On the Number of Periodic Orbits to Odd Order Differential Delay Systems

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Abstract: In this paper, we study the periodic orbits of a type of odd order differential delay system with $2k - 1$ lags via the S^1 index theory and the variational method. This type of system has not been studied by others. Our results provide a new and more accurate method for counting the number of periodic orbits.

Keywords: odd order differential delay system; periodic orbits; critical point theory; variational method

1. Introduction

The delay differential equations have useful applications in various fields such as age-structured population growth, life sciences, control theory, and any model involving responses with non-zero delays [1,2].

The problem of periodic solutions for multi-delay differential equations started from the research work of Kaplan and Yorke [3] in 1974. Consider the following differential delay equation

$$x'(t) = -\sum_{i=1}^{n} f(x(t-i)) \tag{1}$$

in which $f(-x) = -f(x), xf(x) > 0, x \neq 0$. They proved the existence of the periodic solutions of (1) when $n = 1$ and $n = 2$, respectively. In the subsequent literatures [4–17], the number of $2(n+1)$-periodic solutions of Equation (1) was studied by using the results of Nussbaum [18]. In 1998, Li and He [15] made some researches for (1) by use of the theory of symmetric group. Especially, in 2006, Fei used the variational method to study the number of periodic solutions of Equation (1). He studied the cases of $n = 2k - 1$ and $n = 2k$ respectively in literature [4,5] since the specific research methods and details vary greatly. Almost at the same time, Guo and Yu [13] applied critical point theories to study the multiplicity of the 4-periodic solutions for (1) when $n = 1$. After then they [12] studied the same problem for (1) when $n = 2k - 1, k \geq 1$, which is an extension of Fei's article [4] but the proof of theorem is more difficult and more complicated. In addition, all of these researches are about first order differential equations and did not give the precise counting method for the number of periodic solutions.

The goal of this paper is to study the periodic orbits to a type of odd order differential delay system in the form

$$\begin{cases} x^{(2m+1)}(t) = -\sum_{p=1}^{2k-1} \nabla F(x(t-p)), \ a.e. \ t \in [0, 4k] \\ x(t) - x(t - 4k) = 0. \end{cases} \tag{2}$$

where

$$x \in \mathbb{R}^N, \ F \in C^1(\mathbb{R}^N, \mathbb{R}), \ \nabla F(-x) = -\nabla F(x), F(0) = 0 \tag{3}$$

and there are real symmetric matrices $A_0, A_\infty \in \mathbb{R}^{N \times N}$ such that

$$\nabla F(x) = A_0 x + \circ(|x|), |x| \to 0, \qquad \nabla F(x) = A_\infty x + \circ(|x|), |x| \to \infty. \tag{4}$$

The method applied in this paper is the S^1 index theory and the variational approach [19–21]. The variation structure we built in this paper is simpler compared with [13]. In addition, the counting method for the number of periodic orbits in our results only depends on the eigenvalues of A_∞ and A_0. So it is more precise and easier to examine.

To facilitate the following discussion, we suppose that

$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_N, \ \beta_1 \leq \beta_2 \leq \cdots \leq \beta_N$$

are respectively the eigenvalues of A_0 and A_∞. Their corresponding unit eigenvectors are respectively $u_1, u_2, \cdots, u_N; \ v_1, v_2 \cdots, v_N$. At the same time, the one-dimensional space corresponding to each eigenvector is denoted as

$$U_j = span\ \{u_j\}, \ \ V_j = span\{v_j\}, \quad j = 1, \ 2, \cdots, N.$$

2. Space X and Functional Φ

Consider the $4k$-periodic orbits of (2), and suppose

$$x(t - 2k) = -x(t), \ k \geq 1. \tag{5}$$

Let

$$\widehat{X} = \{x \in L^2 : x(t - 2k) = -x(t)\} = \left\{ \sum_{j=0}^{\infty} (a_j \cos \frac{(2j+1)\pi t}{2k} + b_j \sin \frac{(2j+1)\pi t}{2k}) : a_j, b_j \in \mathbb{R}^N \right\},$$

$$X = cl \left\{ \sum_{j=0}^{\infty} (a_j \cos \frac{(2j+1)\pi t}{2k} + b_j \sin \frac{(2j+1)\pi t}{2k}) : a_j, b_j \in \mathbb{R}, \sum_{j=0}^{\infty} (2j+1)^{2m+1} (a_j^2 + b_j^2) < \infty \right\} \subset \widehat{X},$$

and define $P : X \to L^2$ by

$$
\begin{aligned}
P^{2m+1} x(t) &= P^{2m+1} \left(\sum_{j=0}^{\infty} (a_j \cos \frac{(2j+1)\pi t}{2k} + b_j \sin \frac{(2j+1)\pi t}{2k}) \right) \\
&= \sum_{j=0}^{\infty} (2j+1)^{2m+1} (a_j \cos \frac{(2j+1)\pi t}{2k} + b_j \sin \frac{(2j+1)\pi t}{2k}).
\end{aligned}
\tag{6}
$$

Let

$$P^{-(2m+1)} x(t) = \sum_{j=0}^{\infty} \frac{1}{(2j+1)^{2m+1}} (a_j \cos \frac{(2i+1)\pi t}{2k} + b_j \sin \frac{(2j+1)\pi t}{2k}).$$

For $x \in X$, define

$$\langle x, y \rangle = \int_0^{4k} (P^{2m+1} x(t), y(t)) dt, \|x\| = \sqrt{\langle x, x \rangle},$$

$$\langle x, y \rangle_0 = \int_0^{4k} (x(t), y(t)) dt, \|x\|_0 = \sqrt{\langle x, x \rangle_0}.$$

Therefore, $(X, \|\cdot\|)$ is an $H^{m+\frac{1}{2}}([0,4k], \mathbb{R}^N)$ space.

In order to facilitate subsequent calculations, we make a more detailed division as follows,

$$X(j) = \{x(t) = a_j \cos \frac{(2j+1)\pi t}{2k} + b_j \sin \frac{(2j+1)\pi t}{2k} : a_j, b_j \in \mathbb{R}^N\},$$

$$X_{0,i}(j) = cl\{x(t) = a_j \cos \frac{(2j+1)\pi t}{2k} + b_j \sin \frac{(2j+1)\pi t}{2k} : a_j, b_j \in U_i\}, \tag{7}$$

$$X_{\infty,i}(j) = cl\{x(t) = a_j \cos \frac{(2j+1)\pi t}{2k} + b_j \sin \frac{(2j+1)\pi t}{2k} : a_j, b_j \in V_i\}, \tag{8}$$

then we have $X(j) = \sum\limits_{i=1}^{N} X_{0,i}(j) = \sum\limits_{i=1}^{N} X_{\infty,i}(j)$, and let

$$X_{0,i} = \bigoplus_{j=0}^{\infty} X_{0,i}(j), \quad X_{\infty,i} = \bigoplus_{j=0}^{\infty} X_{\infty,i}(j), \quad X = \bigoplus_{1\le i\le N} X_{0,i} = \bigoplus_{1\le i\le N} X_{\infty,i}.$$

It is easy to see that, $\forall x \in X(j)$,

$$||x||^2 = (2j+1)^{(2m+1)}||x||_0^2.$$

Then, we can get

$$\int_0^{4k} (A_0 x(t),\ x(t))dt = \alpha_i ||x||_0^2, \ \forall x \in X_{0,i}(j) \subset X(j),$$

and

$$\int_0^{4k} (A_\infty x(t),\ x(t))dt = \beta_i ||x||_0^2, \ \forall x \in X_{\infty,i}(j) \subset X(j).$$

For the system (2), define the following functional

$$\Phi(x) = \frac{1}{2}\langle Lx, x \rangle + \int_0^{4k} F(x(t))dt \tag{9}$$

where

$$Lx = P^{-(2m+1)} \sum_{j=1}^{2k-1} (-1)^j x^{(2m+1)}(t-j).$$

If $x_j(t) = a_j \cos \frac{(2j+1)\pi t}{2k} + b_j \sin \frac{(2j+1)\pi t}{2k} \in X(j)$, we have

$$Lx = (-1)^{m+1}\left(\frac{\pi}{2k}\right)^{2m+1}\left(\sum_{j=0}^{\infty} x_j \tan \frac{(2j+1)\pi}{4k}\right). \tag{10}$$

Obviously $L|_{X(j)} : X(j) \to X(j)$ is invertible.

According to Theorem 1.4 in [21], the differential of functional Φ is

$$P^{-(2m+1)}\Phi'(x) = Lx + \Psi(x) \tag{11}$$

where $\Psi(x) = P^{-(2m+1)}\nabla F(x)$. It is easy to prove that $\Psi : (X, \|x\|^2) \to (X, \|x\|_0^2)$ is compact.

