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Article

A Note on the Solutions for a Higher-Order Convective Cahn–Hilliard-Type Equation

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Abstract: The higher-order convective Cahn-Hilliard equation describes the evolution of crystal surfaces faceting through surface electromigration, the growing surface faceting, and the evolution of dynamics of phase transitions in ternary oil-water-surfactant systems. In this paper, we study the H^3 solutions of the Cauchy problem and prove, under different assumptions on the constants appearing in the equation and on the mean of the initial datum, that they are well-posed.

Keywords: existence; uniqueness; stability; higher-order convective cahn-hilliard type equation; cauchy problem

MSC: 35G25; 35K55

1. Introduction

In this paper, we study the well-posedness of the Cauchy problem:

$$\begin{cases} \partial_t u + \kappa \partial_x u^2 + \gamma \partial_x^2 u - \beta^2 \partial_x^6 u + \alpha \partial_x^4 u + \delta^2 \partial_x^4 \left(u^3\right) = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

with

$$\kappa,\ \gamma,\ \beta,\ \alpha, \delta \in \mathbb{R}, \quad \gamma < 0, \quad \alpha \geq 0, \quad \kappa,\ \beta,\ \delta \neq 0, \quad \text{or,} \tag{2}$$

$$\beta,\ \alpha, \delta \in \mathbb{R}, \quad \kappa = \gamma = 0, \quad \alpha \geq 0, \quad \beta,\ \delta \neq 0, \quad \text{or,} \tag{3}$$

$$\kappa,\ \gamma,\ \beta,\ \alpha, \delta \in \mathbb{R}, \quad \beta \neq 0, \quad \delta = 0, \quad \text{or,} \tag{4}$$

$$\kappa,\ \gamma,\ \beta,\ \alpha, \delta \in \mathbb{R}, \quad \beta,\ \delta \neq 0. \tag{5}$$

On the initial datum, we assume

$$u_0 \in H^3(\mathbb{R}), \quad \text{or,} \tag{6}$$

$$u_0 \in H^3(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \int_{\mathbb{R}} u_0(x) dx = 0. \tag{7}$$

Inspired by [1–12], in light of (7), we define the following function:

$$P_0(x) = \int_{-\infty}^{x} u_0(y) dy, \tag{8}$$

on which we assume

$$\|P_0\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left(\int_{-\infty}^{x} u_0(y)dy\right)^2 dx < \infty. \tag{9}$$

The equation in (1) is derived in [13] to model the evolution of a crystalline surface with small slopes that undergoes faceting. The unknown u gives the surface slope, the constant κ is proportional to the atomic flux deposition strength and the convective term $\kappa\partial_x u^2$ arises from the deposited atoms normal impingement. The sixth-order linear term $\partial_x^6 u$ regularizes the equation, taking into account the surface curvature and the anisotropy of the surface energy under the surface diffusion.

From a mathematical point of view, the existence and uniqueness of weak solutions of (1) with periodic boundary conditions is proven in [14], under the assumptions $\kappa > 0$ and $\gamma = 0$. In the same setting, a similar result is proven in two space dimensions in [15]. In [16], the authors derived the stationary solutions of (1), again assuming $\kappa > 0$ and $\gamma = 0$. In [17], the existence of a global-in-time attractor is studied, while the well-posedness of the classical solutions of (1) is proven in [18], requiring (7)–(9), and $\gamma = 0$. In this paper, we will prove that, if (2) or (3) hold, we have the well-posedness of the classical solutions of (1) assuming (6), while if (5) holds, we have the well-posedness of (1) assuming (7)–(9).

Taking $\kappa = 0$, (1) becomes

$$\partial_t u + \gamma\partial_x^2 u - \beta^2\partial_x^6 u + \alpha\partial_x^4 u + \delta^2\partial_x^4 \left(u^3\right) = 0, \tag{10}$$

which is a Cahn-Hilliard type equation [19–21]. It was deduced in [22] to describe the evolution of crystal surfaces faceting through surface electromigration. It also describes the phase transition development in ternary oil-water-surfactant systems. One part of the surfactant is hydrophilic, and the other one (termed amphiphile) is lipophilic. Oil, water, and microemulsion (i.e., a homogeneous, isotropic mixture of oil and water) can coexist in equilibrium. The unknown u, in (10), gives the local difference between oil and water concentrations.

From a mathematical point of view, in [23] the initial-boundary-value problem for (10) is analyzed, under appropriate assumptions on $\gamma, \beta, \alpha, \delta$. In [24], the authors analyze the existence of a global-in-time attractor. The existence of weak solutions for the initial-boundary-value problem for (10) is proven in the case of degenerate mobility in [25]. Finally, in [18], the well-posedness of the classical solution of the Cauchy problem of (10) is proven, assuming (7)–(9), with $\gamma = 0$. In this paper, we will show that the classical solutions of the Cauchy problem of (10) are well-posed, assuming (6), if $\gamma \le 0$ and $\alpha > 0$, while in the general case, we will prove the same result assuming (7)–(9).

Observe that in [13], it is proven that, as $\kappa \to \infty$, (1) reduces

$$\partial_t u + \partial_x u^2 + a_1\partial_x^2 u - b_1^2\partial_x^6 u + c_1\partial_x^4 u = 0, \tag{11}$$

which is known as the Kuramoto-Sivashinsky equation (see [26–28]). In Section 4, we will prove the well-posedness of the Cauchy problem for (11), assuming (6).

When $\beta = \delta = 0$ and $\alpha = f^2 \neq 0$, (1) reads

$$\partial_t u + \kappa\partial_x u^2 + \gamma\partial_x^2 u + f^2\partial_x^4 u = 0. \tag{12}$$

(12) appears in several physical situations; for example, it models long waves on a viscous fluid flowing down an inclined plane [29] and drift waves in a plasma [30]. (12) was also independently deduced by Kuramoto [27,31,32] to describe the phase turbulence in reaction-diffusion systems, and by Sivashinsky [28] to describe plane flame propagation, taking into account the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front.

Equation (12) can be used to study incipient instabilities in several physical and chemical systems [33–35]. Moreover, (12) is also termed the Benney-Lin equation [36,37], and was deduced by Kuramoto as a model for phase turbulence in the Belousov-Zhabotinsky reaction [38].

The existence and the dynamical properties of the exact solutions for (12) can be found in [39–44]. The control problem for (12) with periodic boundary conditions, and on a bounded interval, are studied in [45–47]. The problem of global-in-time exponential stabilization of (12) with periodic boundary conditions is analyzed in [48]. A generalization of the optimal control theory for (12) is proposed in [49], while the f global boundary controllability of (12) is considered in [50]. The existence of solitonic solutions for (12) is proven in [51]. The well-posedness of the Cauchy problem for (12) is proven in [52–54], using the energy space technique, a priori estimates together with an application of the Cauchy-Kovalevskaya and the fixed point method, respectively. Instead, the initial-boundary value problem for (1) is studied, using a priori estimates together with an application of the Cauchy-Kovalevskaya, and the energy space technique in [55–57]. Inspired by [58–60], the convergence of the solution of (12) to the unique entropy one of the Burgers equation is proven in [61].

Finally, due to its general structure, we conjecture that (1) can have a possible application in machine learning (see [62,63]).

2. Results and Organization of the Paper

In this paper, we improve and complete the results of [14,16–18] working with H^3 initial data and having general assumptions on the constants appearing in (1). The main result of this paper is the following theorem. We prove the global-in-time existence, uniqueness, and stability of the solutions of the Cauchy problem (1).

Theorem 1. *Fix $T > 0$. Assuming one of the following*

(i) (2) *and* (6),
(ii) (3) *and* (6),
(iii) (4) *and* (6),
(iv) (5) *and* (7),

and (9), *there exists a unique solution u of* (1) *such that*

$$u \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^3(\mathbb{R})). \tag{13}$$

In particular, under the Assumptions (7) *and* (9), *we have that*

$$\int_{\mathbb{R}} u(t,x)dx = 0. \tag{14}$$

Moreover, if u_1 and u_2 are two solutions of (1), *we have that*

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R})} \le e^{C(T)t}\,\|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{R})}, \tag{15}$$

for some suitable $C(T) > 0$, and every $0 \le t \le T$.

The well-posedness of (1) is guaranteed for a short time by the Cauchy-Kowaleskaya Theorem [64]. The solutions are indeed global, thanks to suitable a priori estimates.

The paper is organized as follows. In Section 3 we prove Theorem 1, assuming (*i*) or (*ii*). In Section 4 we prove Theorem 1, assuming (*iii*). In Section 5 we prove Theorem 1, assuming (*iv*).

3. Proof of Theorem 1 Assuming (*i*) or (*ii*)

In this section, we prove Theorem 1, assuming (*i*) or (*ii*). For the sake of notational simplicity, define

$$\gamma = -a^2, \quad \alpha = b^2, \tag{16}$$

and then (1) reads

$$\begin{cases} \partial_t u + \kappa \partial_x u^2 - a^2 \partial_x^2 u - \beta^2 \partial_x^6 u + b^2 \partial_x^4 u + \delta^2 \partial_x^4 \left(u^3\right) = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{17}$$

Since the short time well-posedness of (17) is guaranteed by the Cauchy-Kowaleskaya Theorem [64], here we need to prove some suitable global a priori estimates.

From now on, we denote with C_0 the constants which depend only on the initial data, and with $C(T)$, the constants which depend also on T.

Following [65] (Lemma 2.2), we begin with the following energy estimate in the space $L^\infty(0,\infty; H^1(\mathbb{R})) \cap L^2(0,\infty; H^2(\mathbb{R}))$.

Lemma 1. *Assuming (2), for each $t > 0$, we have that*

$$\begin{aligned} &\frac{\beta^2}{2} \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4} \|u(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{b^2}{2} \|u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\qquad + K_1^2 \int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + K_2^2 \int_0^t \|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \\ &\qquad + a^2 b^2 \int_0^t \|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \\ &\qquad + \int_0^t\!\!\int_{\mathbb{R}} \left[b^2 \partial_x^2 u + \delta^2 \partial_x^2 \left(u^3\right) - \beta^2 \partial_x^4 u\right]^2 dsdx \le C_0, \end{aligned} \tag{18}$$

where K_1^2, K_2^2 are two appropriate positive constants.

