \mid). \tag{1}$$

Throughout this paper, let $X = \{x_1, x_2, x_3, \cdots, x_n\}$ be a universal set.

3. Distance and Similarity Measures between Octahedron Sets

Definition 7. *A mapping* $d : \mathcal{O}(X) \times \mathcal{O}(X) \to I$ *is called a distance measure on* $\mathcal{O}(X)$, *if it satisfies the following conditions: For any* $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{O}(X)$,

(DM_1) $0 \le d(\mathcal{A}, \mathcal{B}) \le 1$,
(DM_2) $d(\mathcal{A}, \mathcal{B}) = 0$ *if and only if* $\mathcal{A} = \mathcal{B}$,
(DM_3) $d(\mathcal{A}, \mathcal{B}) = d(\mathcal{B}, \mathcal{A})$,
(DM_4) *if* $\mathcal{A} \subset_1 \mathcal{B} \subset_1 \mathcal{C}$, *then* $d(\mathcal{A}, \mathcal{C}) \ge d(\mathcal{A}, \mathcal{B}) \vee d(\mathcal{B}, \mathcal{C})$.

In this case, $d(\mathcal{A}, \mathcal{B})$ *is called the distance measure between* $\mathcal{A}$ *and* $\mathcal{B}$.

Definition 8. *A mapping* $s : \mathcal{O}(X) \times \mathcal{O}(X) \to I$ *is called a similarity measure on* $\mathcal{O}(X)$, *if it satisfies the following conditions: For any* $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{O}(X)$,

(DM_1) $0 \le s(\mathcal{A}, \mathcal{B}) \le 1$,
(DM_2) $s(\mathcal{A}, \mathcal{B}) = 1$ *if and only if* $\mathcal{A} = \mathcal{B}$,
(DM_3) $s(\mathcal{A}, \mathcal{B}) = s(\mathcal{B}, \mathcal{A})$,
(DM_4) *if* $\mathcal{A} \subset_1 \mathcal{B} \subset_1 \mathcal{C}$, *then* $s(\mathcal{A}, \mathcal{C}) \le s(\mathcal{A}, \mathcal{B}) \wedge s(\mathcal{B}, \mathcal{C})$.

In this case, $d(\mathcal{A}, \mathcal{B})$ *is called the similarity measure between* $\mathcal{A}$ *and* $\mathcal{B}$.

In fact, distance measure and similarity measure from Definitions 7 and 8, we can easily see that $s(\mathcal{A}, \mathcal{B}) = 1 - d(\mathcal{A}, \mathcal{B})$.

Now, we give some types of distance measures between two octahedron sets in the following:

Example 1. *(1) (Generalized normalized distance measure) Let $\delta > 0$. We define $d_{GN} : \mathcal{O}(X) \times \mathcal{O}(X) \to I$ is the mapping defined as: For any $\mathcal{A}$, $\mathcal{B} \in \mathcal{O}(X)$,*

$$\begin{aligned} d_{GN}(\mathcal{A},\mathcal{B}) &= [\tfrac{1}{5n}\Sigma_{i=1}^{n}(\mid A^{-}(x_i) - B^{-}(x_i) \mid^{\delta} + \mid A^{+}(x_i) - B^{+}(x_i) \mid^{\delta} \\ &+ \mid A^{\in}(x_i) - B^{\in}(x_i) \mid^{\delta} + \mid A^{\notin}(x_i) - B^{\notin}(x_i) \mid^{\delta} \\ &+ \mid \lambda(x_i) - \mu(x_i) \mid^{\delta})]^{\frac{1}{\delta}}. \end{aligned}$$

Then, d_{GN} is a distance measure on $\mathcal{O}(X)$ (see Propositions 1 and 2).

In particular, if $\delta = 1$, then d_{GN} reduces an octahedron normalized Hamming distance and denoted by d_{NH}:

$$\begin{aligned} d_{NH}(\mathcal{A},\mathcal{B}) &= \tfrac{1}{5n}\Sigma_{i=1}^{n}(\mid A^{-}(x_i) - B^{-}(x_i) \mid + \mid A^{+}(x_i) - B^{+}(x_i) \mid \\ &+ \mid A^{\in}(x_i) - B^{\in}(x_i) \mid + \mid A^{\notin}(x_i) - B^{\notin}(x_i) \mid \\ &+ \mid \lambda(x_i) - \mu(x_i) \mid). \end{aligned}$$

If $\delta = 2$, then d_{GN} reduces an octahedron normalized Euclidean distance and denoted by d_{NE}:

$$\begin{aligned} d_{NE}(\mathcal{A},\mathcal{B}) &= [\tfrac{1}{5n}\Sigma_{i=1}^{n}(\mid A^{-}(x_i) - B^{-}(x_i) \mid^{2} + \mid A^{+}(x_i) - B^{+}(x_i) \mid^{2} \\ &+ \mid A^{\in}(x_i) - B^{\in}(x_i) \mid^{2} + \mid A^{\notin}(x_i) - B^{\notin}(x_i) \mid^{2} \\ &+ \mid \lambda(x_i) - \mu(x_i) \mid^{2})]^{\frac{1}{2}}. \end{aligned}$$

In fact, d_{GN} can be viewed as a most generalized case of distance measures.

(2) (Generalized octahedron normalized Hausdorff distance) We define $d_{GNH} : \mathcal{O}(X) \times \mathcal{O}(X) \to I$ is the mapping defined as: for any $\mathcal{A}$, $\mathcal{B} \in \mathcal{O}(X)$,

$$\begin{aligned} d_{GNH}(\mathcal{A},\mathcal{B}) &= [\tfrac{1}{n}\Sigma_{i=1}^{n}\max(\mid A^{-}(x_i) - B^{-}(x_i) \mid^{\delta}, \mid A^{+}(x_i) - B^{+}(x_i) \mid^{\delta}, \\ &\mid A^{\in}(x_i) - B^{\in}(x_i) \mid^{\delta}, \mid A^{\notin}(x_i) - B^{\notin}(x_i) \mid^{\delta}, \\ &\mid \lambda(x_i) - \mu(x_i) \mid^{\delta})]^{\frac{1}{\delta}}. \end{aligned}$$

Then, d_{GNH} is a distance measure on $\mathcal{O}(X)$ (see Propositions 3 and 4).