3. Partition of Space X and Symbols

From (10), we have that if

$$x(t) = \sum_{j=0}^{\infty} (a_j \cos \frac{(2j+1)\pi t}{2k} + b_j \sin \frac{(2j+1)\pi t}{2k}) \in X(j),$$

then

$$\langle Lx, \, x \rangle = (-1)^{m+1} \left(\frac{\pi}{2k} \right)^{2m+1} (2j+1)^{2m+1} \tan \frac{(2j+1)\pi}{4k} ||x||_0^2$$

$$= (-1)^{m+1} \left(\frac{\pi}{2k} \right)^{2m+1} \tan \frac{(2j+1)\pi}{4k} ||x||^2.$$

Thereby, $\forall x \in X_{0,i}(2lk+j) \subset X(2lk+j), j = 0, 1, 2, \cdots, 2k-1$, we have

$$\langle (L + P^{-(2m+1)} A_0)x, x \rangle$$

$$= \left[(-1)^{m+1} \left(\frac{\pi}{2k} \right)^{2m+1} (4lk+2j+1)^{2m+1} \tan \frac{(2j+1)\pi}{4k} + \alpha_i \right] ||x||_0^2.$$

Similarly, $\forall x \in X_{\infty,i}(2lk+j) \subset X(2lk+j), j = 0, 1, 2, \cdots, 2k-1$, we have

$$\langle (L + P^{-(2m+1)} A_\infty)x, x \rangle \tag{12}$$

$$= \left[(-1)^{m+1} \left(\frac{\pi}{2k} \right)^{2m+1} (4lk+2j+1)^{2m+1} \tan \frac{(2j+1)\pi}{4k} + \beta_i \right] ||x||_0^2.$$

Hence, define

$$\begin{cases} X_{0,i}^+(2lk+j) = \left\{ X_{0,i}(2lk+j) : \left[(-1)^{m+1} \left(\frac{\pi}{2k} \right)^{2m+1} (4lk+2j+1)^{2m+1} \tan \frac{(2j+1)\pi}{4k} + \alpha_i \right] > 0 \right\}, \\ X_{0,i}^0(2lk+j) = \left\{ X_{0,i}(2lk+j) : (-1)^{m+1} \left(\frac{(4lk+2j+1)\pi}{2k} \right)^{2m+1} \tan \frac{(2j+1)\pi}{4k} = -\alpha_i \right\}, \\ X_{0,i}^-(2lk+j) = \left\{ X_{0,i}(2lk+j) : \left[(-1)^{m+1} \left(\frac{\pi}{2k} \right)^{2m+1} (4lk+2j+1)^{2m+1} \tan \frac{(2j+1)\pi}{4k} + \alpha_i \right] < 0 \right\}, \\ X_{\infty,i}^+(2lk+j) = \left\{ X_{\infty,i}(2lk+j) : \left[(-1)^{m+1} \left(\frac{\pi}{2k} \right)^{2m+1} (4lk+2j+1)^{2m+1} \tan \frac{(2j+1)\pi}{4k} + \beta_i \right] > 0 \right\}, \\ X_{\infty,i}^0(2lk+j) = \left\{ X_{\infty,i}(2lk+j) : (-1)^{m+1} \left(\frac{(4lk+2j+1)\pi}{2k} \right)^{2m+1} \tan \frac{(2j+1)\pi}{4k} = -\beta_i \right\}, \\ X_{\infty,i}^-(2lk+j) = \left\{ X_{\infty,i}(2lk+j) : \left[(-1)^{m+1} \left(\frac{\pi}{2k} \right)^{2m+1} (4lk+2j+1)^{2m+1} \tan \frac{(2j+1)\pi}{4k} + \beta_i \right] < 0 \right\}. \end{cases} \tag{13}$$

Then, we have

$$X_{\infty,i}^+ = \sum_{j=0}^{2k-1} \sum_{l=0}^{\infty} X_{\infty,i}^+(2lk+j), \quad X_{\infty,i}^0 = \sum_{j=0}^{2k-1} \sum_{l=0}^{\infty} X_{\infty,i}^0(2lk+j), \quad X_{\infty,i}^- = \sum_{j=0}^{2k-1} \sum_{l=0}^{\infty} X_{\infty,i}^-(2lk+j),$$

$$X_{0,i}^+ = \sum_{j=0}^{2k-1} \sum_{l=0}^{\infty} X_{0,i}^+(2lk+j), \quad X_{\infty,i}^0 = \sum_{j=0}^{2k-1} \sum_{l=0}^{\infty} X_{0,i}^0(2lk+j), \quad X_{0,i}^- = \sum_{j=0}^{2k-1} \sum_{l=0}^{\infty} X_{0,i}^-(2lk+j).$$

Thus,

$$X_0^+(2lk+j) = \bigoplus_{i=1}^{N} X_{0,i}^+, \quad X_\infty^+(2lk+j) = \bigoplus_{i=1}^{N} X_{\infty,i}^+,$$

$$X_0^0(2lk+j) = \bigoplus_{i=1}^{N} X_{0,i}^0, \quad X_\infty^0(2lk+j) = \bigoplus_{i=1}^{N} X_{\infty,i}^0,$$

$$X_0^-(2lk+j) = \bigoplus_{i=1}^{N} X_{0,i}^-, \quad X_\infty^-(2lk+j) = \bigoplus_{i=1}^{N} X_{\infty,i}^-.$$

It is easy to see that $|(-1)^{m+1}\left(\frac{(2j+1)\pi}{2k}\right)^{2m+1} \tan \frac{(2j+1)\pi}{4k}| \to \infty$ as $j \to \infty$, then we can get that $\dim X_\infty^0 < \infty$ and $\dim X_0^0 < \infty$.

4. Lemmas

Let X be a Hilbert space, $L : X \to X$ be a linear operator, and $\Phi : X \to R$ be a differentiable functional.

Lemma 1 ([4], Lemma 2.4). *Assume that there are two closed s^1-invariant linear subspaces, X^+ and X^-, and $r > 0$ such that*

(a) *$X^+ \bigcup X^-$ is closed and of finite codimensions in X,*
(b) *$\widehat{L}(X^-) \subset X^-, \widehat{L} = L + P^{-1}A_0$ or $\widehat{L} = L + P^{-1}A_\infty$,*
(c) *there exists $c_0 \in R$ such that*
$$\inf_{x \in X^+} \Phi(x) \geq c_0,$$

(d) *there is $c_\infty \in R$ such that*
$$\Phi(x) \leq c_\infty < \Phi(0) = 0, \forall x \in X^- \bigcap S_r = \{x \in X^- : \|x\| = r\},$$

(e) *Φ satisfies $(P.S)_c$-condition, $c_0 < c < c_\infty$, i.e., every sequence $\{x_n\} \subseteq X$ with $\Phi(x_n) \to c$ and $\Phi'(x_n) \to 0$ possesses a convergent subsequence.*

Then Φ has at least $\frac{1}{2}[\dim(X^+ \bigcap X^-) - \text{codim}_X(X^+ \bigcup X^-)]$ generally different critical orbits in $\Phi^{-1}([c_0, c_\infty])$ if $[\dim(X^+ \bigcap X^-) - \text{codim}_X(X^+ \bigcup X^-)] > 0$.