Assuming (3), for each $t > 0$, we have that

$$\begin{aligned} &\frac{\beta^2}{2} \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4} \|u(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{b^2}{2} \|u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\qquad + \int_0^t\!\!\int_{\mathbb{R}} \left[b^2 \partial_x^2 u + \delta^2 \partial_x^2 \left(u^3\right) - \beta^2 \partial_x^4 u\right]^2 dsdx \le C_0. \end{aligned} \tag{19}$$

Moreover, there exists $C_0 > 0$, independent on κ, a, b, such that, for each $t \ge 0$,

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \le C_0. \tag{20}$$

Proof. We begin by proving (18). Assume (2). Multiplying (17) by $-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u$, we have that

$$\begin{aligned} &\left(-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u\right) \partial_t u + 2\kappa \left(-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u\right) u \partial_x u \\ &\quad - a^2 \left(-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u\right) \partial_x^2 u - \beta^2 \left(-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u\right) \partial_x^6 u \\ &\quad + b^2 \left(-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u\right) \partial_x^4 u + \delta^2 \left(-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u\right) \partial_x^4 \left(u^3\right) = 0. \end{aligned} \tag{21}$$

Performing some integration by parts, we gain

$$\begin{aligned} &\int_{\mathbb{R}} \left(-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u\right) \partial_t u dx \\ &\quad = \frac{d}{dt} \left(\frac{\beta^2}{2} \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4} \|u(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{b^2}{2} \|u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right), \\ &2\kappa \int_{\mathbb{R}} \left(-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u\right) u \partial_x u dx = -2\kappa\beta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx, \\ &- a^2 \int_{\mathbb{R}} \left(-\beta^2 \partial_x^2 u + \delta^2 u^3 + b^2 u\right) \partial_x^2 u dx \end{aligned} \tag{22}$$

$$= a^2\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 3a^2\delta^2 \|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + a^2b^2 \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$\begin{aligned}
&-\beta^2 \int_{\mathbb{R}} \left(-\beta^2\partial_x^2 u + \delta^2 u^3 + b^2 u\right) \partial_x^6 u dx \\
&\quad = -\beta^4 \int_{\mathbb{R}} \partial_x^3 u \partial_x^5 u dx + \beta^2\delta^2 \int_{\mathbb{R}} \partial_x \left(u^3\right) \partial_x^5 u dx + \beta^2 b^2 \int_{\mathbb{R}} \partial_x u \partial_x^5 u dx \\
&\quad = \beta^4 \int_{\mathbb{R}} (\partial_x^4 u)^4 dx - \beta^2\delta^2 \int_{\mathbb{R}} \partial_x^2 \left(u^3\right) \partial_x^4 u dx - \beta^2 a^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx, \\
&b^2 \int_{\mathbb{R}} \left(-\beta^2\partial_x^2 u + \delta^2 u^3 + b^2 u\right) \partial_x^4 u dx \\
&\quad = -\beta^2 b^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx - \delta^2 b^2 \int_{\mathbb{R}} \partial_x \left(u^3\right) \partial_x^3 u dx - b^4 \int_{\mathbb{R}} \partial_x u \partial_x^3 u dx \\
&\quad = -\beta^2 b^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx + \delta^2 b^2 \int_{\mathbb{R}} \partial_x^2 \left(u^3\right) \partial_x^2 u dx + b^4 \int_{\mathbb{R}} (\partial_x^2 u)^2 dx, \\
&\delta^2 \int_{\mathbb{R}} \left(-\beta^2\partial_x^2 u + \delta^2 u^3 + b^2 u\right) \partial_x^4 \left(u^3\right) dx \\
&\quad = \beta^2\delta^2 \int_{\mathbb{R}} \partial_x^3 u \partial_x^3 \left(u^3\right) dx - \delta^4 \int_{\mathbb{R}} \partial_x \left(u^3\right) \partial_x^3 \left(u^3\right) dx - b^2\delta^2 \int_{\mathbb{R}} \partial_x u \partial_x^3 \left(u^3\right) dx \\
&\quad = -\beta^2\delta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_x^2 \left(u^3\right) dx + \delta^4 \int_{\mathbb{R}} \left[\partial_x^2 \left(u^3\right)\right]^2 dx + b^2\delta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^2 \left(u^3\right) dx.
\end{aligned}$$

Therefore, thanks to (22), an integration of (21) on $\mathbb{R}$ gives

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\beta^2}{2}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4}\|u(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{b^2}{2}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right) \\
&\quad + a^2\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 3a^2\delta^2 \|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + a^2b^2 \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad + \beta^4 \int_{\mathbb{R}} (\partial_x^4 u)^4 dx - 2\beta^2\delta^2 \int_{\mathbb{R}} \partial_x^2 \left(u^3\right) \partial_x^4 u dx - 2\beta^2 a^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx \\
&\quad + 2\delta^2 b^2 \int_{\mathbb{R}} \partial_x^2 \left(u^3\right) \partial_x^2 u dx + b^4 \int_{\mathbb{R}} (\partial_x^2 u)^2 dx + \delta^4 \int_{\mathbb{R}} \left[\partial_x^2 \left(u^3\right)\right]^2 dx \\
&= 2\kappa\beta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx.
\end{aligned} \tag{23}$$

Since

$$\begin{aligned}
&b^4 \int_{\mathbb{R}} (\partial_x^2 u)^2 dx + \delta^4 \int_{\mathbb{R}} \left[\partial_x^2 \left(u^3\right)\right]^2 dx + \beta^4 \int_{\mathbb{R}} (\partial_x^4 u)^4 dx \\
&\quad + 2\delta^2 b^2 \int_{\mathbb{R}} \partial_x^2 \left(u^3\right) \partial_x^2 u dx - 2\beta^2 a^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx - 2\beta^2\delta^2 \int_{\mathbb{R}} \partial_x^2 \left(u^3\right) \partial_x^4 u dx \\
&\quad = \int_{\mathbb{R}} \left[b^2\partial_x^2 u + \delta^2\partial_x^2 \left(u^3\right) - \beta^2\partial_x^4 u\right]^2 dx,
\end{aligned}$$

(23) becomes

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\beta^2}{2}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4}\|u(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{b^2}{2}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right) \\
&\quad + a^2\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 3a^2\delta^2 \|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + a^2b^2 \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad + \int_{\mathbb{R}} \left[b^2\partial_x^2 u + \delta^2\partial_x^2 \left(u^3\right) - \beta^2\partial_x^4 u\right]^2 dx = 2\kappa\beta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx.
\end{aligned} \tag{24}$$

Due to the Young inequality,

$$2|\kappa|\beta^2 \int_{\mathbb{R}} |u||\partial_x u||\partial_x^2 u| dx = 2\beta^2 \int_{\mathbb{R}} \left|\frac{\kappa u \partial_x u}{\sqrt{D_1}}\right| \left|\sqrt{D_1}\partial_x^2 u\right| dx$$

$$\le \frac{\beta^2\kappa^2}{D_1}\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 D_1\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

where D_1 is a positive constant, which will be specified later. Consequently, by (24),

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\beta^2}{2}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4}\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \frac{b^2}{2}\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad + \beta^2\left(a^2 - D_1\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left(3a^2\delta^2 - \frac{\beta^2\kappa^2}{D_1}\right)\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + a^2b^2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}}\left[b^2\partial_x^2 u + \delta^2\partial_x^2\left(u^3\right) - \beta^2\partial_x^4 u\right]^2 dx \le 0.
\end{aligned} \tag{25}$$

We search D_1, such that

$$a^2 - D_1 > 0, \quad 3a^2\delta^2 - \frac{\beta^2\kappa^2}{D_1} > 0. \tag{26}$$

Since, after a rescaling, $|a|$ can be taken very big, D_1 does exist and (26) holds.
Therefore, by (25) and (26),

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\beta^2}{2}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4}\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \frac{b^2}{2}\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad + K_1^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + K_2^2\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + a^2b^2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + \int_{\mathbb{R}}\left[b^2\partial_x^2 u + \delta^2\partial_x^2\left(u^3\right) - \beta^2\partial_x^4 u\right]^2 dx \le 0,
\end{aligned}$$

where K_1^2, K_2^2 are two appropriate positive constants. Integrating on $(0,t)$, by (6), we have that

$$\begin{aligned}
&\frac{\beta^2}{2}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4}\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \frac{b^2}{2}\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad + K_1^2\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + K_2^2\int_0^t\left\|u(s,\cdot)\partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\
&\qquad + a^2b^2\int_0^t\left\|\partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\
&\qquad + \int_0^t\!\!\int_{\mathbb{R}}\left[b^2\partial_x^2 u + \delta^2\partial_x^2\left(u^3\right) - \beta^2\partial_x^4 u\right]^2 dsdx \le C_0,
\end{aligned}$$

that is, (18).

We continue by proving (19). Assume (3). Since $\kappa = \gamma = 0$, (17) reads

$$\partial_t u - \beta^2\partial_x^6 u + b^2\partial_x^4 u + \delta^2\partial_x^4\left(u^3\right) = 0. \tag{27}$$

Multiplying (27), by $-\beta^2\partial_x^2 u + \delta^2 u^3 + b^2 u$, arguing as in the previous case, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\beta^2}{2}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4}\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \frac{b^2}{2}\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad + \int_{\mathbb{R}}\left[b^2\partial_x^2 u + \delta^2\partial_x^2\left(u^3\right) - \beta^2\partial_x^4 u\right]^2 dx = 0.
\end{aligned}$$

(6) and an integration on $(0,t)$ give (19).

