mathematics

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Surfaces of Revolution and Canal Surfaces with Generalized Cheng–Yau 1-Type Gauss Maps

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Received: 22 September 2020; Accepted: 7 October 2020; Published: 9 October 2020

Abstract: In the present work, the notion of generalized Cheng–Yau 1-type Gauss map is proposed, which is similar to the idea of generalized 1-type Gauss maps. Based on this concept, the surfaces of revolution and the canal surfaces in the Euclidean three-space are classified. First of all, we show that the Gauss map of any surfaces of revolution with a unit speed profile curve is of generalized Cheng–Yau 1-type. At the same time, an oriented canal surface has a generalized Cheng–Yau 1-type Gauss map if, and only if, it is an open part of a surface of revolution or a torus.

Keywords: surface of revolution; canal surface; Cheng–Yau operator; Gauss map

1. Introduction

The finite-type immersion and finite-type Gauss map proposed by B. Y. Chen are of great use in classifying and characterizing submanifolds whether they are in a Euclidean space or in a pseudo-Euclidean space [1,2]. The related research achievements are so numerous due to the continuous generalizations of such ideas on different submanifolds and in different spacetimes [3,4]. Taking the finite-type Gauss map as an example, the simplest type of finite-type Gauss map is the 1-type Gauss map. An oriented submanifold $\mathbb{M}$ is of 1-type Gauss map when its Gauss map $\mathbb{G}$ fulfills $\Delta\mathbb{G} = \lambda(\mathbb{G} + C)$ for some non-zero constant λ and a constant vector C; the Laplace operator Δ is given by

$$\Delta = -\frac{1}{\sqrt{det(g_{ij})}}\sum_{i,j}\frac{\partial}{\partial x^i}(\sqrt{det(g_{ij})}g^{ij}\frac{\partial}{\partial x^j}),$$

where g^{ij} are the components of the inverse matrix of g_{ij}. Spheres, circular cylinders and planes in Euclidean three-space are representatives which have 1-type Gauss maps [5]. Being a development of the 1-type Gauss map, the notion of a pointwise 1-type Gauss map of submanifolds is put forward by one of the present authors and D. W. Yoon [6]. An oriented submanifold $\mathbb{M}$ with a pointwise 1-type Gauss map fulfills $\Delta\mathbb{G} = f(\mathbb{G} + C)$ for a constant vector C and a non-zero smooth function f. Catenoids, helicoids and right cones in Euclidean three-space are typical surfaces with pointwise 1-type Gauss maps [5].

By extending the concept of submanifolds with pointwise 1-type Gauss maps, submanifolds with generalized 1-type Gauss maps can be defined. Namely

Definition 1. *Ref. [5] A submanifold $\mathbb{M}$ in $\mathbb{E}^m$ is of generalized 1-type Gauss map if its Gauss map $\mathbb{G}$ satisfies*

$$\Delta\mathbb{G} = f\mathbb{G} + gC$$

non-zero smooth functions (f, g) and constant vector $C \in \mathbb{E}^m$.

It is not difficult to find that the generalized 1-type Gauss map of submanifolds is a kind of extension of the 1-type Gauss map and pointwise 1-type Gauss map. The authors of [5] completely classified the developable surfaces, in Euclidean three-space, of the generalized 1-type Gauss map. The canal surfaces and the surfaces of revolution of generalized 1-type Gauss maps have been discussed recently [7].

In 1977, S.Y. Cheng and S.-T. Yau introduced a second-order differential and self-adjoint operator $\mathbb{L}_1 = \square$, named the Cheng–Yau operator, which is defined on a closed orientable Riemannian manifold $\mathbb{M}$ with a local orthonormal frame field $\{e_1, e_2, ..., e_n\}$ and a dual coframe field $\{\theta_1, \theta_2, ..., \theta_n\}$, where $\mathbb{M}$ has a symmetric tensor, as follows:

$$\phi = \sum_{i,j} \phi_{ij}\theta_i\theta_j$$

which satisfies the Cheng–Yau condition

$$\sum_{j=1}^{n} \phi_{ij,j} = 0, \quad 1 \leq i \leq n,$$

where $\phi_{ij,k}$ is the covariant derivative of the tensor ϕ_{ij} with respect to the metric g in the direction e_k. Then, the Cheng–Yau operator of any $\mathbb{C}^2$-function f is defined by [8]

$$\square f = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j} (\phi_{ij} f_i)_j = div(\phi \nabla f).$$

In recent years, the concepts of finite-type and pointwise 1-type Gauss maps for the submanifolds in Euclidean space have been extended and have taken the place of the Laplace operator Δ with the Cheng–Yau operator $\square$. A submanifold $\mathbb{M}$ is an $\mathbb{L}_1$-pointwise 1-type Gauss map when its Gauss map can be expressed as $\square\mathbb{G} = f(\mathbb{G} + C)$ for a constant vector C and a non-zero smooth function f. Moreover, when f is a non-zero constant, $\mathbb{M}$ is said to have a $\mathbb{L}_1$-1-type Gauss map. The rotational and helicoidal surfaces of $\mathbb{L}_1$-pointwise 1-type Gauss map have been discussed in [9]. Two authors of this paper classified the canal surfaces of $\mathbb{L}_1$-pointwise 1-type Gauss map [10].