In addition, we make the following assumptions for convenience,
(S_1) f satisfies (3) and (4),
(S_2) $[F(x) - \frac{1}{2}(A_\infty x, x)] \to \infty, \quad |x| \to \infty.$

Lemma 2. *Under the assumptions of (S_1) and (S_2), there exists $\sigma > 0$ such that*

$$\left\langle (L + P^{-(2m+1)}A_\infty)x, x \right\rangle > \sigma\|x\|^2, \quad x \in X_\infty^+ \text{ and } \left\langle (L + P^{-(2m+1)}A_\infty)x, x \right\rangle < -\sigma\|x\|^2, \quad x \in X_\infty^- . \tag{14}$$

Proof of Lemma 2. Let $\Gamma_i : X \to X_i$, $i \in \{1, 2, \cdots, N\}$ be the orthogonal projection, then $id = \sum\limits_{i=1}^{N} \Gamma_i$. Assuming that $x \in X_\infty^+$, then for $x_i = \Gamma_i x \in X_{\infty,i}^+$, we can get

$$x_i(t) = \sum_{l=0}^{\infty} \sum_{j=0}^{2k-1} x_{\infty,i}^+(2lk + j).$$

Then from (12), we can get

$$\left\langle (L + P^{-(2m+1)}A_\infty)x_i, x_i \right\rangle$$
$$= \sum_{l=0}^{\infty} \sum_{j=0}^{2k-1} \left\langle (L + P^{-(2m+1)}A_{\infty,i})x_{\infty,i}^+(2lk + j), x_{\infty,i}^+(2lk + j) \right\rangle$$
$$= \sum_{l=0}^{\infty} \sum_{j=0}^{2k-1} \left[(-1)^{m+1}(\frac{\pi}{2k})^{2m+1}(4lk + 2j + 1)^{2m+1} \tan \frac{(2j+1)\pi}{4k} + \beta_i \right] \|x_{\infty,i}^+(2lk + j)\|_0^2$$
$$= \sum_{l=0}^{\infty} \sum_{j=0}^{2k-1} \left[(-1)^{m+1}(\frac{\pi}{2k})^{2m+1} \tan \frac{(2j+1)\pi}{4k} + \frac{\beta_i}{(4lk + 2j + 1)^{2m+1}} \right] \|x_{\infty,i}^+(2lk + j)\|^2$$

For convenience, let

$$h_{i,j}(l) = (-1)^{m+1}\left(\frac{\pi}{2k}\right)^{2m+1}\tan\frac{(2j+1)\pi}{4k} + \frac{\beta_i}{(4lk+2j+1)^{2m+1}},$$

then

$$\left\langle (L + P^{-(2m+1)}A_\infty)x_i, \, x_i \right\rangle$$
$$= \sum_{l=0}^{\infty}\sum_{j=0}^{2k-1} h_{i,j}(l)||x_{\infty,i}^+(2lk+j)||^2$$

in which $h_{i,j}(l) > 0$ as $x_i \in X_{\infty,i}^+$.

Furthermore, let $\delta = (-1)^{m+1}\left(\frac{\pi}{2k}\right)^{2m+1}\tan\frac{(2j+1)\pi}{4k}$ and $N_{i,j}^+ = \{l : h_{i,j}(l) > 0\}$. If $\delta < 0$, then $\lim\limits_{l\to\infty} h_{i,j}(l) = \delta < 0$. It follows that $N_{i,j}^+$ is a finite set and then there exists $l_0 \in N_{i,j}^+$, such that $h_{i,j}(l_0) = \min\left\{h_{i,j}(l) : l \in N_{i,j}^+\right\} > 0$. In this case, let $\sigma_{i,j,(1')} = h_{i,j}(l_0)$.

On the other hand, if $\delta > 0$, then there exists $L > 0$, such that $h_{i,j}(l) > \frac{\delta}{2}$ when $l > L$. Therefore, $\exists \hat{l} \in N_{i,j}^+ \cap \{0, 1, \cdots, L\}$, such that

$$h_{i,j}\left(\hat{l}\right) = \min\left\{h_{i,j}(l) : l \in N_{i,j}^+ \cap \{0, 1, \cdots, L\}\right\} > 0.$$

In this case, let $\sigma_{i,j,(1'')} = \min\left\{h_{i,j}\left(\hat{l}\right), \frac{\delta}{2}\right\}$. Then let $\sigma_{i,j,(1)} = \min\left\{\sigma_{i,j,(1')}, \sigma_{i,j,(1'')}\right\}$, one has $h_{i,j}(l) > \sigma_{i,j,(1)} > 0$.

Similarly, if $x_i(t) = \sum\limits_{l=0}^{\infty}\sum\limits_{j=0}^{2k-1} x_{\infty,i}^-(2lk+j) \in X_{\infty,i}^-$, one has $\exists \sigma_{i,j,(2)} > 0$, such that

$$h_{i,j}(l) < -\sigma_{i,j,(2)} < 0.$$

Then let $\sigma_{i,j} = \min\left\{\sigma_{i,j,(1)}, \sigma_{i,j,(2)}\right\} > 0$, we can get that

$$\left\langle (L + P^{-(2m+1)}A_\infty x_i, \, x_i \right\rangle = \sum_{l=0}^{\infty}\sum_{j=0}^{2k-1} h_{i,j}(l)||x_{\infty,i}^+(2lk+j)||^2 > \sigma_{i,j}\sum_{l=0}^{\infty}\sum_{j=0}^{2k-1}||x_{\infty,i}^+(2lk+j)||^2$$

when $x_i \in X_{\infty,i}^+$, and

$$\left\langle (L + P^{-(2m+1)}A_\infty x_i, \, x_i \right\rangle = \sum_{l=0}^{\infty}\sum_{j=0}^{2k-1} h_{i,j}(l)||x_{\infty,i}^-(2lk+j)||^2 < -\sigma_{i,j}\sum_{l=0}^{\infty}\sum_{j=0}^{2k-1}||x_{\infty,i}^-(2lk+j)||^2$$

when $x_i \in X_{\infty,i}^-$.

Then let $\sigma = \min\limits_{\substack{0\le j\le 2k-1 \\ 1\le i\le N}} \sigma_{i,j} > 0$. The inequalities in (14) are proved. $\quad\square$

Lemma 3. *Under the assumptions of* (S_1) *and* (S_2)*, the functional* Φ *defined by* (9) *satisfies* $(P.S)$*-condition.*

Proof of Lemma 3. Let $\Gamma_i : X \to X_{\infty,i}$ be the orthogonal projections, then $x_{n,i} = \Gamma_i x_n \in X_i$, $x_n = \sum\limits_{i=1}^{N} x_{n,i}$.

Assume that $\{x_n\} \subset X$ is a subsequence such that $\Phi'(x_n) \to 0$ and $\Phi(x_n)$ is bounded. Then we can get that, $\forall i \in \{1, 2, \cdots, N\}$, $\Phi'(x_{n,i}) \to 0$ and $\Phi(x_{n,i})$ is bounded. Let $x_{n,i}^+ = x_{n,i} \cap X_{\infty,i}^+$, $x_{n,i}^- =$

$x_{n,i} \cap X_{\infty,i}^-$, $x_{n,i}^0 = x_{n,i} \cap X_{\infty,i}^0$. Similarly, $\Phi'(x_{n,i}^+)$, $\Phi'(x_{n,i}^-)$, $\Phi'(x_{n,i}^0) \to 0$ when $n \to \infty$ and $\Phi(x_{n,i}^+)$, $\Phi(x_{n,i}^-)$, $\Phi(x_{n,i}^0)$ are bounded. According to (4), we can get that

$$\left| \left\langle P^{-(2m+1)}(\nabla F(x_{n,i}^+) - A_\infty x_{n,i}^+), \; x_{n,i}^+ \right\rangle \right| < \frac{\sigma}{2}||x_{n,i}^+||^2 + M,$$

for some $M > 0$. Thus, from

$$\left\langle P^{-(2m+1)}\Phi'(\;x_{n,i}^+), \; x_{n,i}^+ \right\rangle = \left\langle Lx_n + P^{-(2m+1)}\nabla F(\;x_{n,i}^+), \; x_{n,i}^+ \right\rangle \to 0,$$

and (14), we can get

$$\begin{aligned}
& \left\langle P^{-(2m+1)}\Phi'(\;x_{n,i}^+), \; x_{n,i}^+ \right\rangle \\
\geq \; & \left| \left\langle (L + P^{-(2m+1)}A_\infty)x_{n,i}^+, \; x_{n,i}^+ \right\rangle \right| - \left| \left\langle P^{-(2m+1)}(\nabla F(x_{n,i}^+) - A_\infty x_{n,i}^+), \; x_{n,i}^+ \right\rangle \right| \\
\geq \; & \frac{\sigma}{2}||x_{n,i}^+||^2 - M,
\end{aligned}$$

which implies the boundedness of $||x_{n,i}^+||$. Then we can get the boundedness of $||x_n^+||$. Similarly, we have the boundedness of $||x_n^-||$. At the same time, $||x_n^0||$ is bounded since X_n^0 is finite-dimensional. Therefore, $||x_n||$ is bounded. It follows from (11) that

$$P^{-(2m+1)}\Phi'(x_n) = Lx_n + K(x_n).$$

Considering the compactness of operator K and the boundedness of x_n, we can get that $K(x_n) \to u$. Then, we have that

$$Lx_n \to -u$$

as $\Phi'(x_n) \to 0$. Therefore,

$$x_n \to -L^{-1}u,$$

which implies $(P.S)$-condition. $\quad\square$

Lemma 4. *If x is a critical point of Φ, then the orbit corresponding to x is a periodic orbit to the system (2).*