Finally, we prove (20). Thanks to (18) or (19) and the Hölder inequality,

$$|u(t,x)|^3 = 3\left|\int_{-\infty}^x u^2\partial_x u dy\right| \le 3\int_{\mathbb{R}} u^2|\partial_x u| dx$$

$$\leq 3\left\|u(t,\cdot)\right\|^2_{L^4(\mathbb{R})}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq C_0.$$

Hence,

$$\|u(t,\cdot)\|^3_{L^\infty(\mathbb{R})} \leq C_0,$$

which gives (20). □

We continue by proving an L^2- estimate, which is independent on κ, a, b.

Lemma 2. *Fix $T > 0$ and assume (2) or (3). There exists a constant $C(T) > 0$, independent on κ, a, b, such that*

$$\begin{aligned}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2a^2\int_{\mathbb{R}}\|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + \beta^2\int_{\mathbb{R}}\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds&\\ +2b^2\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \leq C(T)&,\end{aligned} \tag{28}$$

$$\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \leq C(T), \tag{29}$$

$$\int_0^t\left\|u(s,\cdot)(\partial_x u(s,\cdot))^2\right\|^2_{L^2(\mathbb{R})}ds \leq C(T), \tag{30}$$

$$\int_0^t\|\partial_x u(s,\cdot)\|^4_{L^4(\mathbb{R})}ds \leq C(T), \tag{31}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. We begin by observing that

$$\partial_x u^3 = 3u^2\partial_x u, \quad \partial_x^2\left(u^3\right) = 6u(\partial_x u)^2 + 3u^2\partial_x^2 u. \tag{32}$$

Multiplying (17) by $2u$, an integration on $\mathbb{R}$ and several integrations by part give

$$\begin{aligned}\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} =& 2\int_{\mathbb{R}} u\partial_t u dx\\ =& -4\kappa\int_{\mathbb{R}} u^2\partial_x dx + 2a^2\int_{\mathbb{R}} u\partial_x^2 u dx + 2\beta^2\int_{\mathbb{R}} u\partial_x^6 u dx\\ &- 2b^2\int_{\mathbb{R}} u\partial_x^4 u dx - 2\delta^2\int_{\mathbb{R}} u\partial_x^4\left(u^3\right)dx\\ =& -2a^2\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\beta^2\int_{\mathbb{R}}\partial_x u\partial_x^5 u dx + 2b^2\int_{\mathbb{R}}\partial_x u\partial_x^3 u dx\\ &+ 2\delta^2\int_{\mathbb{R}}\partial_x u\partial_x^3\left(u^3\right)dx\\ =& -2a^2\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\int_{\mathbb{R}}\partial_x^2 u\partial_x^4 u dx - 2b^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &- 2\delta^2\int_{\mathbb{R}}\partial_x^2 u\partial_x^2\left(u^3\right)dx\\ =& -2a^2\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2b^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &- 2\delta^2\int_{\mathbb{R}}\partial_x^2 u\partial_x^2\left(u^3\right)dx,\end{aligned}$$

that is,

$$\begin{aligned}\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2a^2\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}&\\ +2b^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=-2\delta^2\int_{\mathbb{R}}\partial_x^2 u\partial_x^2\left(u^3\right)dx.&\end{aligned}\tag{33}$$

Due to the Young inequality, we can estimate the right-hand side of (33), as follows:

$$2\delta^2\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^2\left(u^3\right)|dx\le\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\delta^4\int_{\mathbb{R}}\left[\partial_x^2\left(u^3\right)\right]^2dx.$$

Consequently, (33) becomes

$$\begin{aligned}\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2a^2\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}&\\ +2b^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\delta^4\int_{\mathbb{R}}\left[\partial_x^2\left(u^3\right)\right]^2dx.&\end{aligned}\tag{34}$$

Observe that, by (32),

$$\begin{aligned}\int_{\mathbb{R}}\left[\partial_x^2\left(u^3\right)\right]^2dx=&36\int_{\mathbb{R}}u^2(\partial_x u)^4dx+9\left\|u^2(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+36\int_{\mathbb{R}}u^3(\partial_x u)^2\partial_x^2 udx\\ =&36\int_{\mathbb{R}}u^2(\partial_x u)^4dx+9\left\|u^2(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-36\int_{\mathbb{R}}u^2(\partial_x u)^4dx\\ =&9\left\|u^2(t,\cdot)\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=9\int_{\mathbb{R}}u^4(\partial_x^2 u)^2dx.\end{aligned}\tag{35}$$

Using (20),

$$\int_{\mathbb{R}}\left[\partial_x^2\left(u^3\right)\right]^2dx\le 9\|u(t,\cdot)\|^4_{L^\infty(\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\tag{36}$$

It follows from (34) and (36) that

$$\begin{aligned}\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2a^2\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}&\\ +2b^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.&\end{aligned}\tag{37}$$

Since

$$C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=C_0\int_{\mathbb{R}}\partial_x^2 u\partial_x^2 udx=-C_0\int_{\mathbb{R}}\partial_x u\partial_x^3 udx,$$

Lemma 1 and the Young inequality give

$$\begin{aligned}C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le&2\int_{\mathbb{R}}\left|\frac{C_0\partial_x u}{2\beta}\right|\left|\beta\partial_x^3 u\right|dx\\ \le&C_0\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \le&C_0+\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}\tag{38}$$

Consequently, by (37),

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2a^2\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+ 2b^2 \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \leq C(T).$$

Integrating on $(0, t)$, by (6), we have that

$$\begin{aligned} \|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2a^2 \int_0^t \|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds + \beta^2 \int_0^t \left\| \partial_x^3 u(s,\cdot) \right\|^2_{L^2(\mathbb{R})} ds \\ + 2b^2 \int_0^t \left\| \partial_x^2 u(s,\cdot) \right\|^2_{L^2(\mathbb{R})} ds \leq C_0 + C(T)t \leq C(T), \end{aligned}$$

which gives (28).

(29) follows from (28), (38) and an integration on $(0, t)$.

We prove (30). We begin by observing that, by (35) and (36), we have that

$$\begin{aligned} 36 \int_{\mathbb{R}} u^2 (\partial_x u)^4 dx \leq & C_0 \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} - 9 \left\| u^2(t,\cdot) \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\ & - 36 \int_{\mathbb{R}} u^3 (\partial_x u)^2 \partial_x^2 u dx. \end{aligned} \tag{39}$$

Thanks to the Young inequality,

$$36 \int_{\mathbb{R}} |u|^3 (\partial_x u)^2 |\partial_x^2 u| dx \leq 18 \int_{\mathbb{R}} u^2 (\partial_x u)^4 dx + 18 \left\| u^2(t,\cdot) \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}.$$

Consequently, by (39),

$$18 \int_{\mathbb{R}} u^2 (\partial_x u)^4 dx \leq C_0 \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + 9 \left\| u^2(t,\cdot) \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}. \tag{40}$$

Observe that, by (20),

$$9 \left\| u^2(t,\cdot) \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} = 9 \int_{\mathbb{R}} u^4 (\partial_x^2 u)^2 dx \leq C_0 \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}.$$

Therefore, by (40),

$$18 \left\| u(t,\cdot)(\partial_x u(t,\cdot))^2 \right\|^2_{L^2(\mathbb{R})} \leq C_0 \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}. \tag{41}$$

Integrating (41) on $(0, t)$, by (29), we have (30).

Finally, we prove (31). We begin by observing that [66] (Lemma 2.3) says that

$$\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} \leq 6 \left(\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} \right) \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}.$$

Therefore, by Lemma 1 and (28), we have that

$$\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} \leq C(T) \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0, t)$, by (29), we have (31). □

We continue with an a priori estimate in the space $L^2(0, \infty; H^4(\mathbb{R}))$.

Lemma 3. *Fix $T > 0$ and assume (2) or (3). There exists a constant $C(T) > 0$, independent on κ, a, b, such that*

$$\begin{aligned} \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2a^2 \int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + \beta^2 \int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\ + 2b^2 \int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T) \end{aligned} \tag{42}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Multiplying (17) by $-2\partial_x^2 u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned} \frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} =& -2\int_{\mathbb{R}} \partial_x^2 u \partial_t u dx \\ =& 4\kappa \int_{\mathbb{R}} u\partial_x u \partial_x^2 u dx - 2a^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\beta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^6 u dx \\ &+ 2b^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx + 2\delta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 \left(u^3\right) dx \\ =& -2\kappa \int_{\mathbb{R}} (\partial_x u)^3 dx - 2a^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2 \int_{\mathbb{R}} \partial_x^3 u \partial_x^5 u dx \\ &- 2b^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\delta^2 \int_{\mathbb{R}} \partial_x^3 u \partial_x^3 \left(u^3\right) dx \\ =& -2\kappa \int_{\mathbb{R}} (\partial_x u)^3 dx - 2a^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &- 2b^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\delta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_x^2 \left(u^3\right) dx, \end{aligned}$$

that is,

$$\begin{aligned} \frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2a^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ + 2b^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = 2\kappa \int_{\mathbb{R}} (\partial_x u)^3 dx + 2\delta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_x^2 \left(u^3\right) dx. \end{aligned} \tag{43}$$

Due to Lemma 1, (36) and the Young inequality,

$$\begin{aligned} 2|\kappa| \int_{\mathbb{R}} |\partial_x u|^3 dx \le& \kappa^2 \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} \\ \le& C_0 + \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})}, \\ 2\delta^2 \int_{\mathbb{R}} |\partial_x^4 u||\partial_x^2 \left(u^3\right)| dx =& 2\int_{\mathbb{R}} \left|\beta \partial_x^4 u\right| \left|\frac{\delta^2 \partial_x^2 \left(u^3\right)}{\beta}\right| dx \\ \le& \beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\delta^4}{\beta^2} \int_{\mathbb{R}} \left[\partial_x^2 \left(u^3\right)\right]^2 dx \\ \le& \beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Therefore, by (43),

$$\begin{aligned} \frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2a^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ + 2b^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + C_0 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Integrating on $(0,t)$, by (6), (29) and (31),

$$\begin{aligned}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ 2a^2\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + \beta^2\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &+ 2b^2\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &\le C_0 + \int_0^t \|\partial_x u(s,\cdot)\|^4_{L^4(\mathbb{R})} ds + C_0\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T),\end{aligned}$$

which gives (42). □

We continue with an a priori estimate in the space $L^2(0,\infty;H^5(\mathbb{R}))$.