In particular, if $\delta = 1$, then d_{GNH} reduces an octahedron normalized Hamming-Hausdorff distance and denoted by d_{NHH}:

$$\begin{aligned} d_{NHH}(\mathcal{A},\mathcal{B}) &= \tfrac{1}{n}\Sigma_{i=1}^{n}\max(\mid A^{-}(x_i) - B^{-}(x_i) \mid, \mid A^{+}(x_i) - B^{+}(x_i) \mid, \\ &\mid A^{\in}(x_i) - B^{\in}(x_i) \mid, \mid A^{\notin}(x_i) - B^{\notin}(x_i) \mid, \\ &\mid \lambda(x_i) - \mu(x_i) \mid). \end{aligned}$$

If $\delta = 2$, then d_{GNH} reduces an octahedron normalized Euclidean-Hausdorff distance and denoted by d_{NEH}:

$$\begin{aligned} d_{NEH}(\mathcal{A},\mathcal{B}) &= [\tfrac{1}{n}\Sigma_{i=1}^{n}\max(\mid A^{-}(x_i) - B^{-}(x_i) \mid^{2}, \mid A^{+}(x_i) - B^{+}(x_i) \mid^{2}, \\ &\mid A^{\in}(x_i) - B^{\in}(x_i) \mid^{2}, \mid A^{\notin}(x_i) - B^{\notin}(x_i) \mid^{2}, \\ &\mid \lambda(x_i) - \mu(x_i) \mid^{2})]^{\frac{1}{2}}. \end{aligned}$$

In many practical situations, the weight of each element $x_i \in X$ should be taken into account. For example, in MADM problems, since the considered attribute has different importance in general, we need to be assigned with different weights. Since an octahedron set has three types of degree (an interval-valued fuzzy membership degree, an intuitionistic fuzzy membership degree and a fuzzy membership degree) and each degree may have different importance according to a decision-maker, different weights can be assigned to each element in each degree. We assume that the weights

$\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ with $\omega_i \in I$, $\Sigma_{i=1}^{n}\omega_i = 1$; $\eta = (\eta_1, \eta_2, \cdots, \eta_n)^T$ with $\eta_i \in I$, $\Sigma_{i=1}^{n}\eta_i = 1$; $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^T$ with $\xi_i \in I$, $\Sigma_{i=1}^{n}\xi_i = 1$ denote the weights assigned to interval-valued fuzzy membership degree, intuitionistic fuzzy membership degree and fuzzy membership degree of an octahedron set.

Now, we give some types of weighted distance measures between two octahedron sets in the following:

Example 2. *(1) (Generalized octahedron weighted distance measure) Let $\delta > 0$. We define $d_{GW} : \mathcal{O}(X) \times \mathcal{O}(X) \to I$ is the mapping defined as: for any $\mathcal{A}$, $\mathcal{B} \in \mathcal{O}(X)$,*

$$\begin{aligned} d_{GW}(\mathcal{A},\mathcal{B}) \ &= [\tfrac{1}{5n}\Sigma_{i=1}^{n}(\omega_i \mid A^-(x_i) - B^-(x_i) \mid^\delta + \omega_i \mid A^+(x_i) - B^+(x_i) \mid^\delta \\ &+\eta_i \mid A^\in(x_i) - B^\in(x_i) \mid^\delta + \eta_i \mid A^\notin(x_i) - B^\notin(x_i) \mid^\delta \\ &+\xi_i \mid \lambda(x_i) - \mu(x_i) \mid^\delta)]^{\frac{1}{\delta}}. \end{aligned}$$

Then, d_{GW} is a weighted distance measure on $\mathcal{O}(X)$ (The proof is omitted).

In particular, if $\delta = 1$, then d_{GW} reduces an octahedron weighted Hamming distance and denoted by d_{WH}:

$$\begin{aligned} d_{WH}(\mathcal{A},\mathcal{B}) \ &= \tfrac{1}{5n}\Sigma_{i=1}^{n}(\omega_i \mid A^-(x_i) - B^-(x_i) \mid + \omega_i \mid A^+(x_i) - B^+(x_i) \mid \\ &+\eta_i \mid A^\in(x_i) - B^\in(x_i) \mid + \eta_i \mid A^\notin(x_i) - B^\notin(x_i) \mid \\ &+\xi_i \mid \lambda(x_i) - \mu(x_i) \mid). \end{aligned}$$

If $\delta = 2$, then d_{GW} reduces an octahedron weighted Euclidean distance and denoted by d_{WE}:

$$\begin{aligned} d_{WE}(\mathcal{A},\mathcal{B}) \ &= [\tfrac{1}{5n}\Sigma_{i=1}^{n}(\omega_i \mid A^-(x_i) - B^-(x_i) \mid^2 + \omega_i \mid A^+(x_i) - B^+(x_i) \mid^2 \\ &+\eta_i \mid A^\in(x_i) - B^\in(x_i) \mid^2 + \eta_i \mid A^\notin(x_i) - B^\notin(x_i) \mid^2 \\ &+\xi_i \mid \lambda(x_i) - \mu(x_i) \mid^2)]^{\frac{1}{2}}. \end{aligned}$$

(2) (Generalized octahedron weighted Hausdorff distance) We define $d_{GWH} : \mathcal{O}(X) \times \mathcal{O}(X) \to I$ is the mapping defined as: For any $\mathcal{A}$, $\mathcal{B} \in \mathcal{O}(X)$,

$$\begin{aligned} d_{GWH}(\mathcal{A},\mathcal{B}) \ &= [\tfrac{1}{n}\Sigma_{i=1}^{n}\max(\omega_i \mid A^-(x_i) - B^-(x_i) \mid^\delta, \omega_i \mid A^+(x_i) - B^+(x_i) \mid^\delta, \\ &\eta_i \mid A^\in(x_i) - B^\in(x_i) \mid^\delta, \eta_i \mid A^\notin(x_i) - B^\notin(x_i) \mid^\delta, \\ &\xi_i \mid \lambda(x_i) - \mu(x_i) \mid^\delta)]^{\frac{1}{\delta}}. \end{aligned}$$

Then, d_{GWH} is a weighted distance measure on $\mathcal{O}(X)$ (The proof is omitted).