Proof of Lemma 4. Suppose x is a critical point of Φ given by (9). Then $x(t)$ should satisfy

$$-\sum_{j=1}^{2k-1}(-1)^{j+1}x^{(2m+1)}(t-j) + \nabla F(x(t)) = 0, \; a.e. \; t \in [0, 4k]. \tag{15}$$

Consequently,

$$-\sum_{j=1}^{2k-1}(-1)^{j+1}x^{(2m+1)}(t-j-1) + \nabla F(x(t-1)) = 0, \tag{15a}$$

$$-\sum_{j=1}^{2k-1}(-1)^{j+1}x^{(2m+1)}(t-j-2) + \nabla F(x(t-2)) = 0, \tag{15b}$$

$$-\sum_{j=1}^{2k-1}(-1)^{j+1}x^{(2m+1)}(t-j-3) + \nabla F(x(t-3)) = 0, \tag{15c}$$

$$\vdots$$

$$-\sum_{j=0}^{2k-1}(-1)^{j+1}x^{(2m+1)}(t-j-(2k-1)) + \nabla F(x(t-(2k-1))) = 0. \tag{15(2k-1)}$$

Calculating (15a) + (15b) + (15c) + $\cdots$ + (15(2k − 1)), we have

$$x^{(2m+1)}(t) + \sum_{j=1}^{2k-1} \nabla F(x(t-j)) = 0, \ a.e. \ t \in [0, 4k],$$

namely,

$$x^{(2m+1)}(t) = - \sum_{j=1}^{2k-1} \nabla F(x(t-j)), \ a.e. \ t \in [0, 4k],$$

hence, the orbit corresponding to x is a periodic orbit of (2). $\quad\square$

5. Main Results

From the definition of (13), we can get that there exists $d > 0$, such that

$$\begin{cases} X^+_{\infty,i}(2lk+j) = X_{\infty,i}(2lk+j), \\ X^-_{0,i}(2lk+j) = \phi, \\ X^-_{\infty,i}(2lk+j) = \phi, \\ X^+_{0,i}(2lk+j) = X_{0,i}(2lk+j), \end{cases}$$

when $l > d$ and $(-1)^{m+1} \tan \frac{(2j+1)\pi}{4k} > 0$. Meanwhile,

$$\begin{cases} X^-_{\infty,i}(2lk+j) = X_{\infty,i}(2lk+j), \\ X^+_{0,i}(2lk+j) = \phi, \\ X^+_{\infty,i}(2lk+j) = \phi, \\ X^-_{0,i}(2lk+j) = X_{0,i}(2lk+j), \end{cases}$$

when $l > d$ and $(-1)^{m+1} \tan \frac{(2j+1)\pi}{4k} < 0$.

So,

$$X^+_\infty(2lk+j) \bigcap X^-_0(2lk+j) = C_{X(2lk+j)}(X^+_\infty(2lk+j) \bigcup X^-_0(2lk+j)) = \phi,$$

$$X^-_\infty(2lk+j) \bigcap X^+_0(2lk+j) = C_{X(2lk+j)}(X^-_\infty(2lk+j) \bigcup X^+_0(2lk+j)) = \phi,$$

when $l > d$. Thus, we have

$$X^+_\infty \bigcap X^-_0 = \overset{2k-1}{\underset{j=0}{\oplus}} \overset{d}{\underset{l=0}{\oplus}} (X^+_\infty(2lk+j) \bigcap X^-_0(2lk+j)),$$

$$X^-_\infty \bigcap X^+_0 = \overset{2k-1}{\underset{j=0}{\oplus}} \overset{d}{\underset{l=0}{\oplus}} (X^-_\infty(2lk+j) \bigcap X^+_0(2lk+j)),$$

$$C_X(X^+_\infty \bigcup X^-_0) = \overset{2k-1}{\underset{j=0}{\oplus}} \overset{d}{\underset{l=0}{\oplus}} (C_{X(2lk+j)}(X^+_\infty(2lk+j) \bigcup X^-_0(2lk+j))),$$

$$C_X(X^-_\infty \bigcup X^+_0) = \overset{2k-1}{\underset{j=0}{\oplus}} \overset{d}{\underset{l=0}{\oplus}} (C_{X(2lk+j)}(X^-_\infty(2lk+j) \bigcup X^+_0(2lk+j))).$$

Then, we can get

$$\begin{cases} dim(X^+_\infty \bigcap X^-_0) < \infty, \\ cod(X^-_\infty \bigcup X^+_0) = dim \, C_X(X^-_\infty \bigcup X^+_0) < \infty, \\ dim(X^-_\infty \bigcap X^+_0) < \infty, \\ cod(X^+_\infty \bigcup X^-_0) = dim \, C_X(X^+_\infty \bigcup X^-_0) < \infty. \end{cases} \quad (16)$$

Denote

$$K = \dim(X_\infty^+ \cap X_0^-), \quad \bar{K}_c = cod(X_\infty^+ \cup X_0^-),$$
$$K_- = \dim(X_\infty^- \cap X_0^+), \quad \bar{K}_{-c} = cod(X_\infty^- \cup X_0^+).$$

Now we give the main results of this paper.

Theorem 1. *Suppose that* (S_1) *and* (S_2) *hold. Then system* (2) *possesses at least*

$$\tilde{n} = \frac{1}{2} \max\{0, \ K - \bar{K}_c, \ K_- - \bar{K}_{-c}\}$$

4k-periodic orbits satisfying $x(t - 2k) = -x(t)$.

Proof of Theorem 1. Suppose without loss of generality that

$$\tilde{n} = \frac{1}{2}[K - \bar{K}_c] > 0.$$

Firstly, it can be seen from (16) that K, $\bar{K}_c$, K_- and $\bar{K}_{-c}$ are finite numbers, which implies that condition (a) in Lemma 1 holds. For each $i \in N$, we have that $x \in X(i)$ yields $(L + P^{-(2m+1)} A_\infty)x \in X(i)$ which implies that the condition (b) in Lemma 1 holds.

Moreover, Lemma 3 gives the $(P.S)$-condition.

In the following part, we just need to show that conditions (c) and (d) in Lemma 1 hold.

Let $\Pi_i : \mathbb{R}^N \to W_i$, $i = 1, 2, \cdots, N$ be the orthogonal projection. $\forall x \in X^+ = X_\infty^+$, let $x_i = \Pi_i x \in X^+$, then $x = \sum\limits_{i=1}^{N} x_i$, $x_i \in X_{\infty,i}^+$.

The second condition in (4) implies that $|F(x) - \frac{1}{2}(A_\infty x, x)| < \frac{1}{4}\sigma|x|^2 + M_1, x \in R$ for some $M_1 > 0$. Meanwhile, taking Lemma 2 into account, we can get

$$\begin{aligned}
\Phi(x) &= \frac{1}{2} \langle Lx, x \rangle + \int_0^{4k} F(x(t)) dt \\
&= \frac{1}{2} \left\langle (L + P^{-(2m+1)} A_\infty)x, \ x \right\rangle + \int_0^{4k} \left[F(x(t)) - \frac{1}{2}(A_\infty x, x)\right] dt \\
&\geq \frac{1}{2}\sigma\|x\|^2 - \frac{1}{4}\sigma\|x\|^2 - 4kM_1 \\
&\geq \frac{1}{4}\sigma\|x\|^2 - 4kM_1
\end{aligned}$$

if $x \in X^+$. Clearly, there is $c_0 \in R$ such that

$$\inf_{x \in X^+} \Phi(x) \geq c_0.$$

On the other hand,

$$\lim_{|x| \to 0} \frac{|F(x) - \frac{1}{2}(A_0 x, x)|}{|x|^2} = \frac{1}{2} \lim_{|x| \to 0} \frac{|\nabla F(x) - A_0 x|}{|x|} = 0,$$

so there exists $\delta > 0$, such that

$$\left|F(x) - \frac{1}{2}(A_0 x, x)\right| < \frac{\sigma}{4}|x|^2, \ 0 < |x| \leq \delta.$$

Thus, $\exists r > 0$, when $x \in X$, $\|x\| = r$, we have

$$|F(x) - \frac{1}{2}(A_0 x(t), x(t))| < \frac{\sigma}{4}\|x\|^2.$$

So

$$\begin{aligned}
\Phi(x) &= \frac{1}{2}\langle Lx, x\rangle + \int_0^{4k} F(x(t))dt \\
&= \frac{1}{2}\left\langle (L + P^{-(2m+1)}A_0)x,\, x\right\rangle + \int_0^{4k}\left[F(x(t)) - \frac{1}{2}(A_0 x, x)\right]dt \\
&\le -\frac{1}{2}\sigma\|x\|^2 + \frac{1}{4}\sigma\|x\|^2 \\
&\le -\frac{1}{4}\sigma\|x\|^2.
\end{aligned}$$