Lemma 4. *Fix $T>0$ and assume (2) or (3). There exists a constant $C(T)>0$, independent on κ, a, b,*

$$\|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T), \tag{44}$$

$$\begin{aligned}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ 2a^2\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ +\beta^2\int_0^t \left\|\partial_x^5 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds &+ 2b^2\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T),\end{aligned} \tag{45}$$

for every $0\le t\le T$, where $C(T)$ is independent on κ, a, b.

Proof. Let $0\le t\le T$. We begin by observing that, by (32), we have that

$$\partial_x^3\left(u^3\right) = 6(\partial_x u)^3 + 18u\partial_x u\partial_x^2 u + 3u^2\partial_x^3 u. \tag{46}$$

Multiplying (17) by $2\partial_x^4 u$, thanks to (46), an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& 2\int_{\mathbb{R}} \partial_x^4 u\partial_t u dx\\ =& -2\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^4 u dx + 2a^2\int_{\mathbb{R}} \partial_x^2 u\partial_x^4 u dx + 2\beta^2\int_{\mathbb{R}} \partial_x^4 u\partial_x^6 u dx\\ &- 2b^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\delta^2\int_{\mathbb{R}} \partial_x^4 u\partial_x^4\left(u^3\right) dx\\ =& -2\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^4 u dx - 2a^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\beta^2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &- 2b^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\delta^2\int_{\mathbb{R}} \partial_x^5 u\partial_x^3\left(u^3\right) dx\\ =& -2\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^4 u dx - 2a^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\beta^2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &- 2b^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 12\delta^2\int_{\mathbb{R}} (\partial_x u)^3\partial_x^5 u dx - 36\int_{\mathbb{R}} u\partial_x u\partial_x^2 u\partial_x^5 u dx\\ &- 6\int_{\mathbb{R}} u^2\partial_x^3 u\partial_x^5 u dx,\end{aligned}$$

that is,

$$\begin{aligned}\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ 2a^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &+ 2\beta^2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2b^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &= -2\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^4 u dx - 12\delta^2\int_{\mathbb{R}} (\partial_x u)^3\partial_x^5 u dx\end{aligned} \tag{47}$$

$$-36\delta^2\int_{\mathbb{R}} u\partial_x u\partial_x^2 u\partial_x^5 u dx - 6\delta^2\int_{\mathbb{R}} u^2\partial_x^3 u\partial_x^5 u dx.$$

Due to Lemma 1 and the Young inequality,

$$\begin{aligned}
2|\kappa|\int_{\mathbb{R}}|u||\partial_x u||\partial_x^4 u|dx &\le \kappa^2\int_{\mathbb{R}} u^2(\partial_x u)^2dx + \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \kappa^2\|u(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C_0\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C_0 + \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
12\delta^2\int_{\mathbb{R}}|\partial_x u|^3|\partial_x^5 u|dx &= 2\int_{\mathbb{R}}\left|\frac{6\delta^2(\partial_x u)^3}{\beta\sqrt{D_2}}\right|\left|\beta\sqrt{D_2}\partial_x^5 u\right|dx\\
&\le \frac{36\delta^4}{\beta^2 D_2}\int_{\mathbb{R}}(\partial_x u)^6dx + \beta^2 D_2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{36\delta^4}{\beta^2 D_2}\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + \beta^2 D_2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
36\delta^2\int_{\mathbb{R}}|u||\partial_x u\partial_x^2 u||\partial_x^5 u|dx &\le 36\delta^2\|u(t,\cdot)\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|\partial_x u\partial_x^2 u||\partial_x^5 u|dx\\
&\le 2C_0\int_{\mathbb{R}}|\partial_x u\partial_x^2 u||\partial_x^5 u|dx = 2\int_{\mathbb{R}}\left|\frac{C_0\partial_x u\partial_x^2 u}{\beta\sqrt{D_2}}\right|\left|\beta\sqrt{D_2}\partial_x^5 u\right|dx\\
&\le \frac{C_0}{D_2}\int_{\mathbb{R}}(\partial_x u)^2(\partial_x^2 u)^2dx + \beta^2 D_2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{C_0}{D_2}\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 D_2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
6\delta^2\int_{\mathbb{R}}u^2|\partial_x^3 u||\partial_x^5 u|dx &\le 6\delta^2\|u(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|\partial_x^3 u||\partial_x^5 u|dx\\
&\le 2C_0\int_{\mathbb{R}}|\partial_x^3 u||\partial_x^5 u|dx \le 2\int_{\mathbb{R}}\left|\frac{C_0\partial_x^3 u}{\beta\sqrt{D_2}}\right|\left|\beta\sqrt{D_2}\partial_x^5 u\right|dx\\
&\le \frac{C_0}{D_2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 D_2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

where D_2 is a positive constant, which will be specified later. Therefore, by (47),

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2a^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad + \beta^2(2-3D_2)\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2b^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C_0 + \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{36\delta^4}{\beta^2 D_2}\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})}\\
&\qquad + \frac{C_0}{D_2}\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{C_0}{D_2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

and taking $D_2 = \frac{1}{3}$

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2a^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad + \beta^2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2b^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C_0 + \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{108\delta^4}{\beta^2}\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})}
\end{aligned}$$

$$+ C_0 \|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0,t)$, by (6), (29), (31) and (42), we obtain that

$$\begin{aligned}
&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2a^2 \int_0^t \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad + \beta^2 \int_0^t \left\|\partial_x^5 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + 2b^2 \int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\
&\le C_0 + C_0 t + \int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\
&\quad + \frac{108\delta^4}{\beta^2} \|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})} \int_0^t \|\partial_x u(s,\cdot)\|^4_{L^4(\mathbb{R})} ds \\
&\quad + C_0 \|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})} \int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + C_0 \int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\
&\le C(T)\left(1 + \|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned} \tag{48}$$

We prove (44). Thanks to Lemma 1, (48) and the Hölder inequality,

$$\begin{aligned}
(\partial_x u(t,x))^2 =& 2\int_{-\infty}^x \partial_x u \partial_x^2 dx \le 2\int_{\mathbb{R}} |\partial_x||\partial_x^2 u| dx \le 2\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})} \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})} \\
\le& C(T)\sqrt{\left(1 + \|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)}.
\end{aligned}$$

Hence,

$$\|\partial_x u\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})} - C(T) \le 0,$$

which gives (44).

Finally, (45) follows from (44) and (48). □

We continue with an a priori estimate in the space $L^4(0,\infty; W^{2,4}(\mathbb{R}))$.

Lemma 5. *Fix $T > 0$ and assume (2) or (3). There exists a constant $C(T) > 0$, independent on κ, a, b, such that*

$$\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^4_{L^4(\mathbb{R})} ds \le C(T), \tag{49}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. We begin by observing that

$$\int_{\mathbb{R}} (\partial_x^2 u)^4 dx = \int_{\mathbb{R}} (\partial_x^2 u)^3 \partial_x^2 u dx = -3\int_{\mathbb{R}} \partial_x u (\partial_x^2 u)^2 \partial_x^3 u dx. \tag{50}$$

Due to the Young inequality,

$$3\int_{\mathbb{R}} |u|\partial_x u(\partial_x^2 u)^2 |\partial_x^3 u| dx \le \frac{9}{2}\int_{\mathbb{R}} (\partial_x u)^2 (\partial_x^3 u)^2 dx + \frac{1}{2}\int_{\mathbb{R}} (\partial_x^2 u)^4 dx.$$

It follows from (50) that

$$\int_{\mathbb{R}} (\partial_x^2 u)^4 dx \le 9\int_{\mathbb{R}} (\partial_x u)^2 (\partial_x^3 u)^2 dx.$$

By (44), we have that

$$\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} \le 9\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})} \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0,t)$, by (28), we have that

$$\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^4_{L^4(\mathbb{R})} ds \le C(T)\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T),$$

which gives (49). □

We continue with an a priori estimate in the space $L^2(0,\infty;H^6(\mathbb{R}))$.

Lemma 6. *Fix $T>0$ and assume (2) or (3). There exists a constant $C(T)>0$, independent on κ, a, b, such that*

$$\begin{aligned}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ 2a^2\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\ &+ \beta^2\int_0^t \left\|\partial_x^6 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + 2b^2\int_0^t \left\|\partial_x^5 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T)\end{aligned} \tag{51}$$

In particular, we have that

$$\left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T), \tag{52}$$

where $C(T)$ is independent on κ, a, b.