In particular, if $\delta = 1$, then d_{GWH} reduces an octahedron weighted Hamming-Hausdorff distance and denoted by d_{WHH}:

$$\begin{aligned} d_{WHH}(\mathcal{A},\mathcal{B}) \ &= \tfrac{1}{n}\Sigma_{i=1}^{n}\max(\omega_i \mid A^-(x_i) - B^-(x_i) \mid, \omega_i \mid A^+(x_i) - B^+(x_i) \mid, \\ &\eta_i \mid A^\in(x_i) - B^\in(x_i) \mid, \eta_i \mid A^\notin(x_i) - B^\notin(x_i) \mid, \\ &\xi_i \mid \lambda(x_i) - \mu(x_i) \mid). \end{aligned}$$

If $\delta = 2$, then d_{GWH} reduces an octahedron weighted Euclidean-Hausdorff distance and denoted by d_{WEH}:

$$\begin{aligned} d_{WEH}(\mathcal{A},\mathcal{B}) \ &= [\tfrac{1}{n}\Sigma_{i=1}^{n}\max(\omega_i \mid A^-(x_i) - B^-(x_i) \mid^2, \omega_i \mid A^+(x_i) - B^+(x_i) \mid^2, \\ &\eta_i \mid A^\in(x_i) - B^\in(x_i) \mid^2, \eta_i \mid A^\notin(x_i) - B^\notin(x_i) \mid^2, \\ &\xi_i \mid \lambda(x_i) - \mu(x_i) \mid^2)]^{\frac{1}{2}}. \end{aligned}$$

Proposition 1. *The mapping d_{NH} defined in Example 1 (1) is a distance measure on $\mathcal{O}(X)$.*

Proof. Since the proofs of (DM_2) and (DM_3) are easy from the definition of d_{NH}, we will show only (DM_1) and (DM_4).

(DM_1) Let $\mathcal{A}, \mathcal{B} \in \mathcal{O}(X)$. Then, by the definition of d_{NH},

$$\begin{array}{l} \mid A^-(x_i) - B^-(x_i) \mid \geq 0, \ \mid A^+(x_i) - B^+(x_i) \mid \geq 0, \\ \mid A^{\in}(x_i) - B^{\in}(x_i) \mid \geq 0, \ \mid A^{\notin}(x_i) - B^{\notin}(x_i) \mid \geq 0, \mid \lambda(x_i) - \mu(x_i) \mid \geq 0. \end{array}$$

Thus, we have

$$\begin{array}{rl} d_{NH}(\mathcal{A}, \mathcal{B}) & = \frac{1}{5n}\Sigma_{i=1}^n(\mid A^-(x_i) - B^-(x_i) \mid + \mid A^+(x_i) - B^+(x_i) \mid \\ & + \mid A^{\in}(x_i) - B^{\in}(x_i) \mid + \mid A^{\notin}(x_i) - B^{\notin}(x_i) \mid \\ & + \mid \lambda(x_i) - \mu(x_i) \mid) \\ & \geq 0. \end{array}$$

In addition, by the definition of d_{NH},

$$\begin{array}{l} \mid A^-(x_i) - B^-(x_i) \mid \leq 1, \ \mid A^+(x_i) - B^+(x_i) \mid \leq 1, \\ \mid A^{\in}(x_i) - B^{\in}(x_i) \mid \leq 1, \ \mid A^{\notin}(x_i) - B^{\notin}(x_i) \mid \leq 1, \mid \lambda(x_i) - \mu(x_i) \mid \leq 1. \end{array}$$

So, we get

$$\begin{array}{rl} d_{NH}(\mathcal{A}, \mathcal{B}) & = \frac{1}{5n}\Sigma_{i=1}^n(\mid A^-(x_i) - B^-(x_i) \mid + \mid A^+(x_i) - B^+(x_i) \mid \\ & + \mid A^{\in}(x_i) - B^{\in}(x_i) \mid + \mid A^{\notin}(x_i) - B^{\notin}(x_i) \mid \\ & + \mid \lambda(x_i) - \mu(x_i) \mid) \\ & \leq 1. \end{array}$$

Hence, $0 \leq d_{NH}(\mathcal{A}, \mathcal{B}) \leq 1$.

(DM_4) Suppose $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle$, $\mathcal{C} = \langle \mathbf{C}, C, \nu \rangle \in \mathcal{O}(X)$ such that $\mathcal{A} \subset_1 \mathcal{B} \subset_1 \mathcal{C}$, and let $x_i \in X$. Then, we have

$$\mathcal{A}(x_i) \leq_1 \mathcal{B}(x_i) \leq_1 \mathcal{C}(x_i), \text{ i.e.,}$$

$$\begin{array}{l} A^-(x_i) \leq B^-(x_i) \leq C^-(x_i), \ A^+(x_i) \leq B^+(x_i) \leq C^+(x_i), \\ A^{\in}(x_i) \leq B^{\in}(x_i) \leq C^{\in}(x_i), \ A^{\notin}(x_i) \geq B^{\notin}(x_i) \geq C^{\notin}(x_i), \\ \lambda(x_i) \leq \mu(x_i) \leq \nu(x_i). \end{array}$$

Thus, we get

$$\begin{array}{rl} d_{NH}(\mathcal{A}, \mathcal{C}) & = \frac{1}{5n}\Sigma_{i=1}^n(\mid A^-(x_i) - C^-(x_i) \mid + \mid A^+(x_i) - C^+(x_i) \mid \\ & + \mid A^{\in}(x_i) - C^{\in}(x_i) \mid + \mid A^{\notin}(x_i) - C^{\notin}(x_i) \mid \\ & + \mid \lambda(x_i) - \nu(x_i) \mid) \\ & \geq \frac{1}{n}\Sigma_{i=1}^n(\mid A^-(x_i) - B^-(x_i) \mid + \mid A^+(x_i) - B^+(x_i) \mid \\ & + \mid A^{\in}(x_i) - B^{\in}(x_i) \mid + \mid A^{\notin}(x_i) - B^{\notin}(x_i) \mid \\ & + \mid \lambda(x_i) - \mu(x_i) \mid) \\ & = d_{NH}(\mathcal{A}, \mathcal{B}). \end{array}$$

Similarly, we have $d_{NH}(\mathcal{A}, \mathcal{C}) \geq d_H(\mathcal{B}, \mathcal{C})$. So, $d_{NH}(\mathcal{A}, \mathcal{C}) \geq d_{NH}(\mathcal{A}, \mathcal{C}) \vee d_{NH}(\mathcal{B}, \mathcal{C})$. This completes the proof. □

Proposition 2. *Two mappings d_{GN} and d_{NE} defined in Example 1 (1) are distance measure on $\mathcal{O}(X)$.*

Proof. The proofs are similar to the proof of Proposition 1. □

Proposition 3. *A mapping d_G and d_{GNH} defined in Example 1 (2) are distance measure on $\mathcal{O}(X)$.*

Proof. Since the proofs of (DM_2) and (DM_3) are easy from the definition of d_{GNH}, we will show only (DM_1) and (DM_4).