Similar to the idea of generalized 1-type Gauss map, we could define and discuss the submanifolds of generalized Cheng–Yau 1-type Gauss maps. In Section 2, the gradient of a smooth function f is defined on a submanifold and some fundamental elements of canal surfaces are recalled. In Section 3, the surfaces of revolution and the canal surfaces of generalized Cheng–Yau 1-type Gauss maps are surveyed, respectively. Last but not least, some typical examples are presented via the Mathemtica programme.

The surfaces discussed here are regular, smooth and topologically connected.

2. Preliminaries

Let $\mathbb{M}$ be an oriented surface in the Euclidean three-space $\mathbb{E}^3$. Then, the gradient of a smooth function f, which is defined in $\mathbb{M}$, can be expressed by

$$\nabla f = \frac{1}{g_{11}g_{22} - (g_{12})^2}\{(g_{22}f_x - g_{12}f_y)\partial_x + (-g_{12}f_x + g_{11}f_y)\partial_y\}, \tag{1}$$

where $\{x, y\}$ is a local coordinate system of $\mathbb{M}$, s.t. $\langle \partial x, \partial x\rangle = g_{11}$, $\langle \partial x, \partial y\rangle = g_{12}$ and $\langle \partial y, \partial y\rangle = g_{22}$, f_x, and f_y are the partial derivatives of f, respectively [9].

According to the definition of the Cheng–Yau operator of a function f [8], the following conclusion is straightforward and useful.

Lemma 1. *Ref. [11] Let $\mathbb{M}$ be an oriented surface whose Gaussian curvature and mean curvature are denoted by K and H in $\mathbb{E}^3$. Then, the Cheng–Yau operator acting on its Gauss map $\mathbb{G}$ can be expressed by*

$$\Box\mathbb{G} = -\nabla K - 2HK\mathbb{G}. \tag{2}$$

Remark 1. *From Lemma 1, an oriented surface $\mathbb{M}$ has an $\mathbb{L}_1$-harmonic Gauss map if it is flat; $\mathbb{M}$ is of the first kind of $\mathbb{L}_1$-pointwise 1-type Gauss map if its Gaussian curvature is a non-zero constant.*

Motivated by the submanifolds of the generalized 1-type Gauss map in Euclidean space, the following definition is natural.

Definition 2. *An oriented submanifold $\mathbb{M}$ is of a generalized Cheng–Yau 1-type Gauss map in the Euclidean space $\mathbb{E}^m$ if its Gauss map $\mathbb{G}$ satisfies*

$$\Box\mathbb{G} = f\mathbb{G} + gC \tag{3}$$

for non-zero smooth functions (f, g) and constant vector $C \in \mathbb{E}^m$.

Remark 2. *Obviously, when f and g are non-zero constants, the Gauss map is just an $\mathbb{L}_1$-1-type Gauss map; when the function f is equal to g, it is a Gauss map of the $\mathbb{L}_1$-pointwise 1-type. Furthermore, the $\mathbb{L}_1$-pointwise 1-type Gauss map is called the first kind for $C = 0$ and, otherwise, the second kind. When f and g vanish, $\mathbb{G}$ is called the $\mathbb{L}_1$-harmonic.*

In $\mathbb{E}^3$, there exist important and useful surfaces called canal surfaces, which are swept out by moving spheres along space curves. Based on previous works about such surfaces [10,12,13], we focus on the canal surfaces of generalized Cheng–Yau 1-type Gauss maps in this work.

Assuming $c(s)$ be a space curve in $\mathbb{E}^3$ with an arc-length parameter s and Frenet frame $\{T, N, B\}$, according to the generating procedure of canal surfaces, a canal surface $\mathbb{M}$ can be expressed as

$$x(s, \theta) = c(s) + r(s)\{\cos\varphi T + \sin\varphi\cos\theta N + \sin\varphi\sin\theta B\}, \tag{4}$$

where $-r'(s) = \cos\varphi$, $(\varphi = \varphi(s))$ and $\theta \in [0, 2\pi), \varphi \in [0, \pi)$. The curve $c(s)$ is said to be the center curve, $r(s)$ is said to be the radial function of $\mathbb{M}$. In sequence, T, N, B are called the unit tangent, and the principal, normal and binormal vector fields of $c(s)$, respectively.