That is, there are $r > 0$ and $c_\infty < 0$ such that

$$\Phi(x) \le c_\infty < 0 = \Phi(0), \forall x \in X^- \bigcap S_r = \{x \in X : \|x\| = r\}.$$

Then, according to Lemma 1, $\Phi(x)$ has at least $\tilde{n}$ different $4k-$periodic orbits satisfying $x(t - 2k) = -x(t)$. □

Furthermore, let

$$n_{\infty,i}^+(2lk + j) = \begin{cases} 1, & (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} > -\beta_i, \\ 0, & (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} \le -\beta_i, \end{cases}$$

$$n_{\infty,i}^-(2lk + j) = \begin{cases} 1, & (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} < -\beta_i, \\ 0, & (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} \ge -\beta_i, \end{cases}$$

$$n_{0,i}^+(2lk + j) = \begin{cases} 1, & (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} > -\alpha_i, \\ 0, & (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} \le -\alpha_i, \end{cases}$$

$$n_{0,i}^-(2lk + j) = \begin{cases} 1, & (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} < -\alpha_i, \\ 0, & (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} \ge -\alpha_i, \end{cases}$$

According to the definition of (13), we have

$$n_{\infty,i}^+(2lk + j) = \begin{cases} 1, & X_{\infty,i}^+(2lk + j) = X_{\infty,i}(2lk + j), \\ 0, & X_{\infty,i}^+(2lk + j) = \phi, \end{cases}$$

$$n_{\infty,i}^-(2lk + j) = \begin{cases} 1, & X_{\infty,i}^-(2lk + j) = X_{\infty,i}(2lk + j), \\ 0, & X_{\infty,i}^-(2lk + j) = \phi, \end{cases}$$

$$n_{0,i}^+(2lk + j) = \begin{cases} 1, & X_{0,i}^+(2lk + j) = X_{0,i}(2lk + j), \\ 0, & X_{0,i}^+(2lk + j) = \phi, \end{cases}$$

$$n_{0,i}^-(2lk + j) = \begin{cases} 1, & X_{0,i}^-(2lk + j) = X_{0,i}(2lk + j), \\ 0, & X_{0,i}^-(2lk + j) = \phi, \end{cases}$$

Let

$$n_0^\pm(2lk + j) = \sum_{i=1}^N n_{0,i}^\pm(2lk + j), \quad n_\infty^\pm(2lk + j) = \sum_{i=1}^N n_{\infty,i}^\pm(2lk + j).$$

Then, $\forall j \in \{0, 1, 2, \cdots, 2k-1\}$ and $\forall i \in \{1, 2, \cdots, N\}$,

$$\begin{cases} n^+_{\infty,i}(2lk+j) = 1, & n^-_{0,i}(2lk+j) = 0, \\ n^-_{\infty,i}(2lk+j) = 0, & n^+_{0,i}(2lk+j) = 1, \end{cases} \quad (-1)^{m+1} \tan \frac{(2j+1)\pi}{4k} > 0$$

when $l > d > 0$. Therefore,

$$\begin{cases} n^+_{\infty}(2lk+j) = N, & n^-_0(2lk+j) = 0, \\ n^-_{\infty}(2lk+j) = 0, & n^+_0(2lk+j) = N, \end{cases} \quad (-1)^{m+1} \tan \frac{(2j+1)\pi}{4k} > 0$$

when $l > d$.

On the other hand, $X^+_\infty \cap X^-_0 = \overset{2k-1}{\underset{j=0}{\oplus}} \overset{d}{\underset{l=0}{\oplus}} (X^+_\infty(2lk+j) \cap X^-_0(2lk+j))$, so

$$\dim(X^+_\infty \cap X^-_0) - cod_X(X^+_\infty \cup X^-_0)$$

$$= \sum_{j=0}^{2k-1} \sum_{l=0}^{d} [\dim X^+_\infty(2lk+j) + \dim X^-_0(2lk+j) - 2N]$$

$$= 2 \sum_{j=0}^{2k-1} \sum_{l=0}^{d} [n^+_\infty(2lk+j) + n^-_0(2lk+j) - N].$$

Similarly, $X^-_\infty \cap X^+_0 = \overset{2k-1}{\underset{j=0}{\oplus}} \overset{d}{\underset{l=0}{\oplus}} (X^-_\infty(2lk+j) \cap X^+_0(2lk+j))$, so

$$dim(X^-_\infty \cap X^+_0) - cod_X(X^-_\infty \cup X^+_0)$$

$$= 2 \sum_{j=0}^{2k-1} \sum_{l=0}^{d} [n^-_\infty(2lk+j) + n^+_0(2lk+j) - N].$$

Hence,

$$K - \bar{K}_c = \dim(X^+_\infty \cap X^-_0) - cod_X(X^+_\infty \cup X^-_0)$$

$$= \sum_{j=0}^{2k-1} \sum_{l=0}^{d} [2n^+_\infty(2lk+j) + 2n^-_0(2lk+j) - 2N],$$

$$K_- - \bar{K}_{-c} = \sum_{j=0}^{2k-1} \sum_{l=0}^{d} [2n^-_\infty(2lk+j) + 2n^+_0(2lk+j) - 2N].$$

However, the existence of integer d is proved only in theory in the above formulas, and the calculation method is not given, so it is difficult to calculate the specific values of $K - \bar{K}_c$ and $K_- - \bar{K}_{-c}$, causing inconvenience in practical application. So, define

$$d_j = \begin{cases} \min\{l \geq 0: \ (-1)^{m+1}(\frac{(4lk+2j+1)\pi}{2k})^{2m+1} \tan \frac{(2j+1)\pi}{4k} > -\gamma_m\}, & (-1)^{m+1} \tan \frac{(2j+1)\pi}{4k} > 0 \\ \min\{l \geq 0: \ (-1)^{m+1}(\frac{(4lk+2j+1)\pi}{2k})^{2m+1} \tan \frac{(2j+1)\pi}{4k} < -\gamma_M\}, & (-1)^{m+1} \tan \frac{(2j+1)\pi}{4k} < 0 \end{cases} \quad (17)$$

in which $\gamma_m = \min\{\min_{1 \leq i \leq N} \alpha_i, \min_{1 \leq i \leq N} \beta_i\}, \quad \gamma_M = \max\{\max_{1 \leq i \leq N} \alpha_i, \max_{1 \leq i \leq N} \beta_i\}$.

Obviously, $d_j \leq d$. Then when $l > d_j$, we can get

$$
\begin{cases} n_\infty^+(2lk+j) = N, \ n_0^-(2lk+j) = 0, \\ n_\infty^-(2lk+j) = 0, \ n_0^+(2lk+j) = N, \end{cases} \quad (-1)^{m+1} \tan \frac{(2j+1)\pi}{4k} > 0,
$$

$$
\begin{cases} n_\infty^+(2lk+j) = 0, \ n_0^-(2lk+j) = N, \\ n_\infty^-(2lk+j) = N, \ n_0^+(2lk+j) = 0, \end{cases} \quad (-1)^{m+1} \tan \frac{(2j+1)\pi}{4k} < 0.
$$

Hence,

$$
K - \bar{K}_c = \dim(X_\infty^+ \bigcap X_0^-) - cod(X_\infty^+ \bigcup X_0^-)
$$

$$
= 2 \sum_{j=0}^{2k-1} \sum_{l=0}^{d_j} [n_\infty^+(2lk+j) + n_0^-(2lk+j) - N],
$$

$$
K_- - \bar{K}_{-c} = 2 \sum_{j=0}^{2k-1} \sum_{l=0}^{d_j} [n_\infty^-(2lk+j) + n_0^+(2lk+j) - N].
$$

This leads to the following corollary,

Corollary 1. *Suppose that* (S_1) *and* (S_2) *hold. Then system* (2) *possesses at least*

$$
\tilde{n} = \max\{ 0, \ \sum_{j=0}^{2k-1} \sum_{l=0}^{d_j} [n_\infty^+(2lk+j) + n_0^-(2lk+j) - N], \ \sum_{j=0}^{2k-1} \sum_{l=0}^{d_j} [n_\infty^-(2lk+j) + n_0^+(2lk+j) - N]\}
$$

geometrically different $4k$-*periodic orbits satisfying* $x(t - 2k) = -x(t)$.