(DM_1) Let $\mathcal{A},\ \mathcal{B} \in \mathcal{O}(X)$. Then, by the definition of d_{GNH}, we can easily obtain the following:

$$\max(\mid A^-(x_i) - B^-(x_i) \mid^\delta,\ \mid A^+(x_i) - B^+(x_i) \mid^\delta,\ \mid A^\in(x_i) - B^\in(x_i) \mid^\delta,$$
$$\mid A^\notin(x_i) - B^\notin(x_i) \mid^\delta,\ \mid \lambda(x_i) - \mu(x_i) \mid^\delta) \geq 0.$$

Thus, $d_{GNH}(\mathcal{A},\mathcal{B}) \geq 0$. Similarly, we can easily prove that $d_{GNH}(\mathcal{A},\mathcal{B}) \leq 1$. So, $0 \leq d_{GNH}(\mathcal{A},\mathcal{B}) \leq 1$.

(DM_4) Suppose $\mathcal{A},\ \mathcal{B},\ \mathcal{C} \in \mathcal{O}(X)$ such that $\mathcal{A} \subset_1 \mathcal{B} \subset_1 \mathcal{C}$, and let $x_i \in X$. Then, we have

$$A^-(x_i) \leq B^-(x_i) \leq C^-(x_i),\ A^+(x_i) \leq B^+(x_i) \leq C^+(x_i),$$
$$A^\in(x_i) \leq B^\in(x_i) \leq C^\in(x_i),\ A^\notin(x_i) \geq B^\notin(x_i) \geq C^\notin(x_i),$$
$$\lambda(x_i) \leq \mu(x_i) \leq \nu(x_i).$$

Thus, we get

$$\max(\mid A^-(x_i) - C^-(x_i) \mid^\delta,\ \mid A^+(x_i) - C^+(x_i) \mid^\delta,\ \mid A^\in(x_i) - C^\in(x_i) \mid^\delta,$$
$$\mid A^\notin(x_i) - C^\notin(x_i) \mid^\delta,\ \mid \lambda(x_i) - \nu(x_i) \mid^\delta)$$
$$\geq \max(\mid A^-(x_i) - B^-(x_i) \mid^\delta,\ \mid A^+(x_i) - B^+(x_i) \mid^\delta,\ \mid A^\in(x_i) - B^\in(x_i) \mid^\delta,$$
$$\mid A^\notin(x_i) - B^\notin(x_i) \mid^\delta,\ \mid \lambda(x_i) - \mu(x_i) \mid^\delta).$$

So, $d_{GNH}(\mathcal{A},\mathcal{C}) \geq d_H(\mathcal{A},\mathcal{B})$. Similarly, we get $d_{GNH}(\mathcal{A},\mathcal{C}) \geq d_{GNH}(\mathcal{B},\mathcal{C})$. Hence, $d_{GNH}(\mathcal{A},\mathcal{C}) \geq d_{GNH}(\mathcal{A},\mathcal{B}) \vee d_{GNH}(\mathcal{B},\mathcal{C})$. This completes the proof. □

Proposition 4. *Two mappings d_{NHH} and d_{NEH} defined in Example 1 (2) are distance measure on $\mathcal{O}(X)$.*

Proof. The proofs are similar to the proof of Proposition 3. □

Remark 1. *In Definition 7, although the condition (DM_4) is changed into the following:*
(DM_4') if $\mathcal{A} \subset_i \mathcal{B} \subset_i \mathcal{C}$ ($i = 1,\ 2,\ 3,\ 4$), then $d(\mathcal{A},\mathcal{C}) \geq d(\mathcal{A},\mathcal{B}) \vee d(\mathcal{B},\mathcal{C})$,
we can easily see that all the distance measures given in Examples 1 and 2 satisfy the condition (DM_4').

Now, from the relationships between distance measures and similarity measures, we can give some examples of similarity measures on $\mathcal{O}(X)$.

Example 3. *Let $\mathcal{A},\ \mathcal{B} \in \mathcal{O}(X)$.*

(1) (Generalized octahedron similarity measure corresponding to d_{GN})

$$s_{GN}(\mathcal{A},\mathcal{B}) = 1 - d_{GN}(\mathcal{A},\mathcal{B}). \tag{2}$$

In fact,

$$\begin{aligned} s_{GN}(\mathcal{A},\mathcal{B}) &= 1 - [\tfrac{1}{5n}\Sigma_{i=1}^n(\mid A^-(x_i) - B^-(x_i) \mid^\delta + \mid A^+(x_i) - B^+(x_i) \mid^\delta \\ &+ \mid A^\in(x_i) - B^\in(x_i) \mid^\delta + \mid A^\notin(x_i) - B^\notin(x_i) \mid^\delta \\ &+ \mid \lambda(x_i) - \mu(x_i) \mid^\delta)]^{\frac{1}{\delta}}. \end{aligned}$$

(Octahedron similarity measure corresponding to d_{NH})

$$s_{NH}(\mathcal{A},\mathcal{B}) = 1 - d_{NH}(\mathcal{A},\mathcal{B}). \tag{3}$$

In fact,

$$\begin{aligned} s_{NH}(\mathcal{A},\mathcal{B}) &= \tfrac{1}{n}\Sigma_{i=1}^{n}[1-\tfrac{1}{5}(\mid A^{-}(x_i)-B^{-}(x_i)\mid + \mid A^{+}(x_i)-B^{+}(x_i)\mid \\ &+ \mid A^{\in}(x_i)-B^{\in}(x_i)\mid + \mid A^{\notin}(x_i)-B^{\notin}(x_i)\mid \\ &+ \mid \lambda(x_i)-\mu(x_i)\mid)]. \end{aligned}$$

(Octahedron similarity measure corresponding to d_{NE})

$$s_{NE}(\mathcal{A},\mathcal{B}) = 1 - d_{NE}(\mathcal{A},\mathcal{B}). \tag{4}$$

In fact,

$$\begin{aligned} s_{NE}(\mathcal{A},\mathcal{B}) &= 1-[\tfrac{1}{5n}\Sigma_{i=1}^{n}(\mid A^{-}(x_i)-B^{-}(x_i)\mid^2 + \mid A^{+}(x_i)-B^{+}(x_i)\mid^2 \\ &+ \mid A^{\in}(x_i)-B^{\in}(x_i)\mid^2 + \mid A^{\notin}(x_i)-B^{\notin}(x_i)\mid^2 \\ &+ \mid \lambda(x_i)-\mu(x_i)\mid^2)]^{\frac{1}{2}}. \end{aligned}$$

(2) (Generalized octahedron similarity measure corresponding to d_{GNH})

$$s_{GNH}(\mathcal{A},\mathcal{B}) = 1 - d_{GNH}(\mathcal{A},\mathcal{B}). \tag{5}$$