Remark 3. *In particular, when $c(s)$ is a straight line, $\mathbb{M}$ is just a surface of revolution; $\mathbb{M}$ is a tube (or pipe surface) when $r(s)$ is a constant.*

To serve the following discussions, we prepare some basic elements of canal surfaces. Initially, by the aid of the Frenet formula of $c(s)$, from (4), we have

$$x_s = \frac{\partial x}{\partial s} = x_s^1 T + x_s^2 N + x_s^3 B, \quad x_\theta = \frac{\partial x}{\partial \theta} = x_\theta^1 N + x_\theta^2 B, \tag{5}$$

where

$$\begin{aligned}
x_s^1 &= -r\kappa\sin\varphi\cos\theta - rr'' + \sin^2\varphi, \\
x_s^2 &= -rr'\kappa - r\tau\sin\varphi\sin\theta - rr'\varphi'\cos\theta + r'\sin\varphi\cos\theta, \\
x_s^3 &= -rr'\varphi'\sin\theta + r'\sin\varphi\sin\theta + r\tau\sin\varphi\cos\theta, \\
x_\theta^1 &= -r\sin\varphi\sin\theta, \\
x_\theta^2 &= r\sin\varphi\cos\theta.
\end{aligned} \tag{6}$$

Meanwhile, the Gauss map $\mathbb{G}$ of $\mathbb{M}$ is given by

$$\mathbb{G} = \frac{x_s \times x_\theta}{\|x_s \times x_\theta\|} = \cos\varphi T + \sin\varphi\cos\theta N + \sin\varphi\sin\theta B, \tag{7}$$

from which we have

$$\begin{aligned} \mathbb{G}_s &= -(\kappa\sin\varphi\cos\theta + r'')T - (r'\kappa + \tau\sin\varphi\sin\theta + r'\varphi'\cos\theta)N - (r'\varphi'\sin\theta - \tau\sin\varphi\cos\theta)B, \\ \mathbb{G}_\theta &= -\sin\varphi\sin\theta N + \sin\varphi\cos\theta B. \end{aligned} \tag{8}$$

By (5), (6) and (8), the first fundamental form g_{ij} and the second fundamental form h_{ij} are

$$g_{11} = \frac{P^2 + r^2R^2}{\sin^2\varphi}, \quad g_{12} = r^2R, \quad g_{22} = r^2\sin^2\varphi \tag{9}$$

and

$$h_{11} = \frac{-rR^2 - PQ}{\sin^2\varphi}, \quad h_{12} = -rR, \quad h_{22} = -r\sin^2\varphi, \tag{10}$$

where

$$\begin{aligned} P &= rr'' + r\kappa\sin\varphi\cos\theta - \sin^2\varphi, \\ Q &= \kappa\sin\varphi\cos\theta + r'', \\ R &= r'\kappa\sin\varphi\sin\theta + \tau\sin^2\varphi. \end{aligned} \tag{11}$$

By (9) and (10), we have

$$K = \frac{Q}{rP}, \quad H = -\frac{1}{r} - \frac{\sin^2\varphi}{2rP}, \tag{12}$$

where K and H are the Gaussian curvature and the mean curvature of $\mathbb{M}$.

Remark 4. *From $g_{11}g_{22} - g_{12}^2 = r^2P^2$, due to the regularity of $\mathbb{M}$, $P \neq 0$.*

Simultaneously, we observe the following conclusion.

Proposition 1. *Ref. [12] The Gaussian curvature K and the mean curvature H of a canal surface $\mathbb{M}$ in $\mathbb{E}^3$ are related by*

$$H = -\frac{1}{2}(Kr + \frac{1}{r}).$$

Next, we focus on the surfaces of revolution and the canal surfaces that have generalized Cheng–Yau 1-type Gauss maps, respectively.

3. Surfaces of Revolution with Generalized Cheng–Yau 1-Type Gauss Map

Let $\mathbb{M}$ be a surface of revolution in $\mathbb{E}^3$ parameterized by

$$x(s,\theta) = (\psi, \phi\cos\theta, \phi\sin\theta) \tag{13}$$

for some smooth functions, $\psi = \psi(s)$ and $\phi = \phi(s)$. Assuming that the profile curve is of unit speed, i.e., $\phi'^2 + \psi'^2 = 1$, a direct computation shows that

$$x_s = (\psi', \phi'\cos\theta, \phi'\sin\theta), \quad x_\theta = (0, -\phi\sin\theta, \phi\cos\theta). \tag{14}$$

At the same time, the Gauss map $\mathbb{G}$ of $\mathbb{M}$ is

$$\mathbb{G} = (\phi', -\psi'\cos\theta, -\psi'\sin\theta), \tag{15}$$

from which we have

$$\mathbb{G}_s = (\phi'', -\psi'' \cos\theta, -\psi'' \sin\theta), \quad \mathbb{G}_\theta = (0, \psi' \sin\theta, -\psi' \cos\theta).$$