Proof. Let $0\le t\le T$. We begin by observing that, by (46),

$$\partial_x^4\left(u^3\right) = 36(\partial_x u)^2\partial_x^2 u + 18u(\partial_x^2 u)^2 + 24u\partial_x u\partial_x^3 u + 3u^2\partial_x^4 u \tag{53}$$

Multiplying (17) by $-2\partial_x^6 u$, thanks to (46), an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{d}{dt}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& -2\int_{\mathbb{R}} \partial_x^6 u\partial_t u dx\\
=&4\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^6 u dx - 2a^2\int_{\mathbb{R}}\partial_x^2 u\partial_x^6 u dx - 2\beta^2\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&+2b^2\int_{\mathbb{R}}\partial_x^4 u\partial_x^6 u dx + 2\delta^2\int_{\mathbb{R}}\partial_x^5 u\partial_x^4\left(u^3\right)dx\\
=&4\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^6 u dx + 2a^2\int_{\mathbb{R}}\partial_x^3 u\partial_x^5 u dx - 2\beta^2\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&-2b^2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 72\delta^2\int_{\mathbb{R}}(\partial_x u)^2\partial_x^2 u\partial_x^6 u dx\\
&+36\delta^2\int_{\mathbb{R}} u(\partial_x^2 u)^2\partial_x^6 u dx + 48\delta^2\int_{\mathbb{R}} u\partial_x u\partial_x^3 u\partial_x^6 u dx\\
&+6\delta^2\int_{\mathbb{R}} u^2\partial_x^4 u\partial_x^6 u dx\\
=&4\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^6 u dx - 2a^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\beta^2\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&-2b^2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 72\delta^2\int_{\mathbb{R}}(\partial_x u)^2\partial_x^2 u\partial_x^6 u dx\\
&+36\delta^2\int_{\mathbb{R}} u(\partial_x^2 u)^2\partial_x^6 u dx + 48\delta^2\int_{\mathbb{R}} u\partial_x u\partial_x^3 u\partial_x^6 u dx\\
&+6\delta^2\int_{\mathbb{R}} u^2\partial_x^4 u\partial_x^6 u dx.\end{aligned}$$

Therefore, we have that

$$\frac{d}{dt}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2a^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2b^2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \tag{54}$$

$$= 4\kappa \int_{\mathbb{R}} u\partial_x u\partial_x^6 u dx + 72\delta^2 \int_{\mathbb{R}} (\partial_x u)^2 \partial_x^2 u \partial_x^6 u dx$$
$$+ 36\delta^2 \int_{\mathbb{R}} u(\partial_x^2 u)^2 \partial_x^6 u dx + 48\delta^2 \int_{\mathbb{R}} u\partial_x u \partial_x^3 u \partial_x^6 u dx + 6\delta^2 \int_{\mathbb{R}} u^2 \partial_x^4 u \partial_x^6 u dx.$$

Due to Lemma 1, (44), (45) and the Young inequality,

$$\begin{aligned}
4|\kappa| \int_{\mathbb{R}} |u\partial_x u||\partial_x^6 u| dx &\le 4|\kappa| \, \|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\partial_x u||\partial_x^6 u| dx \\
&\le 2C_0 \int_{\mathbb{R}} |\partial_x u||\partial_x^6 u| dx = 2\int_{\mathbb{R}} \left| \frac{C_0 \partial_x u}{\beta\sqrt{D_3}} \right| \left| \beta\sqrt{D_3}\partial_x^6 u dx \right| dx \\
&\le \frac{C_0}{D_3} \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 D_3 \left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\le \frac{C_0}{D_3} + \beta^2 D_3 \left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
72\delta^2 \int_{\mathbb{R}} (\partial_x u)^2 |\partial_x^2 u||\partial_x^6 u| dx &\le 72\delta^2 \|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^2 u||\partial_x^6 u| dx \\
&\le 2C(T) \int_{\mathbb{R}} |\partial_x^2 u||\partial_x^6 u| dx = 2\int_{\mathbb{R}} \left| \frac{C(T)\partial_x^2 u}{\beta\sqrt{D_3}} \right| \left| \beta\sqrt{D_3}\partial_x^6 u \right| dx \\
&\le \frac{C(T)}{D_3} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 D_3 \left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\le \frac{C(T)}{D_3} + \beta^2 D_3 \left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
36\delta^2 \int_{\mathbb{R}} |u|(\partial_x^2 u)^2 |\partial_x^6 u| dx &= 36\delta^2 \|u\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} (\partial_x^2 u)^2 |\partial_x^6 u| dx \\
&\le 2C(T) \int_{\mathbb{R}} (\partial_x^2 u)^2 |\partial_x^6 u| dx = 2\int_{\mathbb{R}} \left| \frac{C(T)(\partial_x^2 u)^2}{\beta\sqrt{D_3}} \right| \left| \beta\sqrt{D_3}\partial_x^6 u \right| dx \\
&\le \frac{C(T)}{D_3} \left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \beta^2 D_3 \left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
48\delta^2 \int_{\mathbb{R}} |u||\partial_x u||\partial_x^3 u||\partial_x^6 u| dx &\le 48\delta^2 \|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\partial_x u||\partial_x^3 u||\partial_x^6 u| dx \\
&\le 2C_0 \int_{\mathbb{R}} |\partial_x u||\partial_x^3 u||\partial_x^6 u| dx \le 2C_0 \|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^3 u||\partial_x^6 u| dx \\
&\le 2C(T) \int_{\mathbb{R}} |\partial_x^3 u||\partial_x^6 u| dx \le 2\int_{\mathbb{R}} \left| \frac{C(T)\partial_x^3 u}{\beta\sqrt{D_3}} \right| \left| \beta\sqrt{D_3}\partial_x^6 u \right| dx \\
&\le \frac{C(T)}{D_3} \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 D_3 \left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
6\delta^2 \int_{\mathbb{R}} u^2 |\partial_x^4 u||\partial_x^6 u| dx &\le 6\delta^2 \|u(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\partial_x^4 u||\partial_x^6 u| dx \\
&\le 2C_0 \int_{\mathbb{R}} |\partial_x^4 u||\partial_x^6 u| dx = 2\int_{\mathbb{R}} \left| \frac{C_0 \partial_x^4 u}{\beta\sqrt{D_3}} \right| \left| \beta\sqrt{D_3}\partial_x^6 u \right| dx \\
&\le \frac{C_0}{D_3} \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 D_3 \left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

where D_3 is a positive constant, which will be specified later. It follows from (54) that

$$\begin{aligned}
&\frac{d}{dt} \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2a^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad + \beta^2 (2 - 5D_3) \left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2b^2 \left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\le \frac{C(T)}{D_3} + \frac{C(T)}{D_3} \left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \frac{C(T)}{D_3} \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{C_0}{D_3} \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Taking $D_3 = \frac{1}{5}$, we have that

$$\begin{aligned}&\frac{d}{dt}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2a^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\qquad + \beta^2\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2b^2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\qquad \le C(T) + C(T)\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Integrating on $(0,t)$, by (6), (28), (42), (49), we have that

$$\begin{aligned}&\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2a^2\int_0^t\left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &\qquad + \beta^2\int_0^t\left\|\partial_x^6 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds + 2b^2\int_0^t\left\|\partial_x^5 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &\quad \le C_0 + C(T)\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^4_{L^4(\mathbb{R})}ds + C(T)\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds + C_0\int_0^t\left\|\partial_x^4 u(s,\cdot)\right\|ds\\ &\quad \le C(T),\end{aligned}$$

which gives (51).

Finally, we prove (52). Thanks to (45), (51) and the Hölder inequality,

$$\begin{aligned}(\partial_x^2 u(t,x))^2 =& 2\int_{-\infty}^x \partial_x^2 u\partial_x^3 u dy \le 2\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^3 u|dx\\ \le& 2\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^3 u(t,\cdot)\right\|_{L^2(\mathbb{R})} \le C(T).\end{aligned}$$

Hence,

$$\left\|\partial_x^2 u\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \le C(T),$$

which gives (52). □

We continue with an a priori estimate in the space $H^1((0,\infty)\times\mathbb{R})$.

Lemma 7. *Fix $T > 0$ and assume (2) or (3). There exists a constant $C(T) > 0$, independent on κ, a, b, such that*

$$a^2\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + b^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t\|\partial_t u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \le C(T), \tag{55}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Multiplying (17) by $2\partial_t u$, we have that

$$2(\partial_t u)^2 + 4\kappa u\partial_x u\partial_t u - 2a^2\partial_x^2 u\partial_t u - 2\beta^2\partial_x^6 u\partial_t u + 2b^2\partial_x^4 u\partial_t u + 2\delta^2\partial_t u\partial_x^4\left(u^3\right) = 0. \tag{56}$$

Since,

$$\begin{aligned}-2a^2\int_{\mathbb{R}}\partial_x^2 u\partial_t u =& a^2\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ -2\beta^2\int_{\mathbb{R}}\partial_x^6 u\partial_t u dx =& \beta^2\frac{d}{dt}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ 2b^2\int_{\mathbb{R}}\partial_x^4 u\partial_t u dx =& b^2\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\end{aligned}$$

thanks to (53), an integration of (56) on $\mathbb{R}$ gives

$$\begin{aligned}
&\frac{d}{dt}\left(a^2\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)+b^2\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad=-4\kappa\int_{\mathbb{R}}u\partial_x u\partial_t u dx-72\delta^2\int_{\mathbb{R}}(\partial_x u)^2\partial_x^2 u\partial_t u dx \\
&\qquad+36\delta^2\int_{\mathbb{R}}u(\partial_x^2 u)^2\partial_t u dx-48\delta^2\int_{\mathbb{R}}u\partial_x u\partial_x^2 u\partial_t u dx-6\delta^2\int_{\mathbb{R}}u^2\partial_x^4 u\partial_t u dx.
\end{aligned}\tag{57}$$