In fact,

$$\begin{aligned} s_{GNH}(\mathcal{A},\mathcal{B}) &= 1-[\tfrac{1}{n}\Sigma_{i=1}^{n}\max(\mid A^{-}(x_i)-B^{-}(x_i)\mid^{\delta}, \mid A^{+}(x_i)-B^{+}(x_i)\mid^{\delta}, \\ &\mid A^{\in}(x_i)-B^{\in}(x_i)\mid^{\delta}, \mid A^{\notin}(x_i)-B^{\notin}(x_i)\mid^{\delta} \\ &\mid \lambda(x_i)-\mu(x_i)\mid^{\delta})]^{\frac{1}{\delta}}. \end{aligned}$$

(Octahedron similarity measure corresponding to d_{NHH})

$$s_{NHH}(\mathcal{A},\mathcal{B}) = 1 - d_{NHH}(\mathcal{A},\mathcal{B}). \tag{6}$$

In fact,

$$\begin{aligned} s_{NHH}(\mathcal{A},\mathcal{B}) &= \tfrac{1}{n}\Sigma_{i=1}^{n}[1-\max(\mid A^{-}(x_i)-B^{-}(x_i)\mid, \mid A^{+}(x_i)-B^{+}(x_i)\mid, \\ &\mid A^{\in}(x_i)-B^{\in}(x_i)\mid, \mid A^{\notin}(x_i)-B^{\notin}(x_i)\mid, \\ &\mid \lambda(x_i)-\mu(x_i)\mid)]. \end{aligned}$$

(Octahedron similarity measure corresponding to d_{NEH})

$$s_{NEH}(\mathcal{A},\mathcal{B}) = 1 - d_{NEH}(\mathcal{A},\mathcal{B}). \tag{7}$$

In fact,

$$\begin{aligned} s_{NEH}(\mathcal{A},\mathcal{B}) &= 1-[\tfrac{1}{n}\Sigma_{i=1}^{n}\max(\mid A^{-}(x_i)-B^{-}(x_i)\mid^2, \mid A^{+}(x_i)-B^{+}(x_i)\mid^2, \\ &\mid A^{\in}(x_i)-B^{\in}(x_i)\mid^2, \mid A^{\notin}(x_i)-B^{\notin}(x_i)\mid^2, \\ &\mid \lambda(x_i)-\mu(x_i)\mid^2)]^{\frac{1}{2}}. \end{aligned}$$

(3) (Generalized octahedron similarity measure corresponding to d_{GW})

$$s_{GW}(\mathcal{A},\mathcal{B}) = 1 - d_{GW}(\mathcal{A},\mathcal{B}). \tag{8}$$

In fact,

$$\begin{aligned} s_{GW}(\mathcal{A},\mathcal{B}) \;&= 1-[\tfrac{1}{5n}\Sigma_{i=1}^{n}(\omega_i \mid A^-(x_i)-B^-(x_i) \mid^\delta +\omega_i \mid A^+(x_i)-B^+(x_i) \mid^\delta \\ &+\eta_i \mid A^\in(x_i)-B^\in(x_i) \mid^\delta +\eta_i \mid A^\notin(x_i)-B^\notin(x_i) \mid^\delta \\ &+\xi_i \mid \lambda(x_i)-\mu(x_i) \mid^\delta)]^{\frac{1}{\delta}}. \end{aligned}$$

(Octahedron similarity measure corresponding to d_{WH})

$$s_{WH}(\mathcal{A},\mathcal{B}) = 1 - d_{WH}(\mathcal{A},\mathcal{B}). \tag{9}$$

In fact,

$$\begin{aligned} s_{WH}(\mathcal{A},\mathcal{B}) \;&= \tfrac{1}{n}\Sigma_{i=1}^{n}[1-\tfrac{1}{5}(\omega_i \mid A^-(x_i)-B^-(x_i) \mid +\omega_i \mid A^+(x_i)-B^+(x_i) \mid \\ &+\eta_i \mid A^\in(x_i)-B^\in(x_i) \mid +\eta_i \mid A^\notin(x_i)-B^\notin(x_i) \mid \\ &+\xi_i \mid \lambda(x_i)-\mu(x_i) \mid)]. \end{aligned}$$

(Octahedron similarity measure corresponding to d_{WE})

$$s_{WE}(\mathcal{A},\mathcal{B}) = 1 - d_{WE}(\mathcal{A},\mathcal{B}). \tag{10}$$

In fact,

$$\begin{aligned} s_{WE}(\mathcal{A},\mathcal{B}) \;&= 1-[\tfrac{1}{5n}\Sigma_{i=1}^{n}(\omega_i \mid A^-(x_i)-B^-(x_i) \mid^2 +\omega_i \mid A^+(x_i)-B^+(x_i) \mid^2 \\ &+\eta_i \mid A^\in(x_i)-B^\in(x_i) \mid^2 +\eta_i \mid A^\notin(x_i)-B^\notin(x_i) \mid^2 \\ &+\xi_i \mid \lambda(x_i)-\mu(x_i) \mid^2)]^{\frac{1}{2}}. \end{aligned}$$

(4) (Generalized octahedron similarity measure corresponding to d_{GWH})

$$s_{GWH}(\mathcal{A},\mathcal{B}) = 1 - d_{GWH}(\mathcal{A},\mathcal{B}). \tag{11}$$

In fact,

$$\begin{aligned} s_{GWH}(\mathcal{A},\mathcal{B}) \;&= 1-[\tfrac{1}{n}\Sigma_{i=1}^{n}\max(\omega_i \mid A^-(x_i)-B^-(x_i) \mid^\delta, \omega_i \mid A^+(x_i)-B^+(x_i) \mid^\delta, \\ &\eta_i \mid A^\in(x_i)-B^\in(x_i) \mid^\delta, \eta_i \mid A^\notin(x_i)-B^\notin(x_i) \mid^\delta, \\ &\xi_i \mid \lambda(x_i)-\mu(x_i) \mid^\delta)]^{\frac{1}{\delta}}. \end{aligned}$$

(Octahedron similarity measure corresponding to d_{WHH})

$$s_{WHH}(\mathcal{A},\mathcal{B}) = 1 - d_{WHH}(\mathcal{A},\mathcal{B}). \tag{12}$$

In fact,

$$\begin{aligned} s_{WHH}(\mathcal{A},\mathcal{B}) \;&= \tfrac{1}{n}\Sigma_{i=1}^{n}[1-\max(\omega_i \mid A^-(x_i)-B^-(x_i) \mid, \omega_i \mid A^+(x_i)-B^+(x_i) \mid, \\ &\eta_i \mid A^\in(x_i)-B^\in(x_i) \mid, \eta_i \mid A^\notin(x_i)-B^\notin(x_i) \mid, \\ &\xi_i \mid \lambda(x_i)-\mu(x_i) \mid)]. \end{aligned}$$