By some calculations, the first fundamental form g_{ij} and the second fundamental form h_{ij} are

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = \phi^2 \tag{16}$$

and

$$h_{11} = \phi'\psi'' - \psi'\phi'', \quad h_{12} = 0, \quad h_{22} = \phi\psi'. \tag{17}$$

From (16) and (17), the Gaussian curvature K and the mean curvature H can be expressed as

$$K = -\frac{\phi''}{\phi}, \quad H = \frac{\phi\psi'' + \phi'\psi'}{2\phi\phi'}. \tag{18}$$

By (14), (16), (18) and (1), we obtain

$$\nabla K(x) = \frac{\phi'\phi'' - \phi\phi'''}{\phi^2}(\psi', \phi' \cos\theta, \phi' \sin\theta). \tag{19}$$

From (15), (18), (19) and Lemma 1, the Cheng–Yau operator of the Gauss map $\mathbb{G}$ is

$$\Box\mathbb{G} = \frac{1}{\phi^2}(\phi\phi'''\psi' + \phi\phi''\psi'', (\phi\phi'\phi''' + \phi\phi''^2 - \phi'') \cos\theta, (\phi\phi'\phi''' + \phi\phi''^2 - \phi'') \sin\theta). \tag{20}$$

If $\mathbb{M}$ has a generalized Cheng–Yau 1-type Gauss map, i.e., $\Box\mathbb{G} = f\mathbb{G} + gC$, where $C = (C_1, C_2, C_3)$ is a constant vector, by substituting (15) and (20) into (3), we obtain

$$\begin{cases} f\phi' + gC_1 = \frac{\phi'''\psi' + \phi''\psi''}{\phi}, \\ f(-\psi' \cos\theta) + gC_2 = \frac{\phi\phi'\phi''' + \phi\phi''^2 - \phi''}{\phi^2} \cos\theta, \\ f(-\psi' \sin\theta) + gC_3 = \frac{\phi\phi'\phi''' + \phi\phi''^2 - \phi''}{\phi^2} \sin\theta. \end{cases} \tag{21}$$

The second and third equations of (21) imply that $C_2 = C_3 = 0$, obviously. Moreover,

$$\begin{cases} f(s) = \frac{\phi'' - \phi\phi'\phi''' - \phi\phi''^2}{\phi^2\psi'}, \\ g(s) = \frac{\phi\phi''' - \phi'\phi''}{C_1\phi^2\psi'}, \end{cases} \tag{22}$$

where $C_1 \neq 0$ is a constant.

Conversely, when we make use of the given functions ψ and ϕ, a surface of revolution with a unit speed profile curve satisfies $\Box\mathbb{G} = f\mathbb{G} + gC$ for such functions (f, g) given by (22) and constant vector $C = (C_1, 0, 0)$. Thus, we have the following result.

Theorem 1. *Any surface of revolution $\mathbb{M}$ with a unit speed profile curve in $\mathbb{E}^3$ has a generalized Cheng–Yau 1-type Gauss map. Explicitly, the Gauss map $\mathbb{G}$ of $\mathbb{M}$ fulfills*

$$\Box\mathbb{G} = f\mathbb{G} + gC$$

for some non-zero smooth functions $(f(s), g(s))$ given by (22) and the constant vector $C = (C_1, 0, 0)$, where C_1 is a non-zero constant.

4. Canal Surfaces with Generalized Cheng–Yau 1-Type Gauss Map

Assuming that an oriented canal surface $\mathbb{M}$ is of the generalized Cheng–Yau 1-type Gauss map kind, then, by Lemma 1, we have

$$\square\mathbb{G} = -\nabla K - 2HK\mathbb{G} = f\mathbb{G} + gC. \tag{23}$$

We decompose the constant vector C as follows:

$$C = C_1 T + C_2 N + C_3 B, \tag{24}$$

where $C_1 = \langle C, T\rangle, C_2 = \langle C, N\rangle, C_3 = \langle C, B\rangle$. By (5), (9) and (1), we obtain

$$\nabla K(x) = \frac{1}{r^2 P^2}[(Ux_s^1)T + (Ux_s^2 + Vx_\theta^1)N + (Ux_s^3 + Vx_\theta^2)B], \tag{25}$$

where $U = g_{22}K_s - g_{12}K_\theta$, $V = -g_{12}K_s + g_{11}K_\theta$.

