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On Global Well-Posedness and Temporal Decay for 3D Magnetic Induction Equations with Hall Effect

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Abstract: The main purpose of this paper is to study the global existence and uniqueness of solutions for three-dimensional incompressible magnetic induction equations with Hall effect provided that $\|u_0\|_{H^{\frac{3}{2}+\varepsilon}} + \|b_0\|_{H^2}$ $(0 < \varepsilon < 1)$ is sufficiently small. Moreover, using the Fourier splitting method and the properties of decay character r^*, one also shows the algebraic decay rate of a higher order derivative of solutions to magnetic induction equations with the Hall effect.

Keywords: magnetic induction equations with Hall effect; global existence; uniqueness; decay character; decay rate

1. Introduction

Supposing that ρ denotes the density, u describes the velocity field of the fluid, b means the magnetic field and π is the pressure, a high-resolution, non-oscillatory, central scheme for the Hall–MHD model [1,2] can be introduced in the following:

$$\partial_t \rho = -\nabla \cdot (\rho u), \tag{1}$$

$$\partial_t (\rho u) = -\nabla \cdot \left\{ \rho u u^t + \left(\pi + \frac{b^2}{2} \right) I_{3\times 3} - b b^t \right\}, \tag{2}$$

$$\partial_t b = -\nabla \times E, \tag{3}$$

$$\partial_t U = -\nabla \cdot \left\{ \left(U + \pi - \frac{b^2}{2} \right) u + E \times B \right\}, \tag{4}$$

$$\nabla \cdot u = \nabla \cdot b = 0. \tag{5}$$

Systems (1)–(5) follow from the MHD equations after normalizing as the Geospace Environment Modeling (GEM) challenge. The total energy, U, momentum, ρu and magnetic field, b, can be coupled through the following state equation:

$$U = \frac{\pi}{\gamma - 1} + \frac{\rho u^2}{2} + \frac{b^2}{2}. \tag{6}$$

Moreover, one expresses the electric field in the generalized Ohm's law [1,2]:

$$E = -u \times b + \eta j + \frac{\delta_i}{L_0} \frac{j \times b}{\rho} - \frac{\delta_i}{L_0} \frac{\nabla \pi}{\rho} + \left(\frac{\delta_e}{L_0} \right)^2 \frac{1}{\rho} \left[\partial_t j + (u \cdot \nabla) j \right], \tag{7}$$

$$j = \nabla \times b,, \tag{8}$$

where L_0, δ_e and δ_i denotes the normalizing length limit, electron inertia and ion inertia, respectively. For the simulations considered in the work, the electron pressure tensor $-\frac{\delta_i}{L_0} \frac{\nabla \pi}{\rho}$ will be ignored [2].

Considering the incompressible case, denoting $\rho \equiv 1$, combining (1)–(8) together yields the following three-dimensional system [1,3–5]:

$$\mathrm{div} u = \mathrm{div} b = 0, \tag{9}$$

$$\partial_t u + u \cdot \nabla u + \nabla \left(\pi + \frac{1}{2}|b|^2\right) - \Delta u = b \cdot \nabla b, \tag{10}$$

$$\partial_t b - \left(\frac{\delta_e}{L_0}\right)^2 \partial_t \Delta b + u \cdot \nabla b - b \cdot \nabla u - \Delta b = \frac{\delta_i}{L_0}\mathrm{rot}(b \times \mathrm{rot} b) - \left(\frac{\delta_e}{L_0}\right)^2 \mathrm{rot}((u \cdot \nabla)\mathrm{rot} b), \tag{11}$$

$$(u, b)(\cdot, 0) = (u_0, b_0)(\cdot) \quad \text{in } \mathbb{R}^3. \tag{12}$$

For simplicity, $\delta_i = 1$ and $\left(\frac{\delta_e}{L_0}\right)^2 = 1$ in this paper.