Due to Lemma 1, (44), (45), (52) and the Young inequality,

$$\begin{aligned}
&4|\kappa|\int_{\mathbb{R}}|u||\partial_x u||\partial_t u|dx\le 4|\kappa|\|u(t,\cdot)\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|\partial_x u||\partial_t u|dx\\
&\quad\le 2C_0\int_{\mathbb{R}}|\partial_x u||\partial_t u|dx=2\int_{\mathbb{R}}\left|\frac{C_0\partial_x u}{\sqrt{D_3}}\right|\left|\sqrt{D_3}\partial_t u\right|dx\\
&\quad\le\frac{C_0}{D_3}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+D_3\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}\le\frac{C_0}{D_3}+D_3\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
&72\delta^2\int_{\mathbb{R}}(\partial_x u)^2|\partial_x^2 u||\partial_t u|dx\le 72\delta^2\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|dx\\
&\quad\le 2C(T)\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|dx=2\int_{\mathbb{R}}\left|\frac{C(T)\partial_x^2 u}{\sqrt{D_3}}\right|\left|\sqrt{D_3}\partial_t u\right|dx\\
&\quad\le\frac{C(T)}{D_3}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+D_3\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}\le\frac{C(T)}{D_3}+D_3\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
&36\delta^2\int_{\mathbb{R}}|u|(\partial_x^2 u)^2|\partial_t u|dx\le 36\delta^2\|u(t,\cdot)\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}(\partial_x^2 u)^2|\partial_t u|dx\\
&\quad\le 2C_0\int_{\mathbb{R}}(\partial_x^2 u)^2|\partial_t u|dx=2C_0\left\|\partial_x^2 u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|dx\\
&\quad\le 2C(T)\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|dx=2\int_{\mathbb{R}}\left|\frac{C(T)\partial_x^2 u}{\sqrt{D_3}}\right|\left|\sqrt{D_3}\partial_t u\right|dx\\
&\quad\le\frac{C(T)}{D_3}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+D_3\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}\le\frac{C(T)}{D_3}+D_3\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
&48\delta^2\int_{\mathbb{R}}|u||\partial_x u||\partial_x^2 u||\partial_t u|dx\le 48\delta^2\|u(t,\cdot)\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|\partial_x u||\partial_x^2 u||\partial_t u|dx\\
&\quad\le 2C_0\int_{\mathbb{R}}|\partial_x u||\partial_x^2 u||\partial_t u|dx\le 2C_0\|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|dx\\
&\quad\le 2C(T)\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|dx=2\int_{\mathbb{R}}\left|\frac{C(T)\partial_x^2 u}{\sqrt{D_3}}\right|\left|\sqrt{D_3}\partial_x^2 u\right|dx\\
&\quad\le\frac{C(T)}{D_3}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+D_3\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}\le\frac{C(T)}{D_3}+D_3\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
&6\delta^2\int_{\mathbb{R}}u^2|\partial_x^4 u||\partial_t u|dx=6\delta^2\|u(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|\partial_x^4 u||\partial_t u|dx\\
&\quad\le 2C_0\int_{\mathbb{R}}|\partial_x^4 u||\partial_t u|dx=2\int_{\mathbb{R}}\left|\frac{C_0\partial_x^4 u}{\sqrt{D_3}}\right|\left|\sqrt{D_3}\partial_t u\right|dx\\
&\quad\le\frac{C_0}{D_3}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+D_3\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

where D_3 is a positive constant, which will be specified later. As a consequence, (57) becomes

$$\begin{aligned}
&\frac{d}{dt}\left(a^2\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\qquad+b^2\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2(1-2D_3)\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}
\end{aligned}$$

$$\leq \frac{C(T)}{D_3} + \frac{C_0}{D_3} \left\| \partial_x^4 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})},$$

and taking $D_3 = \frac{1}{2}$,

$$\begin{aligned} \frac{d}{dt}\left(a^2 \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right) + b^2 \frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ \leq C(T) + C_0 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Integrating on $(0,t)$, by (6) and (42), we have that

$$\begin{aligned} &a^2 \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + b^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t \|\partial_t u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \\ &\leq C_0 + C(T)t + C_0 \int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C(T), \end{aligned}$$

which gives (55). □

We are finally ready to prove Theorem 1, assuming (i) or (ii).

Proof of Theorem 1 assuming *(i)* or *(ii)*. The well-posedness of (1) is guaranteed for a short time by the Cauchy-Kowaleskaya Theorem [64]. Thanks to the a priori estimates proved in Lemmas 1–7, we have that the global-in-time existence of a is the solution of (1) that satisfies (13).

The stability estimates (15) can be proved using the same arguments of [18] (Theorem 1). □

4. Proof of Theorem 1 Assuming *(iii)*

In this section, we prove Theorem 1 assuming (iii). Due to (4), here, (1) becomes

$$\begin{cases} \partial_t u + \kappa\partial_x u^2 + \gamma\partial_x^2 u - \beta^2\partial_x^6 u + \alpha\partial_x^4 u = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{58}$$

The argument of this section is analogous to that of the previous one. We deduce the local-in-time well-posedness from the Cauchy-Kowaleskaya Theorem [64], and we improve the local-in-time existence to the global-in-time one, proving some suitable a priori estimates on u.

We begin with an energy estimate in the space $L^\infty_{loc}(0,\infty; L^2(\mathbb{R})) \cap L^2_{loc}(0,\infty; H^3(\mathbb{R}))$.

Lemma 8. *Fix $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 e^{C_0 t} \int_0^t e^{-C_s} \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C(T), \tag{59}$$

for every $0 \leq t \leq T$. In particular, (29) holds. Moreover, we have that

$$\int_0^t \|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \leq C(T), \tag{60}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying (58) by $2u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned} \frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} =& 2\int_{\mathbb{R}} u\partial_t u dx \\ =& 4\kappa \int_{\mathbb{R}} u^2 \partial_x u dx - 2\gamma \int_{\mathbb{R}} u\partial_x^2 u dx + 2\beta^2 \int_{\mathbb{R}} u\partial_x^6 u dx - 2\alpha \int_{\mathbb{R}} u\partial_x^4 u dx \end{aligned}$$

$$\begin{aligned}
&= -2\gamma\int_{\mathbb{R}} u\partial_x^2 u dx - 2\beta^2\int_{\mathbb{R}}\partial_x u\partial_x^5 u dx + 2\alpha\int_{\mathbb{R}}\partial_x u\partial_x^3 u dx\\
&= -2\gamma\int_{\mathbb{R}} u\partial_x^2 u dx + 2\beta^2\int_{\mathbb{R}}\partial_x^2 u\partial_x^4 u dx - 2\alpha\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&= -2\gamma\int_{\mathbb{R}} u\partial_x^2 u dx - 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\alpha\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

that is,

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = -2\gamma\int_{\mathbb{R}} u\partial_x^2 u dx - 2\alpha\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \tag{61}$$

Since, using the Young inequality,

$$2|\gamma|\int_{\mathbb{R}}|u||\partial_x^2 u|dx \le \gamma^2\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

we can pass from (61) to

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C_0\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \tag{62}$$

Observe that

$$C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = C_0\int_{\mathbb{R}}\partial_x^2 u\partial_x^2 u dx = -C_0\int_{\mathbb{R}}\partial_x u\partial_x^3 u dx.$$

Therefore, by the Young inequality,

$$\begin{aligned}
C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &\le 2\int_{\mathbb{R}}\left|\frac{C_0\partial_x u}{2\sqrt{D_4}}\right|\left|\sqrt{D_4}\partial_x^3 u\right| dx\\
&\le \frac{C_0}{D_4}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + D_4\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned} \tag{63}$$

where D_4 is a positive constant, which will be specified later. Observe again that

$$\frac{C_0}{D_4}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} = \frac{C_0}{D_4}\int_{\mathbb{R}}\partial_x u\partial_x u dx = -\frac{C_0}{D_4}\int_{\mathbb{R}} u\partial_x^2 u dx.$$

Consequently, by the Young inequality,

$$\begin{aligned}
\frac{C_0}{D_4}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} &\le 2\int_{\mathbb{R}}\left|\frac{C_0 D_4 u}{2\sqrt{D_5}}\right|\left|\sqrt{D_5}\partial_x^2 u\right| dx\\
&\le \frac{C_0 D_4^2}{D_5}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + D_5\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned} \tag{64}$$

where D_5 is a positive constant, which will be specified later. Il follows from (63) and (64) that

$$(C_0 - D_5)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le \frac{C_0 D_4^2}{D_5}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + D_4\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Choosing

$$D_5 = \frac{C_0}{2}, \tag{65}$$

we have that

$$\frac{C_0}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le 2D_4^2\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + D_4\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

that is

$$C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le 4D_4^2\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2D_4\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \tag{66}$$

It follows from (62) and (66) that

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\left(\beta^2-D_4\right)\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le\left(C_0+4D_4^2\right)\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Choosing

$$D_4=\frac{\beta^2}{2},\tag{67}$$

we have that

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C_0\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

By the the Gronwall Lemma and (6), we get

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2e^{C_0t}\int_0^t e^{-C_0s}\left\|\partial_x^3u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C_0e^{C_0t}\le C(T),$$

which gives (59).

We prove (29). Thanks to (59), (66) and (67),

$$C_0\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C(T)+\frac{\beta^2}{2}\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0,t)$, by (59), we have that

$$C_0\int_0^t\left\|\partial_x^2u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T)t+\frac{\beta^2}{2}\int_0^t\left\|\partial_x^3u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T),$$

which gives (29).

Finally, we prove (60). Thanks to (59), (64) and (67),

$$C_0\|\partial_xu(t,\cdot)\|^2_{L^2(\mathbb{R})}\le C(T)+C_0\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0,t)$, by (29), we have that

$$C_0\int_0^t\|\partial_xu(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\le C(T)t+C_0\int_0^t\left\|\partial_x^2u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T),$$

which gives (60). □

We continue with an energy estimate in the space $L^\infty_{loc}(0,\infty;H^1(\mathbb{R}))\cap L^2_{loc}(0,\infty;H^4(\mathbb{R}))$.

Lemma 9. *Fix $T>0$. There exists a constant $C(T)>0$, such that*

$$\|u\|_{L^\infty((0,T)\times\mathbb{R})}\le C(T),\tag{68}$$

$$\|\partial_xu(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\int_0^t\left\|\partial_x^4u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T),\tag{69}$$

for every $0\le t\le T$.