Note that, from (11) and (12), the partial derivatives of the Gaussian curvature K are

$$\begin{aligned} K_s =& \frac{-2rr'\kappa^2\sin^2\varphi\cos^2\theta - 5rr'r''\kappa\sin\varphi\cos\theta + (r'\kappa - r\kappa')\sin^3\varphi\cos\theta}{r^2P^2} \\ &+ \frac{r'r''\sin^2\varphi - 4rr'r''^2 - rr'''\sin^2\varphi}{r^2P^2}, \\ K_\theta =& \frac{\kappa\sin^3\varphi\sin\theta}{rP^2}. \end{aligned} \tag{26}$$

By substituting (7), (24) and (25) into (23), we get

$$\begin{cases} Ux_s^1 = -r^2P^2(2HK\cos\varphi + f\cos\varphi + gC_1), \\ Ux_s^2 + Vx_\theta^1 = -r^2P^2(2HK\sin\varphi\cos\theta + f\sin\varphi\cos\theta + gC_2), \\ Ux_s^3 + Vx_\theta^2 = -r^2P^2(2HK\sin\varphi\sin\theta + f\sin\varphi\sin\theta + gC_3). \end{cases} \tag{27}$$

According to the above equation system, we have the following two cases.

Case 1: $r' = -\cos\varphi \neq 0$. From the first equation of (27), we have

$$f = -\frac{Ux_s^1 + r^2P^2(2HK\cos\varphi + gC_1)}{r^2P^2\cos\varphi}; \tag{28}$$

by substituting (28) into the last two equations of (27), we obtain

$$g = \frac{Ux_s^1\sin\varphi\cos\theta - \cos\varphi(Ux_s^2 + Vx_\theta^1)}{r^2P^2(C_2\cos\varphi - C_1\sin\varphi\cos\theta)} = \frac{Ux_s^1\sin\varphi\sin\theta - \cos\varphi(Ux_s^3 + Vx_\theta^2)}{r^2P^2(C_3\cos\varphi - C_1\sin\varphi\sin\theta)}. \tag{29}$$

From (29), we have

$$\begin{aligned} &U[\sin\varphi(C_2\sin\theta - C_3\cos\theta)x_s^1 + (C_3\cos\varphi - C_1\sin\varphi\sin\theta)x_s^2 + (C_1\sin\varphi\cos\theta - C_2\cos\varphi)x_s^3] \\ =&Vr\sin\varphi[C_1\sin\varphi - \cos\varphi(C_2\cos\theta + C_3\sin\theta)]. \end{aligned} \tag{30}$$

Since $\{\cos(n\theta), \sin(n\theta) | n \in \mathbb{N}\}$ constitutes a linearly independent function system, when analyzing the coefficients of $\cos 4\theta$ and $\sin 4\theta$ in (30) by the aid of (5), (9) and (26), we have

$$\begin{cases} r^2\kappa^3\sin^4\varphi\cos^3\varphi C_2 = 0, \\ r^2\kappa^3\sin^4\varphi\cos^3\varphi C_3 = 0. \end{cases} \tag{31}$$

Based on Equation (31), we think of a non-empty subset $\mathcal{O} = \{p \in \mathbb{M} \mid \kappa(p) \neq 0\}$. Because $\sin\varphi \neq 0, r \neq 0$, we know $C_2 = C_3 = 0$ on $\mathcal{O}$. By substituting them into (30), we have

$$r(UR + \sin^2\varphi V)C_1 = 0. \tag{32}$$

Furthermore, by contrasting the coefficient of the highest degree of $\sin 3\theta$ in (32), we obtain that $C_1 = 0$, then $C = (0,0,0)$. In this situation, $\mathbb{M}$ is of the first kind of $\mathbb{L}_1$- pointwise 1-type Gauss map, i.e., $\Box\mathbb{G} = f\mathbb{G}$. From the Theorem 3.2 of [10], $\mathbb{M}$ is an open part of a surface of revolution, i.e., $\kappa = 0$. Thus, $\mathcal{O}$ is empty; $\kappa \equiv 0$ when $r' \neq 0$. In this case, $\mathbb{M}$ is a surface of revolution.

By simplifying (30) with the help of $\kappa = 0$, we have

$$(C_3\cos\theta - C_2\sin\theta)(\sin\varphi - r\varphi')K_s = 0. \tag{33}$$

Note that $\sin\varphi - r\varphi' \neq 0$ or else $P = 0$ and $\mathbb{M}$ is degenerate. If $K_s = 0$, then $\mathbb{M}$ has constant Gaussian curvature due to $K_\theta = 0$ when $\kappa = 0$. From Reamrk 1, $\mathbb{M}$ is of the first kind of $\mathbb{L}_1$- pointwise 1-type Gauss map. Therefore, $K_s \neq 0$ and (33) follow that $C_2 = C_3 = 0$. Furthermore, from (27) we have

$$f = \frac{r'K_s}{P} - 2HK, \quad g = \frac{K_s}{C_1 P}, \tag{34}$$

where C_1 is a non-zero constant. As $\kappa = 0$, $P, HandK$ are all functions of s, (34) yields $f = f(s), g = g(s)$. Explicitly, we have

$$\begin{aligned} f(s) &= \frac{(r'' - rr'r''')(1 - r'^2) + rr''^2(2rr'' - r'^2 - 3)}{r^2(rr'' + r'^2 - 1)^3}, \\ g(s) &= \frac{(r'r'' - rr''')(1 - r'^2) - 4rr'r''^2}{C_1 r^2(rr'' + r'^2 - 1)^3}. \end{aligned} \tag{35}$$