If $\delta_e = 0$, systems (9)–(12) reduce to the three-dimensional incompressible Hall–MHD system, whose applications cover a very wide range of physical objects, for example, magnetic reconnection in space plasmas, star formation, neutron stars and geo-dynamo. The global well-posedness, regularity criterion and decay characterization of solutions to 3D incompressible Hall–MHD system were studied by many authors [6–14]. It is worth pointing out that Wan et al. [15] assumed that the initial data $(u_0, b_0) \in H^m(\mathbb{R}^3)$ with $m > \frac{5}{2}$, $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, and $\|u_0\|_{\dot{B}^{\frac{1}{2}+\varepsilon}_{2,\infty}} + \|b_0\|_{\dot{B}^{\frac{3}{2}}_{2,\infty}}$ are sufficiently small, proving that the 3D Hall–MHD system admits a unique global solution $(u, b) \in C(0, \infty; H^m(\mathbb{R}^3))$, which may be the latest result on the small initial data global well-posedness for the Hall–MHD system.

For systems (9)–(12), Fan et al. [16] established the existence of global weak solutions, existence of local strong solutions and some blow-up criteria. They pointed out that if $u_0 \in L^2(\mathbb{R}^3)$, $b_0 \in H^1(\mathbb{R}^3)$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, then there exists a weak solution (u, b) for systems (9)–(12), which satisfies the energy inequality

$$\int_{\mathbb{R}^3} (|u|^2 + |b|^2 + |\nabla b|^2)dx + 2\int_0^T \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla b|^2)dxdt \le \int_{\mathbb{R}^3} (|u_0|^2 + |b_0|^2 + |\nabla b_0|^2)dx.$$

Latterly, Ma et al. [17] proved the global existence of strong solutions to 3D two-fluid MHD equations provided that $\|u_0\|_{\dot{H}^{\frac{1}{2}}} + \|b_0\|_{\dot{H}^{\frac{1}{2}}} + \|b_0\|_{\dot{H}^{\frac{3}{2}}}$ is sufficiently small. The main difference between systems (9)–(12), the Hall–MHD system and the two-fluid MHD system is the nonlinear term $\mathrm{rot}\,((u \cdot \nabla)\mathrm{rot} b)$. Because of the existence of this nonlinear term, it is difficult to obtain the global well-posedness of systems (9)–(12) under the same assumption as Wan et al. [15] and Ma et al. [17].

The first purpose of this paper is to to prove the following theorem on the global well-posedness of systems (9)–(12).

Theorem 1. *Let $\varepsilon \in (0, 1)$ and $m, p, K \in \mathbb{N}$, $K \ge \max\{m, 3\}$. Assume that the initial data $(u_0, b_0) \in H^K(\mathbb{R}^3) \times H^{K+1}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ such that*

$$\|u_0\|_{H^{\frac{3}{2}+\varepsilon}} + \|b_0\|_{H^2} < \eta(\alpha), \tag{13}$$

for some small enough constant $\eta(\alpha) > 0$. Then, there exists a unique global solution (u, b) for systems (9)–(12), such that

$$\|\partial_t^p \Lambda^m u\|_{L^2}^2 + \|\partial_t^p \Lambda^m b\|_{L^2}^2 + \|\partial_t^p \Lambda^{m+1} b\|_{L^2}^2 \le C(\|u_0\|_{H^K}^2 + \|b_0\|_{H^K}^2), \tag{14}$$

for all $m + 2p \le K$.

Remark 1. *In the above and the following, Λ^m is defined by*

$$\Lambda^m f(x) = (-\Delta)^{\frac{m}{2}} f(x) = \int_{\mathbb{R}^3} |\xi|^m \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi. \tag{15}$$

The temporal decay rate of solutions is also an interesting topic in the study of dissipative equations. One of the main tools to study the temporal decay rate is the Fourier splitting method, which was introduced by Schonbek in [18,19]. Laterly, this method was well extended to investigate the decay for the solutions of PDE from mathematical physics, see, e.g., Schonbek et al. [20] for the MHD sysem, Brandolese et al. [21] for the viscous Boussinesq system, Dai et al. [22] for liquid crystal systems, Weng [14] and Chae et al. [8] for the Hall–MHD system, Niche [23] for the Navier–Stokes–Voigt equations, Ferreira et al. [24] for quasi-geostrophic equations, Zhao et al. [25] for third grade fluids, etc.