Especially, when the system (2) can be decomposed into n systems, let

$$
z = (y_1, y_2, \cdots, y_n), \ r_0 = 0,
$$

and

$$
y_i^{(2m+1)}(t) = - \sum_{p=1}^{2k-1} \nabla \tilde{H}(y_i(t-p)), \quad y_i \in \mathbb{R}_i^{r_i - r_{i-1}}, \ i = 1, 2, \cdots, n \tag{18}
$$

in which $y_i = (x_{r_{i-1}+1}, x_{r_{i-1}+2}, \cdots, x_{r_i})$, $\nabla \tilde{H}(y_i) = \nabla_{y_i} H(0, \cdots, 0, y_i, 0, \cdots, 0)$.
It is easy to see that (18) is a $(r_i - r_{i-1})$-dimensional system. From (4), we can get

$$
\begin{aligned}
|\nabla \tilde{H}(y_i) - B_{0,i} y_i| &= \circ(|y_i|), \ |y_i| \to 0, \\
|\nabla \tilde{H}(y_i) - B_{\infty,i} y_i| &= \circ(|y_i|), \ |y_i| \to \infty.
\end{aligned}
$$

Assume that

$$
\sigma(B_{0,i}) = (\alpha_{r_{i-1}+1}, \alpha_{r_{i-1}+2}, \cdots, \alpha_{r_i}), \quad and \quad \sigma(B_{\infty,i}) = (\beta_{r_{i-1}+1}, \beta_{r_{i-1}+2}, \cdots, \beta_{r_i})
$$

are respectively the eigenvalues of $B_{0,i}$ and $B_{\infty,i}$. Their corresponding unit eigenvectors in space are respectively $(u_{r_{i-1}+1}, u_{r_{i-1}+2}, \cdots, u_{r_i})$ and $(v_{r_{i-1}+1}, v_{r_{i-1}+2}, \cdots, v_{r_i})$.

For (18), let

$$X^i = cl\{x(t) = \sum_{j=0}^{\infty} a_j \cos \frac{(2j+1)\pi t}{2k} + b_j \sin \frac{(2j+1)\pi t}{2k} \ : \ a_j, b_j \in \mathbb{R}^{r_i - r_{i-1}},$$

$$\sum_{j=0}^{\infty} (2j+1)^{2m+1}(|a_j|^2 + |b_j|^2) < \infty\}$$

and

$$X^i(j) = \{a_j \cos \frac{(2j+1)\pi t}{2k} + b_j \sin \frac{(2j+1)\pi t}{2k} \ : \ a_j, b_j \in \mathbb{R}^{r_i - r_{i-1}}\},$$

then

$$X^i = \bigoplus_{j=0}^{\infty} X^i(j) = \bigoplus_{l=0}^{\infty} \bigoplus_{j=0}^{2k-1} X^i(2lk+j).$$

Further subdivision, let

$$X^i_{0,s}(j) = \{a_j \cos \frac{(2j+1)\pi t}{2k} + b_j \sin \frac{(2j+1)\pi t}{2k} \ : \ a_j, b_j \in U_s\},$$

$$X^i_{\infty,s}(j) = \{a_j \cos \frac{(2j+1)\pi t}{2k} + b_j \sin \frac{(2j+1)\pi t}{2k} \ : \ a_j, b_j \in V_s\},$$

$$X^i = \bigoplus_{j=0}^{\infty} \bigoplus_{s=r_{i-1}+1}^{r_i} X^i_{0,s}(j) = \bigoplus_{j=0}^{\infty} \bigoplus_{s=r_{i-1}+1}^{r_i} X^i_{\infty,s}(j).$$

$\forall s \in \{r_{i-1}+1,\ r_{i-1}+2,\ \cdots,\ r_i\}$, denote

$$X^{i,+}_{0,s} = \oplus \left\{ X^i_{0,s}(2lk+j) : (-1)^{m+1} \left(\frac{\pi}{2k}\right)^{2m+1} (4lk+2j+1)^{2m+1} \tan \frac{(2j+1)\pi}{4k} + \alpha_s > 0 \right\},$$

$$X^{i,0}_{0,s} = \oplus \left\{ X^i_{0,s}(2lk+j) : (-1)^{m+1} \left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan \frac{(2j+1)\pi}{4k} = -\alpha_s \right\},$$

$$X^{i,-}_{0,s} = \oplus \left\{ X^i_{0,s}(2lk+j) : (-1)^{m+1} \left(\frac{\pi}{2k}\right)^{2m+1} (4lk+2j+1)^{2m+1} \tan \frac{(2j+1)\pi}{4k} + \alpha_s < 0 \right\}$$

$$X^{i,+}_{\infty,s} = \oplus \left\{ X^i_{\infty,s}(2k+j) : (-1)^{m+1} \left(\frac{\pi}{2k}\right)^{2m+1} (4k+2j+1)^{2m+1} \tan \frac{(2j+1)\pi}{4k} + \beta_s > 0 \right\},$$

$$X^{i,0}_{\infty,s} = \oplus \left\{ X^i_{\infty,s}(2k+j) : (-1)^{m+1} \left(\frac{(4k+2j+1)\pi}{2k}\right)^{2m+1} \tan \frac{(2j+1)\pi}{4k} = -\beta_s \right\},$$

$$X^{i,-}_{\infty,s} = \oplus \left\{ X^i_{\infty,s}(2k+j) : (-1)^{m+1} \left(\frac{\pi}{2k}\right)^{2m+1} (4k+2j+1)^{2m+1} \tan \frac{(2j+1)\pi}{4k} + \beta_s < 0 \right\},$$

and

$$X^{i,+}_0 = \bigoplus_{s=r_{i-1}+1}^{r_i} X^{i,+}_{0,s}, \ \ X^{i,0}_0 = \bigoplus_{s=r_{i-1}+1}^{r_i} X^{i,0}_{0,s}, \ X^{i,-}_0 = \bigoplus_{s=r_{i-1}+1}^{r_i} X^{i,-}_{0,s},$$

$$X^{i,+}_\infty = \bigoplus_{s=r_{i-1}+1}^{r_i} X^{i,+}_{\infty,s}, \ \ X^{i,0}_\infty = \bigoplus_{s=r_{i-1}+1}^{r_i} X^{i,0}_{\infty,s}, \ X^{i,-}_\infty = \bigoplus_{s=r_{i-1}+1}^{r_i} X^{i,-}_{\infty,s},$$

$$K^i = \dim(X^{i,+}_\infty \cap X^{i,-}_0), \ \ \bar{K}^i_c = cod(X^{i,+}_\infty \cup X^{i,-}_0),$$

$$K^i_- = \dim(X^{i,-}_\infty \cap X^{i,+}_0), \ \ \bar{K}_{-c} = cod(X^{i,-}_\infty \cup X^{i,+}_0).$$

Then we can get

Theorem 2. *Suppose that* (S_1) *and* (S_2) *hold and the differential system* (2) *can be decomposed into the independent systems shown in* (18), *then the system* (2) *possesses at least*

$$\tilde{n} = \frac{1}{2} \sum_{i=1}^{n} \max\{ 0, \ K^i - \bar{K}^i_c, \ K^i_- - \bar{K}^i_{-c} \}$$

geometrically different $4k$-*periodic orbits satisfying* $x(t - 2k) = -x(t)$.

Proof of Theorem 2. If the system (2) can be decomposed into n subsystems as shown in (18), according to Theorem 1, each subsystem has at least

$$\tilde{n}^i = \frac{1}{2} \max\{ 0, \ K^i - \bar{K}^i_c, \ K^i_- - \bar{K}^i_{-c} \}, \ i \in \{1, 2, \cdots, n\}$$

$4k$-periodic orbits satisfying $x(t - 2k) = -x(t)$. However, the periodic orbits of each different system represented by (18) are different and all of them are periodic orbits of (2). Therefore, the conclusion of Theorem 2 is given. $\square$

Furthermore, when $r_{i-1} + 1 \leq s \leq r_i$ $(r_0 = 0, \ r_n = N)$, let

$$n^{i,+}_{\infty,s}(2lk + j) = \begin{cases} 1, & (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} > -\beta_s, \\ 0, & (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} \leq -\beta_s, \end{cases}$$

$$n^{i,-}_{\infty,s}(2lk + j) = \begin{cases} 1, & (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} < -\beta_s, \\ 0, & (-1)^{m+1}\left(\frac{4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} \geq -\beta_s; \end{cases}$$

$$n^{i,+}_{0,s}(2lk + j) = \begin{cases} 1, & (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} > -\alpha_s, \\ 0, & (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} \leq -\alpha_s, \end{cases}$$

$$n^{i,-}_{0,s}(2lk + j) = \begin{cases} 1, & (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} < -\alpha_s, \\ 0, & (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} \geq -\alpha_s, \end{cases}$$

$$n^{i,+}_{\infty}(2lk + j) = \sum_{s=r_{i-1}+1}^{r_i} n^{i,+}_{\infty,s}(2lk + j), \ \ n^{i,-}_{\infty}(2lk + j) = \sum_{s=r_{i-1}+1}^{r_i} n^{i,-}_{\infty,s}(2lk + j),$$

$$n^{i,+}_{0}(2lk + j) = \sum_{s=r_{i-1}+1}^{r_i} n^{i,+}_{0,s}(2lk + j), \ \ n^{i,-}_{0}(2lk + j) = \sum_{s=r_{i-1}+1}^{r_i} n^{i,-}_{0,s}(2lk + j).$$