Therefore, $\mathbb{M}$ is of the generalized Cheng–Yau 1-type Gauss map for functions (f, g) given by (35) and the vector $C = (C_1, 0, 0)$, where $C_1 \neq 0$.

Because $\mathbb{M}$ is a surface of revolution, we can put $c(s) = (s, 0, 0)$ in (4) with Frenet frame $T = (1,0,0), N = (0,1,0), B = (0,0,1)$. Therefore, $\mathbb{M}$ can be expressed by

$$x(s,\theta) = (s + r(s)\cos\varphi(s), r(s)\sin\varphi(s)\cos\theta, r(s)\sin\varphi(s)\sin\theta).$$

Case 2: $r' = -\cos\varphi = 0$, i.e., $\mathbb{M}$ is a tube surface.

First of all, suppose that $C_1 \neq 0$. Then, we get, from the first equation of (27),

$$g = -\frac{x_s^1 U}{C_1 r^2 P^2}. \tag{36}$$

Taking (36) into the last two equations of (27), we obtain

$$f = \frac{(C_2 x_s^1 - C_1 x_s^2)U - C_1 x_\theta^1 V}{r^2 P^2 C_1 \cos\theta} - 2HK = \frac{(C_3 x_s^1 - C_1 x_s^3)U - C_1 x_\theta^2 V}{r^2 P^2 C_1 \sin\theta} - 2HK, \tag{37}$$

according to (37), we have

$$U[(C_2\sin\theta - C_3\cos\theta)(1 - r\kappa\cos\theta) + C_1 r\tau] + C_1 rV = 0. \tag{38}$$

Considering the coefficient of the power of $\sin\theta$ in (38) with the help of (9) and (26), we get $C_1\kappa = 0$; hence, $\kappa = 0$. However, when $r' = 0$ and $\kappa = 0$, $\mathbb{M}$ is part of a circular cylinder. By Remark 1, it has an $\mathbb{L}_1$ harmonic Gauss map. It is a contradiction; therefore, $C_1 = 0$.

Looking back at the first equation of (27) together with $r' = 0$ and $C_1 = 0$, we have $x^1_s U = 0$, i.e.,

$$\kappa' \cos\theta + \kappa\tau \sin\theta = 0;$$

therefore, $\kappa = c_0$, $(0 \neq c_0 \in \mathbb{R})$ and $\tau = 0$, then the center curve $c(s)$ is a circle and $\mathbb{M}$ is a torus.

Furthermore, from the last two equations of (27), we have

$$f = -\frac{V(C_2\cos\theta + C_3\sin\theta)}{rP^2(C_2\sin\theta - C_3\cos\theta)} - 2HK, \quad g = \frac{V}{rP^2(C_2\sin\theta - C_3\cos\theta)}, \tag{39}$$

where $C_2^2 + C_3^2 \neq 0$.

Since V, P, K and H are all functions of θ when $r' = 0$ and $\kappa \neq 0$ is a constant, (39) yields that the functions (f, g) only depend on θ. Explicitly, we have

$$\begin{aligned} f(\theta) =& \frac{\kappa\cos\theta(2r\kappa\cos\theta - 1)}{r^2(r\kappa\cos\theta - 1)^2} - \frac{\kappa\sin\theta(C_2\cos\theta + C_3\sin\theta)}{r^2(r\kappa\cos\theta - 1)^2(C_2\sin\theta - C_3\cos\theta)}, \\ g(\theta) =& \frac{\kappa\sin\theta}{r^2(r\kappa\cos\theta - 1)^2(C_2\sin\theta - C_3\cos\theta)}. \end{aligned} \tag{40}$$

Therefore, $\mathbb{M}$ is of the generalized Cheng–Yau 1-type Gauss map for functions (f, g) given by (40) and the vector $C = (0, C_2, C_3)$, where $C_2^2 + C_3^2 \neq 0$.

Conversely, suppose $\mathbb{M}$ is an open part of a surface of revolution or a torus; we can easily find that $\Box\mathbb{G} = f\mathbb{G} + gC$ is fulfilled for some non-zero smooth functions (f, g) given by (35) and (40) with the constant vectors $C = (C_1, 0, 0)$ and $C = (0, C_2, C_3)$, respectively.