Recently, in order to characterize the decay rate of dissipative equations more profoundly, Bjorland et al. [26] and Niche et al. [27] introduced the idea of decay indicator P_r and decay character r^*. Latterly, Brandolese [28] improved the definition of the decay indicator and the decay character by taking advantage of the insight provided by the Littlewood–Paley analysis and the use of Besov spaces. For more details on P_r and r^*, we refer to Section 2.

In consequence, it is desirable to understand the asymptotic behavior of the magnetic induction equations with the Hall effect. With the aid of the classical Fourier splitting method and the properties of decay character r^*, the decay rate of solutions to systems (9)–(12) has been characterized:

Lemma 1 ([29])**.** *Assume that $(u_0, b_0) \in L^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ and $r^* = r^*(u_0) = r^*(b_0) \in (-\frac{3}{2}, +\infty)$ is the decay character. Let (u, b) be the solution of systems (9)–(12) with initial value (u_0, b_0). Then*

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 \le C(1+t)^{-\min\{\frac{3}{2}+r^*, \frac{5}{2}\}}, \ \forall t > 0,$$

where the constant C depends essentially on $\|u_0\|_{L^2}$, $\|b_0\|_{L^2}$ and $\|\nabla b_0\|_{L^2}$.

Lemma 2 ([30])**.** *Suppose that $m \in \mathbb{N}$, $K_0 \ge \max\{3, m\}$, $(u_0, b_0) \in H^{K_0}(\mathbb{R}^3) \times H^{K_0+1}(\mathbb{R}^3)$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Let $r^* = r^*(u_0) = r^*(b_0) \in (-\frac{3}{2}, +\infty)$ be the decay character. Then, for the small global-in-time solution (u, b), there exists a positive constant $C = C(, \|u_0\|_{H^{K_0}}, \|b_0\|_{H^{K_0+1}})$, such that*

$$\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^m b\|_{L^2}^2 + \|\Lambda^{m+1} b\|_{L^2}^2 \le C(1+t)^{-\min\{\frac{3}{2}+r^*+m, \frac{5}{2}+m\}}, \ \textit{for large } t.$$

On the basis of Lemmas 1 and 2, using the properties of decay character r^* and Fourier splitting method, one can continue to study the decay characterization of solutions to systems (9)–(12), establish the decay rate of higher-order derivative of solutions on both time and space. Note that the global in-time existence and uniqueness can be guaranteed for sufficiently small initial data. The result can be described as follows:

Theorem 2. *Let m, $p \in \mathbb{N}$, $K \ge \max\{3+2p, m+2p\}$ and $r^* = r^*(u_0) = r^*(b_0) \in (-\frac{3}{2}, +\infty)$ be the decay character. Suppose that $(u_0, b_0) \in H^K(\mathbb{R}^3) \times H^{K+1}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then, there exists a positive constant $C = C(, \|u_0\|_{H^K}, \|b_0\|_{H^{K+1}})$, such that*

$$\|\partial_t^p \Lambda^m u\|_{L^2}^2 + \|\partial_t^p \Lambda^m b\|_{L^2}^2 + \|\partial_t^p \Lambda^{m+1} b\|_{L^2}^2 \le C(1+t)^{-\min\{\frac{3}{2}+r^*+m+2p, \frac{5}{2}+m+2p\}}, \tag{16}$$

for large t.

The rest of this paper is organized as follows. In Section 2, we give some preliminary results on the properties of decay character r^*. Section 3 is devoted to the proofs of Theorem 1. The proof of Theorem 2 is given in Section 4. Conclusions are outlined in Section 5.

2. Properties of Decay Character

2.1. Definition and Properties of Decay Character

The definitions of decay indicator $P_r(u_0)$ and decay character r^* was first introduced in [26], Brandolese [28] redefined them, which seems more precise.