Proof. Let $0\le t\le T$. Multiplying (58) by $-2\partial_x^2u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{d}{dt}\|\partial_xu(t,\cdot)\|^2_{L^2(\mathbb{R})}=&-2\int_{\mathbb{R}}\partial_x^2u\partial_tudx\\=&4\kappa\int_{\mathbb{R}}u\partial_xu\partial_x^2udx+2\gamma\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-2\beta^2\int_{\mathbb{R}}\partial_x^6u\partial_x^2udx+2\alpha\int_{\mathbb{R}}\partial_x^4u\partial_x^2udx\end{aligned}$$

$$
\begin{aligned}
=&4\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^2 u dx + 2\gamma\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\int_{\mathbb{R}}\partial_x^5 u\partial_x^3 u dx - 2\alpha\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
=&4\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^2 u dx + 2\gamma\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\alpha\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Consequently, we have that

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad = 4\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^2 u dx + 2\gamma\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\alpha\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{70}
$$

Due to the Young inequality,

$$
\begin{aligned}
4\kappa\int_{\mathbb{R}}|u\partial_x u||\partial_x^2 u|dx \le& 2\kappa^2\int_{\mathbb{R}} u^2(\partial_x u)^2 dx + 2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le& 2\kappa^2\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, by (70),

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le 2\kappa^2\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2(1+|\gamma|)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2|\alpha|\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C_0\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

(6), (29), (59), (60) and an integration on $(0,t)$ give

$$
\begin{aligned}
&\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\int_0^t\left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\
&\quad\le C_0 + C_0\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds\\
&\qquad + C_0\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + C_0\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\
&\quad\le C(T)\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned}
\tag{71}
$$

We prove (68). Thanks to (59), (69), and the Hölder inequality,

$$
\begin{aligned}
u^2(t,x) =& 2\int_{-\infty}^x u\partial_x u dy \le 2\int_{\mathbb{R}}|u||\partial_x u|dx\\
\le& \|u(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})} \le C(T)\sqrt{\left(1+\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)}.
\end{aligned}
$$

Hence,

$$
\|u\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\|u\|^2_{L^\infty((0,T)\times\mathbb{R})} - C(T) \le 0,
$$

which gives (68).

Finally, (69) follows form (68) and (71). □

We continue with an energy estimate in the space $L^\infty_{loc}(0,\infty;H^2(\mathbb{R}))\cap L^2_{loc}(0,\infty;H^5(\mathbb{R}))$.

Lemma 10. *Fix $T>0$. There exists a constant $C(T)>0$, such that*

$$
\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\int_{\mathbb{R}}\left\|\partial_x^5 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T),
\tag{72}
$$

for every $0 \leq t \leq T$. *In particular,* (44) *holds.*

Proof. Let $0 \leq t \leq T$. Multiplying (58) by $2\partial_x^4 u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &= 2\int_{\mathbb{R}} \partial_x^4 u \partial_t u dx \\
&= -4\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^4 u dx - 2\gamma\int_{\mathbb{R}} \partial_x^2 u\partial_x^4 u dx + 2\beta^2\int_{\mathbb{R}} \partial_x^6 u\partial_x^4 u dx - 2\alpha\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&= -4\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^4 u dx + 2\gamma\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\beta^2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\alpha\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad = -4\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^4 u dx + 2\gamma\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\alpha\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{73}$$

Due to (68), (69) and the Young inequality,

$$\begin{aligned}
4|\kappa|\int_{\mathbb{R}} |u\partial_x u||\partial_x^4 u|dx &\leq 4|\kappa|\,\|u\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} |\partial_x u||\partial_x^4 u|dx \\
&\leq C(T)\int_{\mathbb{R}} |\partial_x u||\partial_x^4 u|dx \leq C(T)\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\leq C(T) + C(T)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

It follows from (73) that

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad\leq C(T) + C(T)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2|\gamma|\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2|\alpha|\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad\leq C(T) + C(T)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Integrating on $(0,t)$, by (6), (59) and (69), we have

$$\begin{aligned}
&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\int_0^t \left\|\partial_x^5 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\
&\quad\leq C_0 + C(T)t + C(T)\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + C(T)\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\
&\quad\leq C(T),
\end{aligned}$$

which gives (72).

Finally, by (69), (72) and the Hölder inequality, we have (44). □

We continue with an energy estimate in the space $L^\infty_{loc}(0,\infty;H^3(\mathbb{R}))\cap L^2_{loc}(0,\infty;H^6(\mathbb{R}))$.

Lemma 11. *Fix* $T > 0$. *There exists a constant* $C(T) > 0$, *such that*

$$\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\int_0^t \left\|\partial_x^6 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C(T), \tag{74}$$

for every $0 \leq t \leq T$. *In particular,* (52) *holds.*

Proof. Let $0 \le t \le T$. Multiplying (58) by $-2\partial_x^6 u$, an integration on $\mathbb{R}$ gives

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = -2\int_{\mathbb{R}} \partial_x^6 u \partial_t u dx \\
&= 4\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^6 u dx + 2\gamma\int_{\mathbb{R}} \partial_x^6 u\partial_x^2 u dx - 2\beta^2\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\alpha\int_{\mathbb{R}} \partial_x^6 u\partial_x^4 u dx \\
&= 4\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^6 u dx - 2\gamma\int_{\mathbb{R}} \partial_x^5 u\partial_x^3 u dx - 2\beta^2\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\alpha\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&= 4\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^6 u dx + 2\gamma\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\beta^2\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\alpha\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad = 4\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^6 u dx + 2\gamma\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\alpha\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{75}
$$

Due to (68), (69) and the Young inequality,

$$
\begin{aligned}
4|\kappa|\int_{\mathbb{R}} |u||\partial_x u||\partial_x^6 u|dx &\le 2|\kappa|\,\|u\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} |\partial_x u||\partial_x^6 u|dx \\
&\le 2C(T)\int_{\mathbb{R}} |\partial_x u||\partial_x^6 u|dx = 2\int_{\mathbb{R}}\left|\frac{C(T)\partial_x u}{\beta}\right|\left|\beta\partial_x^6 u\right|dx \\
&\le C(T)\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\le C(T) + \beta^2\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (75) that

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_x^6 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad \le C(T) + 2|\gamma|\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2|\alpha|\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad \le C(T) + C(T)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_x^5 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

By (6), (69), (72), and an integration on $(0,t)$,

$$
\begin{aligned}
&\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\int_0^t\left\|\partial_x^6 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\
&\quad \le C_0 + C(T)t + C(T)\int_0^t\left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + C(T)\int_0^t\left\|\partial_x^5 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\
&\quad \le C(T),
\end{aligned}
$$

which gives (74).

Finally, arguing as in Lemma 6, we have (52). □

We continue with an energy estimate in the space $L^\infty_{loc}(0,\infty;H^3(\mathbb{R}))\cap H^1_{loc}((0,\infty)\times\mathbb{R})$.

Lemma 12. *Fix $T > 0$. There exists a constant $C(T) > 0$, such that*

$$
\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t\|\partial_t u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \le C(T). \tag{76}
$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Multiplying (58) by $2\partial_t u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
&\frac{d}{dt}\left(\beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \alpha \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \gamma \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad = -2\beta^2 \int_{\mathbb{R}} \partial_t u \partial_x^6 u dx + 2\alpha \int_{\mathbb{R}} \partial_t u \partial_x^4 u dx + 2\gamma \int_{\mathbb{R}} \partial_t \partial_x u \partial_x^2 u dx\\
&\quad = -2 \left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 4\kappa \int_{\mathbb{R}} u \partial_x u \partial_t u dx.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\frac{d}{dt}\left(\beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \alpha \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \gamma \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad + 2 \left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = -4\kappa \int_{\mathbb{R}} u \partial_x u \partial_t u dx.
\end{aligned} \tag{77}$$

Due to (68), (69), and the Young inequality,

$$\begin{aligned}
4|\kappa| \int_{\mathbb{R}} |u||\partial_x u||\partial_t u| dx &\le 4|\kappa| \left\|u\right\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x u||\partial_t u| dx\\
&\le 2C(T) \int_{\mathbb{R}} |\partial_x u||\partial_t u| dx \le C(T) \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C(T) + \left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Consequently, by (77),

$$\begin{aligned}
&\frac{d}{dt}\left(\beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \alpha \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \gamma \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad + \left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T).
\end{aligned}$$

(6) and an integration on $(0,t)$ give

$$\begin{aligned}
&\beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \alpha \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \gamma \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + \int_0^t \left\|\partial_t u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C_0 + C(T)t \le C(T).
\end{aligned}$$

Therefore, by (69), (72), we have that

$$\begin{aligned}
&\beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t \left\|\partial_t u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\
&\quad \le C(T) + |\alpha| \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + |\gamma| \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T),
\end{aligned}$$

which gives (76). □

We are finally ready to prove Theorem 1 assuming (*iii*).

Proof of Theorem 1 assuming (*iii*). The well-posedness of (1) is guaranteed for a short time by the Cauchy-Kowaleskaya Theorem [64]. Thanks to the a priori estimates proved in Lemmas 8–12, we have that the global-in-time existence of a is solution of (1) that satisfies (13).

The stability estimates (15) can be proved using the same arguments of [18] (Theorem 1). □

5. Proof of Theorem 1 Assuming (iv)

In this final section, we prove Theorem 1 assuming (iv).

The argument is again analogous to the one of the previous sectionss. We deduce the local-in-time well-posedness from the Cauchy-Kowaleskaya Theorem [64], and we improve the local-in-time existence to the global-in-time one proving some suitable a priori estimates on u.

We begin with the zero mean estimate.

Lemma 13. *For each $t > 0$, we have* (14).

Proof. Integrating (1) on $\mathbb{R}$, we have that

$$\int_{\mathbb{R}} \partial_t u(t,x)dx = \frac{d}{dt}\int_{\mathbb{R}} u(t,x)dx = 0 \tag{78}$$

(14) follows from (6) and (78). □

Remark 1. *In light of* (14), *we can consider the following equation:*

$$P(t,x) = \int_{-\infty}^{x} u(t,y)dy. \tag{79}$$

Moreover, again by (14), *we have that*

$$P(t,-\infty) = P(t,\infty) = 0. \tag{80}$$

We continue by proving some energy estimates on the function P.