According to the above discussion works, we have the following results.

Theorem 2. *An oriented canal surface $\mathbb{M}$ is of the generalized Cheng–Yau 1-type Gauss map if it is a torus or an open part of a surface of revolution with the following form:*

$$x(s, \theta) = (s + r(s)\cos\varphi(s), r(s)\sin\varphi(s)\cos\theta, r(s)\sin\varphi(s)\sin\theta).$$

As immediate consequences of the above theorem, we have

Corollary 1. *Let an oriented canal surface $\mathbb{M}$ with a generalized Cheng–Yau 1-type Gauss map be an open part of a surface of revolution. Then, the Gauss map $\mathbb{G}$ of $\mathbb{M}$ satisfies*

$$\Box\mathbb{G} = f\mathbb{G} + gC$$

for some non-zero smooth functions $(f(s), g(s))$ given by

$$\begin{aligned} f(s) =& \frac{(r'' - rr'r''')(1 - r'^2) + rr''^2(2rr'' - r'^2 - 3)}{r^2(rr'' + r'^2 - 1)^3}, \\ g(s) =& \frac{(r'r'' - rr''')(1 - r'^2) - 4rr'r''^2}{C_1 r^2(rr'' + r'^2 - 1)^3} \end{aligned}$$

and the vector $C = (C_1, 0, 0)$, $(C_1 \in \mathbb{R} - \{0\})$.

In particular, when the canal surface with a generalized Cheng–Yau 1-type Gauss map is an open part of a surface of revolution, which has a profile curve of unit speed, we have the following result.

Corollary 2. *Let an oriented canal surface $\mathbb{M}$ with a generalized Cheng–Yau 1-type Gauss map be an open part of a surface of revolution that has a profile curve of unit speed. Then, the Gauss map $\mathbb{G}$ of $\mathbb{M}$ fulfills*

$$\Box\mathbb{G} = f\mathbb{G} + gC$$

for some non-zero smooth functions $(f(s), g(s))$ given by (42) and the constant vector $C = (C_1, 0, 0)$, where C_1 is a non-zero constant. Moreover, the radius function $r(s)$ of $\mathbb{M}$ is given by (44) explicitly.

Proof. By comparing the parametrization of $\mathbb{M}$, as stated in Theorem 2, with the general form of the surface of revolutionm as stated in (13), we can let $\psi(s) = r(s)\cos\varphi(s) + s, \phi(s) = r(s)\sin\varphi(s)$, s.t. $\psi'^2(s) + \phi'^2(s) = 1$, i.e.,

$$r'^2 + (1 - r'^2 - rr'')^2 = 1. \tag{41}$$

Because of the assumption for canal surfaces, $-r' = \cos\varphi$, from the above equation, we have

$$(r\varphi' - \sin\varphi)^2 = 1.$$

By combining the the expression forms of (f, g) in (35) and (22), we have

$$\begin{aligned} f(s) =& \frac{\varphi'}{r^2\sin^2\varphi} + \varepsilon[\frac{\varphi'^2\cos 2\varphi}{r\sin^2\varphi} + \frac{\varphi''\cos\varphi}{r\sin\varphi}], \\ g(s) =& \frac{\varphi'^2\cos\varphi}{C_1 r\sin^2\varphi} + \frac{\varphi''}{C_1 r\sin\varphi} - \varepsilon\frac{\varphi'\cos\varphi}{C_1 r^2\sin^2\varphi}, \end{aligned} \tag{42}$$

where, $\varepsilon = 1$ for $r\varphi' - \sin\varphi = 1$; $\varepsilon = -1$ for $r\varphi' - \sin\varphi = -1$.

Furthermore, by solving differential Equation (41), we get

$$(r + c)(2cr - c^2)^{\frac{1}{2}} = 3cs + c_0, (c, c_0 \in \mathbb{R}), \tag{43}$$

Then we obtain a real solution of $r(s)$ as follows:

$$r(s) = -\frac{3c^2}{\sqrt[3]{4}B} - \frac{B}{6c\sqrt[3]{2}} - \frac{c}{2}, \tag{44}$$

where $A = -972c^4s^2 - 648c^3c_0s - 108c^2c_0^2 - 54c^6, B = (A + \sqrt{-2916c^{16} + A^2})^{\frac{1}{3}}$. □