Definition 1 ([28]). *Suppose that $v_0 \in L^2(\mathbb{R}^n)$, $B_\rho = \{\xi \in \mathbb{R}^n : |\xi| \le \rho\}$ and $\Lambda = (-\Delta)^{\frac{1}{2}}$. If the following two lower and upper limits exist, they are the lower and upper decay indicators of v_0:*

$$P_r(v_0))_- = \lim_{\rho \to 0^-} \rho^{-2r-n} \int_{B(\rho)} |\widehat{v}_0(\xi)|^2 d\xi,$$

$$P_r(v_0))_+ = \lim_{\rho \to 0^+} \rho^{-2r-n} \int_{B(\rho)} |\widehat{v}_0(\xi)|^2 d\xi.$$

When $P_r(v_0)_- = P_r(v_0)_+$, then $P_r(v_0) = P_r(v_0)_- = P_r(v_0)_+$ can be defined as the decay indicator corresponding to v_0.

Definition 2 ([28]). *The upper and lower decay characters of $v_0 \in L^2(\mathbb{R}^n)$ are defined as*

$$r(v_0)_+ = \sup\{r \in \mathbb{R} : P_r(v_0)_+ < \infty\},$$

$$r(v_0)_- = \sup\{r \in \mathbb{R} : P_r(v_0)_- < \infty\}.$$

Definition 3 ([28]). *If $v_0 \in L^2(\mathbb{R}^n)$ is such that there exists $r^* \in (-\frac{n}{2}, \infty)$ such that*

$$r^*(v_0) = \max\{r \in \mathbb{R} : P_r(v_0)_+ < \infty\} = \min\{r \in \mathbb{R} : P_r(v_0)_- > 0\}.$$

then this number $r^* = r^*(v_0)$ can be called the decay character of v_0. The decay character of v_0 in the two limit situations is defined as follows:

$$r^*(v_0) = +\infty, \quad \text{if } r(v_0)_+ = r(v_0)_- = +\infty,$$

$$r^*(v_0) = -\frac{n}{2}, \quad \text{if } r(v_0)_+ = r(v_0)_- = -\frac{n}{2}.$$

Lemma 3 ([27]). *Let $v_0 \in H^s(\mathbb{R}^n)$, with $s > 0$. Then*

(1) if $-\frac{n}{2} < r^*(v_0) < \infty$, then $-\frac{n}{2} + s < r_s^*(v_0) < \infty$ and $r_s^*(v_0) = s + r^*(v_0)$;
(2) $r_s^*(v_0) = \infty$ if and only if $r^*(v_0) = \infty$;
(3) $r^*(v_0) = -\frac{n}{2}$ if and only if $r_s^*(v_0) = -\frac{n}{2} + s$.

Remark 2. *The decay character $r^* = r^*(v_0)$ measures the "order" of $\widehat{v}_0(\xi)$ at $\xi = 0$ in frequency space. The theory of [26,27] allows defining the decay character only in the following three situations:*

(1) Either, $\exists r \in (-\frac{n}{2} + s, +\infty)$ such that $0 < P_r^s(u_0) < +\infty$, and in this case r is unique,
(2) Or $\forall r \in (-\frac{n}{2} + s, +\infty)$, one has $P_r^s(u_0) = 0$,
(3) Or $\forall r \in (-\frac{n}{2} + s, +\infty)$, one has $P_r^s(u_0) = +\infty$.

However, not in the other cases (e.g., it can happen that

$$\exists r, r' \in (-\frac{n}{2} + s, +\infty) \text{ such that } P_r^s(u_0) = 0 \text{ and } P_{r'}^s(u_0) = +\infty.$$

In addition, it can also happen that the limit-defining $P_r^s(u_0)$ does not exist.