Then for $j \in \{0, 1, 2, \cdots, 2k - 1\}$, $i \in \{1, 2, \cdots, n\}$, $s \in \{r_{i-1} + 1, r_{i-1} + 2, \cdots, r_i\}$, define

$$d^i_j = \begin{cases} \min\{l \geq 0 : (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} > -\gamma^i_m\}, & (-1)^{m+1}\tan\frac{(2j+1)\pi}{4k} > 0 \\ \min\{l \geq 0 : (-1)^{m+1}\left(\frac{(4lk+2j+1)\pi}{2k}\right)^{2m+1} \tan\frac{(2j+1)\pi}{4k} < -\gamma^i_M\}, & (-1)^{m+1}\tan\frac{(2j+1)\pi}{4k} < 0 \end{cases}$$

in which $\gamma^i_m = \min\{ \min\limits_{r_{i-1}+1 \leq i \leq r_i} \alpha_i, \ \min\limits_{r_{i-1}+1 \leq i \leq r_i} \beta_i \}$, $\gamma^i_M = \max\{ \max\limits_{r_{i-1}+1 \leq i \leq r_i} \alpha_i, \ \max\limits_{r_{i-1}+1 \leq i \leq r_i} \beta_i \}$. Obviously, $d^i_j \leq d$. Then when $l > d^i_j$, we have

$$\begin{cases} n_{\infty,s}^{i,+}(2lk+j)=1, & n_{0,s}^{i,-}(2lk+j)=0, \\ n_{\infty,s}^{i,-}(2lk+j)=0, & n_{0,s}^{i,+}(2lk+j)=1, \end{cases} \quad (-1)^{m+1}\tan\frac{(2j+1)\pi}{4k}>0,$$

$$\begin{cases} n_{\infty,s}^{i,+}(2lk+j)=0, & n_{0,s}^{i,-}(2lk+j)=1, \\ n_{\infty,s}^{i,-}(2lk+j)=1, & n_{0,s}^{i,+}(2lk+j)=0, \end{cases} \quad (-1)^{m+1}\tan\frac{(2j+1)\pi}{4k}<0,$$

$$n_{\infty}^{i,+}(2lk+j)+n_0^{i,-}(2lk+j)=r_i-r_{i-1},$$

$$n_{\infty}^{i,-}(2lk+j)+n_0^{i,+}(2lk+j)=r_i-r_{i-1}.$$

Hence,

$$K^i-\bar{K}_c^i=\dim(X_\infty^+\cap X_0^-)-cod(X_\infty^+\cup X_0^-)$$

$$=\sum_{j=0}^{2k-1}\sum_{l=0}^{d_j^i}[2n_\infty^{i,+}(2lk+j)+2n_0^{i,-}(2lk+j)-2(r_i-r_{i-1})]$$

$$K_{-}^i-\bar{K}_{-c}^i=\sum_{j=0}^{2k-1}\sum_{l=0}^{d_j^i}[2n_\infty^{i,-}(2lk+j)+2n_0^{i,+}(2lk+j)-2(r_i-r_{i-1})]$$

Combining theorem 2 with corollary 1, it is easy to get

Corollary 2. *Suppose that* (S_1) *and* (S_2) *hold and the differential system* (2) *can be decomposed into the independent systems shown in* (18), *then the system* (2) *possesses at least*

$$\tilde{n}=\sum_{i=1}^{n}\max\Big\{0,\ \sum_{j=0}^{2k-1}\sum_{l=0}^{d_j^i}[n_\infty^{i,+}(2lk+j)+n_0^{i,-}(2lk+j)-(r_i-r_{i-1})],$$

$$\sum_{j=0}^{2k-1}\sum_{l=0}^{d_j^i}[n_\infty^{i,-}(2lk+j)+n_0^{i,+}(2lk+j)-(r_i-r_{i-1})]\Big\}$$

geometrically different 4k-periodic orbits satisfying $x(t-2k)=-x(t)$.

6. Example

Example 1. *We study the multiplicity of 8-periodic orbits of the system*

$$x^{(3)}(t)=-\sum_{q=1}^{3}\nabla F(x(t-q)),\quad x\in\mathbb{R}^4 \tag{19}$$

in which

$$F(x)=\frac{(A_0x,\ x)+(A_\infty x,\ x)\sum_{i=1}^{4}x_i^2}{2(1+\sum_{i=1}^{4}x_i^2)},\quad x=(x_1,x_2,\cdots,x_4)$$

$$A_0=\pi^3\begin{pmatrix}-12 & 0 & 0 & 0\\ 0 & -10 & 0 & 0\\ 0 & 0 & 33 & 2\\ 0 & 0 & 2 & 36\end{pmatrix},\quad A_\infty=\pi^3\begin{pmatrix}12 & 0 & 0 & 0\\ 0 & 15 & 0 & 0\\ 0 & 0 & -6 & 2\\ 0 & 0 & 2 & -9\end{pmatrix}.$$

In this case,

$$\alpha_1 = -12\pi^3, \ \alpha_2 = -10\pi^3, \ \alpha_3 = 32\pi^3, \ \alpha_4 = 37\pi^3,$$

$$u_1 = (1,0,0,0), \ u_2 = (0,1,0,0), \ u_3 = (0,0,\frac{2}{\sqrt{5}},\frac{-1}{\sqrt{5}}), \ u_4 = (0,0,\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}),$$

$$\beta_1 = 12\pi^3, \ \beta_2 = 15\pi^3, \ \beta_3 = -5\pi^3, \ \beta_4 = -10\pi^3,$$

$$v_1 = (1,0,0,0), \ v_2 = (0,1,0,0), \ v_3 = (0,0,\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}), \ v_4 = (0,0,\frac{1}{\sqrt{5}},\frac{-2}{\sqrt{5}}).$$

Then

$$W_1 = span\{u_1\} = span\{v_1\}, \ W_2 = span\{u_2\} = span\{v_2\}, \ W_3 = span\{u_3, u_4\} = span\{v_3, v_4\},$$

and $\dim W_1 = \dim W_2 = 1, \ \dim W_3 = 2.$

Let

$$F_1(x_1) = \pi^3 \left[\frac{-6x_1^2 + 6x_1^4}{1 + x_1^2}\right], \qquad F_2(x_2) = \pi^3 \left[\frac{-10x_2^2 + 15x_2^4}{2(1 + x_2^2)}\right],$$

$$F_3(x_3, x_4) = \pi^3 \left[\frac{33x_3^2 + 4x_3x_4 + 36x_4^2 + (-6x_3^2 + 4x_3x_4 - 9x_4^2)(x_3^2 + x_4^2)}{2(1 + x_3^2 + x_4^2)}\right].$$

Then

$$\nabla F_1 = \pi^3 \left(\frac{-12x_1 + 36x_1^3}{1 + x_1^2} - \frac{24x_1^5}{(1 + x_1^2)^2}\right), \quad \nabla F_2 = \pi^3 \left(\frac{-10x_1 + 40x_1^3}{1 + x_2^2} - \frac{25x_1^3}{(1 + x_2^2)^2}\right),$$

$$\nabla F_3 = \pi^3 \left(\frac{33x_3 + 2x_4 + (-6x_3 + 2x_4)(x_3^2 + x_4^2)}{1 + x_3^2 + x_4^2} + \frac{x_3(-6x_3^2 + 4x_3x_4 - 9x_4^2)}{(1 + x_3^2 + x_4^2)^2},\right.$$

$$\left.\frac{2x_3 + 36x_4 + (-6x_3 + 2x_4)(x_3^2 + x_4^2)}{1 + x_3^2 + x_4^2} - \frac{x_4(-6x_3^2 + 4x_3x_4 - 9x_4^2)}{(1 + x_3^2 + x_4^2)^2}\right).$$