Corollary 3. *Let an oriented canal surface $\mathbb{M}$ with a generalized Cheng–Yau 1-type Gauss map Gauss map be a torus. Then, the Gauss map $\mathbb{G}$ of $\mathbb{M}$ satisfies*

$$\Box\mathbb{G} = f\mathbb{G} + gC$$

for some non-zero smooth functions $(f(\theta), g(\theta))$ given by

$$\begin{aligned} f(\theta) =& \frac{\kappa\cos\theta(2r\kappa\cos\theta - 1)}{r^2(r\kappa\cos\theta - 1)^2} - \frac{\kappa\sin\theta(C_2\cos\theta + C_3\sin\theta)}{r^2(r\kappa\cos\theta - 1)^2(C_2\sin\theta - C_3\cos\theta)}, \\ g(\theta) =& \frac{\kappa\sin\theta}{r^2(r\kappa\cos\theta - 1)^2(C_2\sin\theta - C_3\cos\theta)} \end{aligned}$$

and the vector $C = (0, C_2, C_3)$ in which $C_2, C_3 \in \mathbb{R}$ and $C_2^2 + C_3^2 \neq 0$.

Remark 5. *The canal surfaces that have $\mathbb{L}_1$-pointwise 1-type Gauss maps and the ones that have $\mathbb{L}_1$-1-type Gauss maps have been discussed in [10]; we do not repeat them here.*

5. Examples

In this section, we present some typical examples of Cheng–Yau generalized 1-type Gauss maps.

Example 1. *Let $\mathbb{M}$ be a surface of revolution, as follows (see Figure 1):*

$$x(s,\theta) = (e^s, s^2\cos\theta, s^2\sin\theta).$$

After calculations, its Gauss map $\mathbb{G}$ is $\mathbb{G} = (2s, -e^s\cos\theta, -e^s\sin\theta)$, whose Cheng–Yau operator can be expressed as

$$\Box\mathbb{G} = \frac{2-4s^2}{s^4e^s}\mathbb{G} - \frac{4}{s^3e^s}(1,0,0).$$

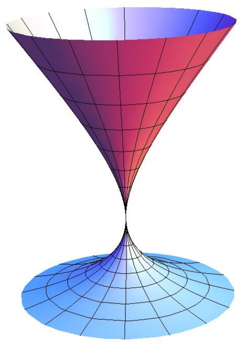

Figure 1. The surface of revolution in Example 1.

Example 2. *Let $\mathbb{M}$ be a surface of revolution that has a profile curve of unit speed and is parameterized by (see Figure 2)*

$$x(s,\theta) = (s + r(s)\cos\varphi, r(s)\sin\varphi\cos\theta, r(s)\sin\varphi\sin\theta),$$

where $r(s)$ is given by

$$r(s) = \frac{1}{2}(1 + \frac{1}{T} - T),$$

in which $T = (1 + 18s^2 - 6\sqrt{s^2 + 9s^4})^{\frac{1}{3}}$.

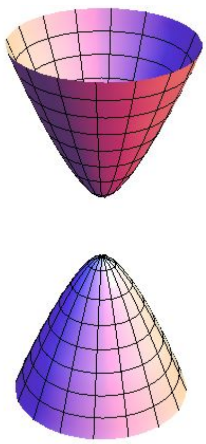

Figure 2. The surface of revolution in Example 2.

Example 3. *Let $\mathbb{M}$ be a torus parameterized by (see Figure 3)*

$$x(s,\theta) = (\sin s - \frac{1}{2}\sin s\cos\theta, \frac{1}{2}\sin\theta, \cos s - \frac{1}{2}\cos s\cos\theta).$$

Through calculations, we find that its Gauss map $\mathbb{G}$ is $\mathbb{G} = (\sin s\cos\theta, -\sin\theta, \cos s\cos\theta)$, whose Cheng–Yau operator can be expressed as

$$\Box\mathbb{G} = \frac{16\cos\theta}{\cos\theta - 2}\mathbb{G} + \frac{16}{(\cos\theta - 2)^2}(0,1,0).$$

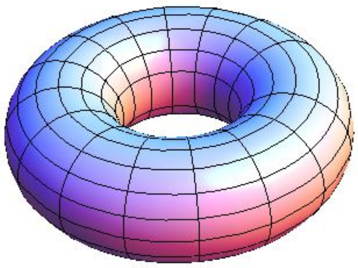

Figure 3. The torus in Example 3.

Based on the definitions of canal surfaces in Minkowski three-space $\mathbb{E}_1^3$ [14,15], the canal surfaces in $\mathbb{E}_1^3$ will be classified in terms of their Gauss maps via the Laplacian operator and the Cheng–Yau operator in the near future.

Author Contributions: J.Q., X.F. and X.T. proposed the problem and calculated the details. Y.H.K. checked and polished the draft. All authors have read and agreed to the published version of the manuscript.

Funding: The first author was supported by National Natural Science Foundation of China (No. 11801065) and the Fundamental Research Funds for the Central Universities (N2005012). The fourth author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF), which is funded by the Ministry of Education (NRF-2020R1I1A3051852).

Acknowledgments: We thank the referee for their careful review and the valuable comments that helped us to improve the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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