Lemma 14. *Let $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\begin{aligned}
\|P(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ 2\beta^2 e^{2t}\int_0^t e^{-2s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\
&+ 2\delta^2 e^{t}\int_0^{2t} e^{-2s}\|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \\
&\le C(T) + C(T)\int_0^t \|u(s,\cdot)\|^4_{L^4(\mathbb{R})} ds + C(T)\int_0^t \|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds,
\end{aligned} \tag{81}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Integrating (1) on $(-\infty, x)$, we have that

$$\int_{-\infty}^{x} \partial_t u dx + \kappa u^2 + \gamma\partial_x u - \beta^2\partial_x^5 u + \alpha\partial_x^3 u + \delta^2\partial_x^3\left(u^3\right) = 0. \tag{82}$$

Differentiating (79) with respect to t, we obtain that

$$\partial_t P(t,x) = \frac{d}{dt}\int_{-\infty}^{x} u(t,y)dy = \int_{-\infty}^{x} \partial_t u(t,y)dy. \tag{83}$$

It follows from (82) and (83) that

$$\partial_t P + \kappa u^2 + \gamma\partial_x u - \beta^2\partial_x^5 u + \alpha\partial_x^3 u + \delta^2\partial_x^3\left(u^3\right) = 0. \tag{84}$$

Arguing as in [18] (Lemma 2), we have that

$$\begin{aligned} -2\beta^2\int_{\mathbb{R}} P\partial_x^5 u dx =& 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ 2\alpha\int_{\mathbb{R}} P\partial_x^3 u dx =& 2\alpha\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ 2\delta^2\int_{\mathbb{R}} P\partial_x^3\left(u^3\right) dx =& 6\delta^2\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{85}$$

Therefore, multiplying (84) by $2P$, thanks to (85), an integration on $\mathbb{R}$ gives

$$\begin{aligned} &\frac{d}{dt}\left\|P(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 6\delta^2\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\qquad = -2\kappa\int_{\mathbb{R}} Pu^2 dx - 2\gamma\int_{\mathbb{R}} P\partial_x u dx - 2\alpha\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{86}$$

Due to the Young inequality,

$$\begin{aligned} 2|\kappa|\int_{\mathbb{R}} |P|u^2 dx \le \left\|P(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \kappa^2\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})},\\ 2|\gamma|\int_{\mathbb{R}} |P||\partial_x u| dx \le \left\|P(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \gamma^2\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

It follows from (86) that

$$\begin{aligned} &\frac{d}{dt}\left\|P(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 6\delta^2\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\qquad \le 2\left\|P(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0\left\|u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + C_0\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Therefore, by the Gronwall Lemma and (9), we have that

$$\begin{aligned} &\left\|P(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2 e^{2t}\int_0^t e^{-2s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &\qquad + 6\delta^2 e^{2t}\int_0^t e^{-2s}\left\|u(s,\cdot)\partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &\qquad \le C_0 + C_0 e^{2t}\int_0^t e^{-2s}\left\|u(s,\cdot)\right\|^4_{L^4(\mathbb{R})} ds + C_0 e^{2t}\int_0^t e^{-2s}\left\|\partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &\qquad \le C(T) + C(T)\int_0^t \left\|u(s,\cdot)\right\|^4_{L^4(\mathbb{R})} ds + C(T)\int_0^t \left\|\partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds, \end{aligned}$$

which gives (81). □

Lemma 15. *Let $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\frac{\beta^2}{6}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4}\left\|u(t,\cdot)\right\|^2_{L^4(\mathbb{R})} \le C(T), \tag{87}$$

for every $0 \le t \le T$. In particular, we have (29), (31),

$$\begin{aligned} \left\|P(t,\cdot)\right\|_{L^2(\mathbb{R})} \le& C(T),\\ \left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})} \le& C(T),\\ \int_0^t \left\|u(s,\cdot)\partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le& C(T),\\ \int_0^t\!\!\int_{\mathbb{R}} \left[\beta^2(\partial_x^4 u)^2 - \delta^2\partial_x^3\left(u^3\right)\right]^2 ds dx \le& C(T), \end{aligned}$$

for every $0 \leq t \leq T$. *Moreover,*

$$\begin{aligned}\|P\|_{L^\infty((0,T)\times\mathbb{R})} \leq & C(T)\\ \|u\|_{L^\infty((0,T)\times\mathbb{R})} \leq & C(T).\end{aligned}$$

Proof. Let $0 \leq t \leq T$. Consider an real constant A, which will be specified later. Observe that

$$\begin{aligned}\delta^2\gamma \int_{\mathbb{R}} u^3\partial_x^2 u dx =& -3\delta^2\gamma \|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ 2A\gamma \int_{\mathbb{R}} u\partial_x^2 u dx =& -2A \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{88}$$

Multiplying (1) by

$$-\beta^2\partial_x^2 u + \delta^2 u^3 + Au,$$

thanks to (88) and arguing as in [18] (Lemma 3), an integration on $\mathbb{R}$ gives

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\beta^2}{2}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4}\|u(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{A}{2}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\ &\quad + \int_{\mathbb{R}}\left[\beta^2(\partial_x^4 u)^2 - \delta^2\partial_x^2\left(u^3\right)\right]^2 dx\\ &= -\beta^2(A+\alpha)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \delta^2(A+\alpha)\int_{\mathbb{R}}\partial_x^2 u\partial_x^2\left(u^3\right)dx\\ &\quad - \left(A\alpha - \gamma\beta^2\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 3\delta^2\gamma\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\quad + 2\beta^2\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^2 u dx.\end{aligned}$$

Taking

$$A = -\alpha,$$

we have that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\beta^2}{2}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4}\|u(t,\cdot)\|^4_{L^4(\mathbb{R})} - \frac{\alpha}{2}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\ &\quad + \int_{\mathbb{R}}\left[\beta^2(\partial_x^4 u)^2 - \delta^2\partial_x^2\left(u^3\right)\right]^2 dx\\ &= \left(\alpha^2 + \gamma\beta^2\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 3\delta^2\gamma\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^2 u dx.\end{aligned} \tag{89}$$

Due to the Young inequality,

$$2\beta^2|\kappa|\int_{\mathbb{R}}|u\partial_x u||\partial_x^2 u|dx \leq \beta^2\kappa^2\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

It follows from (89) that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\beta^2}{2}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4}\|u(t,\cdot)\|^4_{L^4(\mathbb{R})} - \frac{\alpha}{2}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right) + \int_{\mathbb{R}}\left[\beta^2(\partial_x^4 u)^2 - \delta^2\partial_x^2\left(u^3\right)\right]^2 dx\\ &\leq C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Integrating on $(0,t)$, by (7), we have that

$$\frac{\beta^2}{2}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\delta^2}{4}\|u(t,\cdot)\|^4_{L^4(\mathbb{R})} - \frac{\alpha}{2}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

$$
+\int_0^t\int_{\mathbb{R}}\left[\beta^2(\partial_x^4 u)^2-\delta^2\partial_x^2\left(u^3\right)\right]^2 dsdx
\le C_0+C_0\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+C_0\int_0^t\|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds,
$$

that is

$$
\begin{aligned}
&\frac{\beta^2}{2}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\delta^2}{4}\|u(t,\cdot)\|^4_{L^4(\mathbb{R})}+\int_0^t\int_{\mathbb{R}}\left[\beta^2(\partial_x^4 u)^2-\delta^2\partial_x^2\left(u^3\right)\right]^2 dsdx \\
&\quad\le C_0+C_0\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+C_0\int_0^t\|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds+\frac{\alpha}{2}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad\le C_0+C_0\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+C_0\int_0^t\|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds+C_0\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{90}
$$

Observe that, by (79) and (80),

$$
C_0\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}=C_0\int_{\mathbb{R}}uudx=-C_0\int_{\mathbb{R}}P\partial_x udx.
$$

Therefore, by the Young inequality,

$$
\begin{aligned}
C_0\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}\le&2\int_{\mathbb{R}}\left|\frac{\sqrt{3}C_0P}{2\beta}\right|\left|\frac{\beta\partial_x u}{\sqrt{3}}\right|dx \\
\le&C_0\|P(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{3}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{91}
$$

It follows from (81), (90) and (91) that

$$
\begin{aligned}
&\frac{\beta^2}{6}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\delta^2}{4}\|u(t,\cdot)\|^4_{L^4(\mathbb{R})}+\int_0^t\int_{\mathbb{R}}\left[\beta^2(\partial_x^4 u)^2-\delta^2\partial_x^2\left(u^3\right)\right]^2 dsdx \\
&\quad\le C_0+C_0\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+C_0\int_0^t\|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds+C_0\|P(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad\le C(T)+C(T)\int_0^t\|u(s,\cdot)\|^4_{L^4(\mathbb{R})}ds+C(T)\int_0^t\|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \\
&\quad\le C(T)+C(T)\left(\frac{\beta^2}{6}\int_0^t\|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds+\frac{\delta^2}{4}\int_0^t\|u(s,\cdot)\|^4_{L^4(\mathbb{R})}ds\right).
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
&\frac{\beta^2}{6}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\delta^2}{4}\|u(t,\cdot)\|^4_{L^4(\mathbb{R})} \\
&\quad\le C(T)+C(T)\left(\frac{\beta^2}{6}\int_0^t\|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds+\frac{\delta^2}{4}\int_0^t\|u(s,\cdot)\|^4_{L^4(\mathbb{R})}ds\right).
\end{aligned}
$$

The Gronwall Lemma and (7) give (87).

Finally, arguing as in [18] (Lemma 3), the proof is concluded. □

Arguing as in [18] (Theorem 1), we have Theorem 1.

6. Conclusions

This paper is dedicated to the well-posedness of a solution to the Cauchy problem for a higher-order convective Cahn-Hilliard equation. Such an equation models the evolution of crystal surfaces faceting through surface electromigration, the growing surface faceting, and the evolution of dynamics of phase transitions in ternary oil-water-surfactant systems. The well-posedness of (1) is proved for a short time by the Cauchy-Kowaleskaya Theorem [64]. The global-in-time well-posedness is thus proved, proving several a priori estimates.

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