2.2. Decay Characterization of a Linear Equation

Consider the linear part of (11):

$$\begin{cases} v_t - \Delta v_t - \Delta v = 0, \\ v(x,0) = v_0(x), \end{cases} \quad x \in \mathbb{R}^n,\ n \in \mathbb{N}^+. \tag{17}$$

Define the space $H^1(\mathbb{R}^3)$, such that

$$\|v\|^2_{H^1(\mathbb{R}^3)} = \|v\|^2_{L^2(\mathbb{R}^3)} + \|\nabla v\|^2_{L^2}, \quad s \geq 0.$$

Hence,

$$\frac{d}{dt}(\|v\|^2_{L^2} + \|\nabla v\|^2_{L^2}) + 2\|\nabla v\|_{L^2} = 0.$$

In the Fourier space, the solution to (17) is

$$\hat{v}(\xi,t) = e^{-\frac{|\xi|^2}{1+|\xi|^2}t}\hat{v}_0(\xi),$$

and

$$\partial_t^p \hat{v}(\xi,t) = (-1)^p \left(\frac{|\xi|^2}{1+|\xi|^2}\right)^p e^{-\frac{|\xi|^2}{1+|\xi|^2}t}\hat{v}_0(\xi).$$

The L^2-decay characterization of solutions to system (17) was established by Niche [23].

Lemma 4 ([23]). *Assume that $v_0 \in H^1(\mathbb{R}^n)$, which has decay character $r^*(v_0) = r^*$, is a solution to system (17). Then*

(1) If $-\frac{n}{2} < r^* < +\infty$, there exist two positive constants $C_1,\ C_2 > 0$, such that

$$C_1(1+t)^{-(\frac{n}{2}+r^*)} \leq \|v(t)\|^2_{L^2} + \|\nabla v(t)\|^2_{L^2} \leq C_2(1+t)^{-(\frac{n}{2}+r^*)};$$

(2) if $r^* = -\frac{n}{2}$, there exists $C = C(\varepsilon) > 0$, such that

$$\|v(t)\|^2_{L^2} + \|\nabla v(t)\|^2_{L^2} \geq C(1+t)^{-\varepsilon}, \quad \forall \varepsilon > 0,$$

which means the decay of $\|v\|^2_{L^2} + \|\nabla v(t)\|^2_{L^2}$ is slower than any uniform algebraic rate;

(3) if $r^* = +\infty$, there exists a $C > 0$ such that

$$\|v(t)\|^2_{L^2} + \|\nabla v(t)\|^2_{L^2} \leq C(1+t)^{-m}, \quad \forall m > 0,$$

that is, the decay of $\|v\|^2_{L^2} + \|\nabla v(t)\|^2_{L^2}$ is faster than any algebraic rate.

Therei s also a lemma on the $\dot{H}^s$-decay rate of solutions to system (17).

Lemma 5 ([31,32]). *Suppose that $\Lambda^s v_0 \in H^1(\mathbb{R}^n)$ $(s > 0)$ has decay character $r_s^* = r_s^*(v_0)$. Then*

(1) If $-\frac{n}{2} \leq r^* < +\infty$, there exist two positive constants $C_1,\ C_2 > 0$, such that

$$C_1(1+t)^{-(\frac{n}{2}+r^*+s)} \leq \|\Lambda^s v(t)\|^2_{L^2} + \|\Lambda^{s+1} v(t)\|^2_{L^2} \leq C_2(1+t)^{-(\frac{n}{2}+r^*+s)};$$

(2) if $r^* = +\infty$, there exists a $C > 0$ such that

$$\|v(t)\|^2_{L^2} + \|\Lambda^{s+1} v(t)\|^2_{L^2} \leq C(1+t)^{-m}, \quad \forall m > 0,$$

that is, the decay of $\|\Lambda^s v\|^2_{L^2} + \|\Lambda^{s+1} v\|^2_{L^2}$ is faster than any algebraic rate.

The following result regarding the decay characterization of solutions to (17) can be found in [32].