(Octahedron similarity measure corresponding to d_{WEH})

$$s_{WEH}(\mathcal{A},\mathcal{B}) = 1 - d_{WEH}(\mathcal{A},\mathcal{B}). \tag{13}$$

In fact,

$$s_{WEH}(\mathcal{A},\mathcal{B}) = 1-[\frac{1}{n}\Sigma_{i=1}^{n}\max(\omega_i \mid A^-(x_i)-B^-(x_i)\mid^2, \omega_i \mid A^+(x_i)-B^+(x_i)\mid^2, \eta_i \mid A^{\in}(x_i)-B^{\in}(x_i)\mid^2, \eta_i \mid A^{\notin}(x_i)-B^{\notin}(x_i)\mid^2, \xi_i \mid \lambda(x_i)-\mu(x_i)\mid^2)]^{\frac{1}{2}}.$$

From Propositions 1–4, and the duality between distance measures and similarity measures, we can prove that (2)–(13) are similarity measures. But, we will show directly that (3) satisfies the conditions that are defined in Definition 8.

Proposition 5. s_{NH} *is a similarity measure for two octahedron sets* $\mathcal{A}$ *and* $\mathcal{B}$.

Proof. Let $\mathcal{A}, \mathcal{B} \in \mathcal{O}(X)$, and, for each $i = 1, 2, \cdots, n$, let

$$D_i = (\mid A^-(x_i)-B^-(x_i)\mid + \mid A^+(x_i)-B^+(x_i)\mid + \mid A^{\in}(x_i)-B^{\in}(x_i)\mid .$$

(i) $0 \le s_{NH}(\mathcal{A},\mathcal{B}) \le 1$.

Case 1. Suppose $D_i = 0$ or $D_i = 5$. Then, clearly, we have

$$s_{NH}(\mathcal{A},\mathcal{B}) = 1 \text{ or } s_{NH}(\mathcal{A},\mathcal{B}) = 0. \quad (14)$$

Case 2. Suppose $0 < D_i < 5$. Then, clearly, $0 < \frac{D_i}{5} < 1$. Thus, $0 < 1 - \frac{D_i}{5} < 1$. So,

$$0 = \frac{1}{n}\Sigma_{i=1}^{n}0 < \frac{1}{n}\Sigma_{i=1}^{n}(1-\frac{D_i}{5}) < \frac{1}{n}\Sigma_{i=1}^{n}1 = 1.$$

Hence,

$$0 < s_{NH}(\mathcal{A},\mathcal{B}) < 1. \quad (15)$$

Therefore, from (14) and (15), we get $0 \le s_{NH}(\mathcal{A},\mathcal{B}) \le 1$.

(ii) $s_{NH}(\mathcal{A},\mathcal{B}) = 1$ iff $\mathcal{A} = \mathcal{B}$.

$$\begin{aligned} & s_{NH}(\mathcal{A},\mathcal{B}) = 1 \\ \Leftrightarrow\ & \tfrac{1}{n}\Sigma_{i=1}^{n}(1-\tfrac{D_i}{5}) = 1 \\ \Leftrightarrow\ & D_i = 0 \\ \Leftrightarrow\ & \mid A^-(x_i)-B^-(x_i)\mid = 0, \mid A^+(x_i)-B^+(x_i)\mid = 0, \\ \Leftrightarrow\ & \mid A^{\in}(x_i)-B^{\in}(x_i)\mid = 0, \mid A^{\notin}(x_i)-B^{\notin}(x_i)\mid = 0, \mid \lambda(x_i)-\mu(x_i)\mid = 0 \\ \Leftrightarrow\ & \mathcal{A} = \mathcal{B}. \end{aligned}$$

(iii) $s_{NH}(\mathcal{A},\mathcal{B}) = s_{NH}(\mathcal{B},\mathcal{A})$. The proof is obvious from the property of "| |".

(iv) Let $\mathcal{A}, \mathcal{B}, \mathcal{B} \in \mathcal{O}(X)$ such that $\mathcal{A} \subset_1 \mathcal{B} \subset_1 \mathcal{C}$. Then,

$$s_{NH}(\mathcal{A},\mathcal{C}) \le s_{NH}(\mathcal{A},\mathcal{B}) \text{ and } s_{NH}(\mathcal{A},\mathcal{C}) \le s_{NH}(\mathcal{B},\mathcal{C}).$$

For each $i = 1, 2, \cdots, n$, let $x_i \in X$. Since $\mathcal{A} \subset_1 \mathcal{B} \subset_1 \mathcal{C}$, we have

$$\begin{aligned} & A^-(x_i) \le B^-(x_i) \le C^-(x_i),\ A^+(x_i) \le B^+(x_i) \le C^+(x_i), \\ & A^{\in}(x_i) \le B^{\in}(x_i) \le C^{\in}(x_i),\ A^{\notin}(x_i) \ge B^{\notin}(x_i) \ge C^{\notin}(x_i), \\ & \lambda(x_i) \le \mu(x_i) \le \nu((x_i)). \end{aligned}$$

Then, we get $D_i(\mathcal{A},\mathcal{C}) \geq D_i(\mathcal{A},\mathcal{B})$,
where

$$\begin{aligned} D_i(\mathcal{A},\mathcal{B}) &= (\mid A^-(x_i) - B^-(x_i) \mid + \mid A^+(x_i) - B^+(x_i) \mid \\ &+ \mid A^{\in}(x_i) - B^{\in}(x_i) \mid + \mid A^{\notin}(x_i) - B^{\notin}(x_i) \mid + \mid \lambda(x_i) - \mu(x_i) \mid) \end{aligned}$$

and

$$\begin{aligned} D_i(\mathcal{A},\mathcal{C}) &= (\mid A^-(x_i) - C^-(x_i) \mid + \mid A^+(x_i) - C^+(x_i) \mid \\ &+ \mid A^{\in}(x_i) - C^{\in}(x_i) \mid + \mid A^{\notin}(x_i) - C^{\notin}(x_i) \mid + \mid \lambda(x_i) - \nu(x_i) \mid). \end{aligned}$$

Then, clearly we can easily see that $D_i(\mathcal{A},\mathcal{C}) \geq D_i(\mathcal{A},\mathcal{B})$. Thus, we have

$$1 - \frac{1}{5} D_i(\mathcal{A},\mathcal{C}) \leq 1 - \frac{1}{5} D_i(\mathcal{A},\mathcal{B}).$$

So, we get $\frac{1}{n}\Sigma_{i=1}^{n}[1 - \frac{1}{5}D_i(\mathcal{A},\mathcal{C})] \leq \frac{1}{n}\Sigma_{i=1}^{n}[1 - \frac{1}{5}D_i(\mathcal{A},\mathcal{B})]$. Hence, we get

$$s_{NH}(\mathcal{A},\mathcal{C}) \leq s_{NH}(\mathcal{A},\mathcal{B}).$$

Similarly, we can prove that $s_{NH}(\mathcal{A},\mathcal{C}) \leq s_{NH}(\mathcal{B},\mathcal{C})$. Therefore, s_{NH} a similarity measure on $\mathcal{O}(X)$.