System (19) *can be decomposed into independent systems as follows*

$$x_1^{(3)}(t) = -\sum_{q=1}^{3} \nabla F_1(x_1(t-q)), \quad x_1 \in \mathbb{R},$$

$$x_2^{(3)}(t) = -\sum_{q=1}^{3} \nabla F_3(x_2(t-q)), \quad x_2 \in \mathbb{R},$$

$$(x_3^{(3)}(t), x_4^{(3)}(t)) = -\sum_{q=1}^{3} \nabla F_2(x_3(t-q), x_4(t-q)), \quad (x_3, x_4) \in \mathbb{R}^2.$$

Hence, according to corollary 2, *we can get*

$$\tilde{n} = \max\{0, \sum_{j=0}^{3}\sum_{l=0}^{d_j^1} [n_\infty^{1,+}(4l+j) + n_0^{1,-}(4l+j) - 1], \ \sum_{j=0}^{3}\sum_{l=0}^{d_j^1} [n_\infty^{1,-}(4l+j) + n_0^{1,+}(4l+j) - 1]\}$$

$$+ \max\{0, \sum_{j=0}^{3}\sum_{l=0}^{d_j^2} [n_\infty^{2,+}(4l+j) + n_0^{2,-}(4l+j) - 1], \ \sum_{j=0}^{3}\sum_{l=0}^{d_j^2} [n_\infty^{2,-}(4l+j) + n_0^{2,+}(4l+j) - 1]\}$$

$$+ \max\{0, \sum_{j=0}^{3}\sum_{l=0}^{d_j^3} [n_\infty^{3,+}(4l+j) + n_0^{3,-}(4l+j) - 2], \ \sum_{j=0}^{3}\sum_{l=0}^{d_j^3} [n_\infty^{3,-}(4l+j) + n_0^{3,+}(4l+j) - 2]\}.$$

In this case, $m = 1$, $k = 2$, $(-1)^{m+1} = 1$, $j \in \{0, 1, 2, 3\}$, then by calculating, we have

$$\tan \frac{\pi}{8} = 0.4142, \ \tan \frac{3\pi}{8} = 2.4142, \ \tan \frac{5\pi}{8} = -2.4142, \ \tan \frac{7\pi}{8} = -0.4142.$$

$$\gamma_m^1 = -12\pi^3, \ \gamma_M^1 = 12\pi^3, \ \gamma_m^2 = -10\pi^3, \ \gamma_M^2 = 15\pi^3, \ \gamma_m^3 = -10\pi^3, \ \gamma_M^3 = 37\pi^3.$$

$$d_0^1 = \min\{l \geq 0: \ 0.4142 \left(\frac{8l+1}{4}\right)^3 > 12\} = 2,$$

$$d_1^1 = \min\{l \geq 0: \ 2.4142 \left(\frac{8l+3}{4}\right)^3 > 12\} = 1,$$

$$d_2^1 = \min\{l \geq 0: \ -2.4142 \left(\frac{8l+5}{4}\right)^3 < -12\} = 1,$$

$$d_3^1 = \min\{l \geq 0: \ -0.4142 \left(\frac{8l+7}{4}\right)^3 < -12\} = 1,$$

$$d_0^2 = \min\{l \geq 0: \ 0.4142 \left(\frac{8l+1}{4}\right)^3 > 10\} = 2,$$

$$d_1^2 = \min\{l \geq 0: \ 2.4142 \left(\frac{8l+3}{4}\right)^3 > 10\} = 1,$$

$$d_2^2 = \min\{l \geq 0: \ -2.4142 \left(\frac{8l+5}{4}\right)^3 < -15\} = 1,$$

$$d_3^2 = \min\{l \geq 0: \ -0.4142 \left(\frac{8l+7}{4}\right)^3 < -15\} = 1,$$

$$d_0^3 = \min\{l \geq 0: \ 0.4142 \left(\frac{8l+1}{4}\right)^3 > 10\} = 2,$$

$$d_1^3 = \min\{l \geq 0: \ 2.4142 \left(\frac{8l+3}{4}\right)^3 > 10\} = 1,$$

$$d_2^3 = \min\{l \geq 0: \ -2.4142 \left(\frac{8l+5}{4}\right)^3 < -37\} = 1,$$

$$d_3^3 = \min\{l \geq 0: \ -0.4142 \left(\frac{8l+7}{4}\right)^3 < -37\} = 2.$$

and

$j = 0$,

$n_{\infty,1}^{1,+}(0) = n_{\infty,1}^{1,+}(4) = n_{\infty,1}^{1,+}(8) = 1, \quad n_{0,1}^{1,-}(0) = n_{0,1}^{1,-}(4) = 1, \ n_{0,1}^{1,-}(8) = 0,$

$n_{\infty,2}^{2,+}(0) = n_{\infty,2}^{2,+}(4) = n_{\infty,2}^{2,+}(8) = 1, \quad n_{0,2}^{2,-}(0) = n_{0,2}^{2,-}(4) = 1, \ n_{0,2}^{2,-}(8) = 0$

$n_{\infty,3}^{3,+}(0) = n_{\infty,3}^{3,+}(4) = 0, \ n_{\infty,3}^{3,+}(8) = 1, \quad n_{0,3}^{3,-}(0) = n_{0,3}^{3,-}(4) = n_{0,3}^{3,-}(8) = 1$

$n_{\infty,4}^{3,+}(0) = n_{\infty,4}^{3,+}(4) = 0, \ n_{\infty,4}^{3,+}(8) = 1, \quad n_{0,4}^{3,-}(0) = n_{0,4}^{3,-}(4) = n_{0,4}^{3,-}(8) = 0.$

$j = 1$,

$n_{\infty,1}^{1,+}(1) = 1, \ n_{\infty,1}^{1,+}(5) = 1, \quad n_{0,1}^{1,-}(1) = 1, \ n_{0,1}^{1,-}(5) = 0,$

$n_{\infty,2}^{2,+}(1) = 1, \ n_{\infty,2}^{2,+}(5) = 1, \quad n_{0,2}^{2,-}(1) = 1, \ n_{0,2}^{2,-}(5) = 0,$

$n_{\infty,3}^{3,+}(1) = 0, \ n_{\infty,3}^{3,+}(5) = 1, \quad n_{0,3}^{3,-}(1) = 0, \ n_{0,3}^{3,-}(5) = 0,$

$n_{\infty,4}^{3,+}(1) = 0, \ n_{\infty,4}^{3,+}(5) = 1, \quad n_{0,4}^{3,-}(1) = 0, \ n_{0,4}^{3,-}(5) = 0.$

$j = 2$,

$n_{\infty,1}^{1,+}(2) = 1, \ n_{\infty,1}^{1,+}(6) = 0, \quad n_{0,1}^{1,-}(2) = 1, \ n_{0,1}^{1,-}(6) = 1,$

$n_{\infty,2}^{2,+}(2) = 1, \ n_{\infty,2}^{2,+}(6) = 0, \quad n_{0,2}^{2,-}(2) = 1, \ n_{0,2}^{2,-}(6) = 1,$

$n_{\infty,3}^{3,+}(2) = n_{\infty,3}^{3,+}(6) = 0, \quad n_{0,3}^{3,-}(2) = 0, \ n_{0,3}^{3,-}(6) = 1,$

$n_{\infty,4}^{3,+}(2) = n_{\infty,4}^{3,+}(6) = 0, \quad n_{0,4}^{3,-}(2) = 0, \ n_{0,4}^{3,-}(6) = 1.$

$j = 3,$

$n_{\infty,1}^{1,+}(3) = 1,\ n_{\infty,1}^{1,+}(7) = 0, \quad n_{0,1}^{1,-}(3) = 1,\ n_{0,1}^{1,-}(7) = 1,$

$n_{\infty,2}^{2,+}(3) = 1,\ n_{\infty,2}^{2,+}(7) = 0, \quad n_{0,2}^{2,-}(3) = 1,\ n_{0,2}^{2,-}(7) = 1,$

$n_{\infty,3}^{3,+}(3) = n_{\infty,3}^{3,+}(7) = n_{\infty,3}^{3,+}(10) = 0, \quad n_{0,3}^{3,-}(3) = n_{0,3}^{3,-}(7) = 0,\ n_{0,3}^{3,-}(10) = 1,$

$n_{\infty,4}^{3,+}(3) = n_{\infty,4}^{3,+}(7) = n_{\infty,4}^{3,+}(10) = 0, \quad n_{0,4}^{3,-}(3) = n_{0,4}^{3,-}(7) = 0,\ n_{0,4}^{3,-}(10) = 1.$

So, according to corollary 2*, system* (19) *possesses at least* $\tilde{n} = 10 + 10 + 3 = 23$ *geometrically different* 8-*periodic orbits satisfying* $x(t-4) = -x(t)$.

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