Lemma 6. *Let $v_0 \in H^{K+1}(\mathbb{R}^n)$ ($s > 0$) have decay character $r_s^* = r_s^*(v_0)$. Then, for all $0 < m + 2p \le K$, the following decay estimates hold:*

(1) If $-\frac{n}{2} \le r^* < \infty$, then there exists a positive constant C_1 such that

$$\|\partial_t^p \Lambda^m v(t)\|_{L^2}^2 + \|\partial_t^p \Lambda^{m+1} v(t)\|_{L^2}^2 \le C_1(1+t)^{-(r^*+m+2p+\frac{n}{2})};$$

(2) if $r^* = \infty$, given any $s > 0$, there exists a positive constant $C_2 = C_2(s)$ such that

$$\|\partial_t^p \Lambda^m v(t)\|_{L^2}^2 + \|\partial_t^p \Lambda^{m+1} v(t)\|_{L^2}^2 \le C_2(1+t)^{-s},$$

which means the decay is faster than any algebraic rate.

2.3. Decay Characterization of the Linear Part for Systems (9)–(12)

For the linear part of systems (9)–(12):

$$\begin{cases} \bar{u}_t - \Delta \bar{u} = 0, \\ \bar{b}_t - \Delta \bar{b}_t - \Delta \bar{b} = 0, \\ \bar{u}(x,0) = \bar{u}_0(x), \\ \bar{b}(x,0) = \bar{b}_0(x), \end{cases} \quad x \in \mathbb{R}^3. \tag{18}$$

Combining the results of Niche [23], Niche et al. [27], Anh et al. [33], Zhao [30], we obtain the following three lemmas:

Lemma 7 ([32])**.** *Assume that $(\bar{u}_0, \bar{b}_0) \in L^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, which has decay character r^*. Then*

(1) If $-\frac{3}{2} < r^* < +\infty$, there exist two positive constants $C_1,\ C_2 > 0$, such that

$$C_1(1+t)^{-(\frac{3}{2}+r^*)} \le \|\bar{u}(t)\|_{L^2}^2 + \|\bar{b}(t)\|_{L^2}^2 + \|\nabla \bar{b}(t)\|_{L^2}^2 \le C_2(1+t)^{-(\frac{3}{2}+r^*)};$$

(2) if $r^* = -\frac{3}{2}$, there exists $C = C(\varepsilon) > 0$, such that

$$\|\bar{u}(t)\|_{L^2}^2 + \|\bar{b}(t)\|_{L^2}^2 + \|\nabla \bar{b}(t)\|_{L^2}^2 \ge C(1+t)^{-\varepsilon}, \quad \forall \varepsilon > 0,$$

which means the decay of $\|\bar{u}(t)\|_{L^2}^2 + \|\bar{b}(t)\|_{L^2}^2 + \|\nabla \bar{b}(t)\|_{L^2}^2$ is slower than any uniform algebraic rate;

(3) if $r^* = +\infty$, there exists a $C > 0$ such that

$$\|\bar{u}(t)\|_{L^2}^2 + \|\bar{b}(t)\|_{L^2}^2 + \|\nabla \bar{b}(t)\|_{L^2}^2 \le C(1+t)^{-m}, \quad \forall m > 0,$$

that is, the decay of $\|\bar{u}(t)\|_{L^2}^2 + \|\bar{b}(t)\|_{L^2}^2 + \|\nabla \bar{b}(t)\|_{L^2}^2$ is faster than any algebraic rate.

Lemma 8 ([32])**.** *Suppose that $(\Lambda^s \bar{u}_0, \Lambda^s \bar{b}_0) \in L^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ ($s > 0$) has decay character r_s^*. Then*

(1) If $-\frac{3}{2} \le r^* < +\infty$, there exist two positive constants $C_1,\ C_2 > 0$, such that

$$C_1(1+t)^{-(\frac{3}{2}+r^*+s)} \le \|\Lambda^s \bar{u}(t)\|_{L^2}^2 + \|\Lambda^s \bar{b}(t)\|_{L^2}^2 + \|\Lambda^s \nabla \bar{b}(t)\|_{L^2}^2 \le C_2(1+t)^{-(\frac{3}{2}+r^*+s)};$$

(2) if $r^* = +\infty$, there exists a $C > 0$ such that

$$\|\Lambda^s \bar{u}(t)\|_{L^2}^2 + \|\Lambda^s \bar{b}(t)\|_{L^2}^2 + \|\Lambda^s \nabla \bar{b}(t)\|_{L^2}^2 \le C(1+t)^{-m}, \quad \forall m > 0,$$

that is, the decay of $\|\Lambda^s \bar{u}(t)\|_{L^2}^2 + \|\Lambda^s \bar{b}(t)\|_{L^2}^2 + \|\Lambda^s \nabla \bar{b}(t)\|_{L^2}^2$ is faster than any algebraic rate.