□

4. MCGDM Method Based on Similarity Measure in Octahedron Set Environment

In this section, we give a new method based on similarity measure in octahedron set environment. Assume that $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is a set of n alternatives with criteria $\beta = \{\beta_1, \beta_2, \cdots, \beta_m\}$, and let $\gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_r\}$ be the r decision-makers. Let $\delta = \{\delta_1, \delta_2, \cdots, \delta_r\}$ be the weight vector of decision-makers such that $\delta_k > 0$ and $\Sigma_{k=1}^{r}\delta_k = 1$. We propose MCGDM method presented using the following steps.

Step 1. Formation of ideal octahedron set decision matrix. Ideal octahedron set decision matrix is an important matrix for similarity measure of MCGDM given in the following form:

$$\begin{bmatrix} & \beta_1 & \beta_2 & \cdots & \beta_m \\ \alpha_1 & \mathcal{A}_{11} & \mathcal{A}_{12} & \cdots & \mathcal{A}_{1m} \\ \alpha_2 & \mathcal{A}_{21} & \mathcal{A}_{22} & \cdots & \mathcal{A}_{2m} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \alpha_n & \mathcal{A}_{n1} & \mathcal{A}_{n2} & \cdots & \mathcal{A}_{nm} \end{bmatrix}, \tag{16}$$

where $\mathcal{A}_{ij} = \langle \mathbf{A}_{ij}, A_{ij}, \lambda_{ij} \rangle$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$.

Step 2. Construction of octahedron set decision matrix. Since r decision-makers are involved in the decision-making process, the k-th ($k = 1, 2, \cdots, r$) decision-maker gives the evaluation information of the alternative α_i ($i = 1, 2, \cdots, n$) with respect to criteria β_j ($j = 1, 2, \cdots, m$) in terms of octahedron set. The k-th decision matrix, denoted by M^k, is constructed by the following matrix:

$$M^k = \left\langle \mathcal{A}_{ij}^k \right\rangle = \begin{bmatrix} & \beta_1 & \beta_2 & \cdots & \beta_m \\ \alpha_1 & \mathcal{A}_{11}^k & \mathcal{A}_{12}^k & \cdots & \mathcal{A}_{1m}^k \\ \alpha_2 & \mathcal{A}_{21}^k & \mathcal{A}_{22}^k & \cdots & \mathcal{A}_{2m}^k \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \alpha_n^k & \mathcal{A}_{n1}^k & \mathcal{A}_{n2}^k & \cdots & \mathcal{A}_{nm}^k \end{bmatrix}, \tag{17}$$

where $k = 1, 2, \cdots, r$, $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$.

Step 3. Determination of attribute weight. All attributes are not equally important in a decision-making situation. Every decision-maker provides their own opinion regarding to the attribute weight in terms of linguistic variables that can be converted into octahedron set. Let $w_k(\beta_j)$ denote the attribute weight for the attribute β_j given by the k-th decision-maker in terms of octahedron set. We convert into $w_k(\beta_j)$ into fuzzy number as follows:

$$w_k^F(\beta_j) = \begin{cases} [1-(\frac{V_{kj}}{5})^{\frac{1}{2}}] & \text{if } \beta_j \in \beta \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

where

$$\begin{aligned} V_{kj} &= [(1-A^{-}(\beta_j))^2 + (1-A^{+}(\beta_j))^2 + (1-A^{\in}(\beta_j))^2 + (A^{\notin}((\beta_j))^2 \\ &+(1-\lambda(\beta_j))^2]^{\frac{1}{2}} \end{aligned},$$

and each of the above values denote the value of the octahedron set corresponding to (k, β_j).

Then, aggregate weight for the criteria β_j can be determined as follows:

$$W_j = \frac{[1-\Pi_{k=1}^{r}(1-w_k^F(\beta_j))]}{\Sigma_{k=1}^{r}[1-\Pi_{k=1}^{r}(1-w_k^F(\beta_j))]}, \tag{19}$$

where $\Sigma_{k=1}^{r} W_j = 1$.

Step 4. Calculation of weighted similarity measure. We calculate weighted similarity measure between the ideal matrix M and the k-th decision matrix M^k as follows:

$$s_{NH}^{W}(M, M^k) = \left\langle \lambda_i^k \right\rangle = (\lambda_1^k, \lambda_2^k, \cdots, \lambda_n^k)^T = \left[\frac{1}{m}\Sigma_{j=1}^{m}(1-\frac{D_{ij}^k}{5})W_j\right]_{i=1}^{n}, \tag{20}$$

where $D_{ij}^k = \mid A_{ij}^{-}(x_r) - A_{ij}^{k,-}(x_r) \mid + \mid A_{ij}^{+}(x_r) - A_{ij}^{k,+}(x_r) \mid + \mid A_{ij}^{\in}(x_r) - A_{ij}^{k,\in}(x_r) \mid$
$+ \mid A_{ij}^{\notin}(x_r) - A_{ij}^{k,\notin}(x_r) \mid + \mid \lambda(x_r) - \lambda(x_r) \mid$ for each $x_r \in X$ and $k = 1, 2, \cdots, r$.

Step 5. Ranking of alternatives. In order to rank alternatives, we give the following formula:

$$\rho_i = \Sigma_{k=1}^{r} \delta_k \lambda_i^k, \tag{21}$$

where $i = 1, 2, \cdots, n$.

We can arrange alternatives according to the descending order values of ρ_i. The highest value of ρ_i reflects the best alternative.

Example 4 (Numerical example). *In order to solve a MCGDM problem, we adapt "Illustrative example" given by Ye [11] to demonstrate the applicablility and effectiveness of the proposed method. Assume that an investment company wants to invest a sum of money in the best option. The investment company is composed of a decision-making committee comprised of three members, say k_1, k_2, k_3 to make a panel of four alternatives to invest money. The alternatives Car company (α_1), Food company (α_2), Computer company (α_3), and Arm company (α_4). Decision-makers take decision based on the criteria, namely risk analysis (β_1), growth analysis (β_2), environment impact (β_3), and criteria weights, which are given by the decision-makers in terms of linguistic variables that can be converted into octahedron set (see Table 1).*

Table 1. Linguistic term for rating of attribute/criterion.

Linguistic Terms	Octahedron Set
Very important (VI)	$\langle [0.7, 0.9], (0.7, 0.2), 0.9 \rangle$
Important (I)	$\langle [0.6, 0.8], (0.6, 0.3), 0.6 \rangle$
Medium (M)	$\langle [0.4, 0.5], (0.5, 0.4), 0.5 \rangle$
Unimportant (UI)	$\langle [0.2, 0.4], (0.3, 0.6), 0.4 \rangle$
Very unimportant (VUI)	$\langle [0.1, 0.2], (0.2, 0.7), 0.2 \rangle$

Step 1. Formation of ideal octahedron set decision matrix. *Ideal octahedron set decision matrix M is given as follows:*

$$M = \begin{bmatrix} & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle \\ \alpha_2 & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle \\ \alpha_3 & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle \\ \alpha_4 & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle & \langle [1,1],(1,0),1 \rangle \end{bmatrix} \tag{22}$$

Step 2. Construction of octahedron set decision matrix. *The k_i-th decision matrix M^{k_i} $(i = 1,\ 2,\ 3)$ in octahedron set form is constructed for four alternatives with respect to the three criteria by the following matrix:*

$$M^{k_1} = \begin{bmatrix} & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle \\ \alpha_2 & \langle [0.6,0.8],(0.6,0.3),0.8 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle \\ \alpha_3 & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.6,0.8],(0.6,0.3),0.8 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle \\ \alpha_4 & \langle [0.3,0.4],(0.4,0.5),0.4 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle \end{bmatrix},$$

$$M^{k_2} = \begin{bmatrix} & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \langle [0.3,0.4],(0.4,0.5),0.4 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle \\ \alpha_2 & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle \\ \alpha_3 & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle \\ \alpha_4 & \langle [0.6,0.8],(0.6,0.3),0.8 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle \end{bmatrix},$$

$$M^{k_3} = \begin{bmatrix} & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.4,0.5],(.5,.4),0.5 \rangle & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle \\ \alpha_2 & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.7,0.9],(.7,.2),0.9 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle \\ \alpha_3 & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle & \langle [0.6,0.8],(0.6,0.3),0.8 \rangle & \langle [0.6,0.8],(0.6,0.3),0.8 \rangle \\ \alpha_4 & \langle [0.7,0.9],(0.7,0.2),0.9 \rangle & \langle [0.4,0.5],(0.5,0.4),0.5 \rangle & \langle [0.3,0.4],(0.4,0.5),0.4 \rangle \end{bmatrix}.$$

Step 3. Determination of attribute weight. *Linguistic terms given in Table 1 are used to evaluate each attribute. The importance of each attribute for every decision-maker is rated with linguistic terms (see Table 2). Moreover, each linguistic term is converted into octahedron set (see Table 3).*

Table 2. Attribute rating linguistic variables.

	β_1	β_2	β_3
k_1	VI	M	I
k_2	VI	VI	M
k_3	M	VI	M

Table 3. Attribute rating in octahedron set.

	β_1	β_2	β_3
k_1	$\langle [0.7,0.9],(0.7,0.2),0.9 \rangle$	$\langle [0.4,0.5],(0.5,0.4),0.5 \rangle$	$\langle [0.6,0.8],(0.6,0.3),0.8 \rangle$
k_2	$\langle [0.7,0.9],(0.7,0.2),0.9 \rangle$	$\langle [0.7,0.9],(0.7,0.2),0.9 \rangle$	$\langle [0.4,0.5],(0.5,0.4),0.5 \rangle$
k_3	$\langle [0.4,0.5],(0.5,0.4),0.5 \rangle$	$\langle [0.7,0.9],(0.7,0.2),0.9 \rangle$	$\langle [0.4,0.5],(0.5,0.4),0.5 \rangle$

By using Equations (18) and (19), we get the following attribute weights:

$$W_1 = W_2 = W_3 = 0.33. \tag{23}$$

Step 4. Calculation of weighted similarity measures. *By using Formula (20), we obtain weighted similarity measures between the ideal matrix M and the k_s-th decision matrix M^{k_s} $(s = 1, 2, 3)$ as follows:*

$$s^W_{NH}(M, M^{k_1}) = \begin{bmatrix} 0.205 \\ 0.207 \\ 0.187 \\ 0.178 \end{bmatrix}, s^W_{NH}(M, M^{k_2}) = \begin{bmatrix} 0.178 \\ 0.185 \\ 0.218 \\ 0.220 \end{bmatrix}, s^W_{NH}(M, M^{k_3}) = \begin{bmatrix} 0.211 \\ 0.211 \\ 0.229 \\ 0.187 \end{bmatrix}, \quad (24)$$

Step 5. Ranking of alternatives. *In order to rank the alternatives according to the descending value of ρ_i, by using Equations (22)–(24), we obtain ρ_i $(i = 1, 2, 3, 4)$:*

$$\rho_1 = 0.196, \ \rho_2 = 0.199, \ \rho_3 = 0.232, \ \rho_4 = 0.193.$$

Then, $\rho_3 > \rho_2 > \rho_1 > \rho_4$. Thus, the ranking order is as follows:

$$\alpha_3 > \alpha_2 > \alpha_1 > \alpha_4.$$

So, we can see that Computer company (α_3) is the best alternative for money investment.

5. Conclusions

With this paper, we wished to renew an interest in the systematic study of the relationships between multi-criteria group decision-making (MCGDM) method with respect to octahedron set theory. For this purpose, various distance and similarity measures for octahedron sets were defined, and some of their properties were proved. The usefulness and interest of this correspondence of new defined distance and similarity measures will of course be enhanced if there is a way of returning from the transforms, that is to say, if there is a new method that characterize our proposed similarity measure. In Section 4, all the studies came to fruition, and we took up a result, MCGDM method based on a similarity measure for octahedron sets environments, which plays a pivotal role for demonstrating the usefulness of giving numerical examples. The detailed application of MCGDM method was carried out by introducing a a numerical example in the closing of Section 4. It considered some of the new results and consequences, which could be useful from the point of view of octahedron set theory, which were not studied at all. All these findings will provide a base to researchers who want to work in the field of the application of octahedron sets and will help to strengthen the foundations of the other MCGDM problems in octahedron set environment, such as economic policy, foreign policy between countries, trade policy, financial policy, etc., by using big data.

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