Lemma 9 ([32]). *Suppose that $K \in \mathbb{Z}^+$. Let $(\bar{u}_0, \bar{b}_0) \in H^K(\mathbb{R}^3) \times H^{K+1}(\mathbb{R}^3)$ have decay character $r*$. Then, for all $0 < m + 2p \le K$, the following decay estimates hold:*

(1) If $-\frac{3}{2} \le r^* < \infty$, then there exists a positive constant C_1 such that

$$\|\partial_t^p \Lambda^m \bar{u}(t)\|_{L^2}^2 + \|\partial_t^p \Lambda^m \bar{b}(t)\|_{L^2}^2 + \|\partial_t^p \Lambda^{m+1} \bar{b}(t)\|_{L^2}^2 \le C_1(1+t)^{-(r^*+m+2p+\frac{3}{2})};$$

(2) if $r^* = \infty$, given any $s > 0$, there exists a positive constant $C_2 = C_2(s)$ such that

$$\|\partial_t^p \Lambda^m \bar{u}(t)\|_{L^2}^2 + \|\partial_t^p \Lambda^m \bar{b}(t)\|_{L^2}^2 + \|\partial_t^p \Lambda^{m+1} \bar{b}(t)\|_{L^2}^2 \le C_2(1+t)^{-s},$$

that is, the decay is faster than any algebraic rate.

3. Proof of Theorems 1

One first proves that (14) holds for $p = 0$.

Testing by u and b, respectively, adding them together gives

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right) + \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 = 0. \tag{19}$$

Taking Λ to (11), testing by Λb, respectively, it yields that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Lambda b\|_{L^2}^2 + \|\Lambda^2 b\|_{L^2}^2\right) + \|\Lambda^2 b\|_{L^2}^2 \\
=&-\int_{\mathbb{R}^3} \Lambda(u\cdot\nabla b)\Lambda b dx + \int_{\mathbb{R}^3} \Lambda(b\cdot\nabla u)\Lambda b dx \\
&+\int_{\mathbb{R}^3} \Lambda(\mathrm{rot}(b\times \mathrm{rot} b))\Lambda b dx - \int_{\mathbb{R}^3} \Lambda(\mathrm{rot}((u\cdot\nabla)\mathrm{rot} b))\Lambda b dx \\
\le& C\|u\|_{L^3}\|\nabla b\|_{L^6}\|\Lambda^2 b\|_{L^2} + C\|b\|_{L^6}\|\nabla u\|_{L^3}\|\Lambda^2 b\|_{L^2} \\
&+C(\|\nabla b\|_{L^3}\|\nabla b\|_{L^6} + \|b\|_{L^\infty}\|\Lambda^2 b\|_{L^2})\|\Lambda^2 b\|_{L^2} + C\|\nabla u\|_{L^\infty}\|\Lambda^2 b\|_{L^2} \\
\le& C(\|u\|_{H^{\frac{3}{2}+\varepsilon}} + \|b\|_{H^2})(\|\nabla b\|_{H^1}^2 + \|\nabla u\|_{H^{\frac{3}{2}+\varepsilon}}^2),
\end{aligned} \tag{20}$$

where one has used

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} \Lambda(\mathrm{rot}((u\cdot\nabla)\mathrm{rot} b))\Lambda b dx\right| \\
\le&\left|\int_{\mathbb{R}^3} \Lambda((u\cdot\nabla)\mathrm{rot} b)\Lambda \mathrm{rot} b dx - \int_{\mathbb{R}^3} ((u\cdot\nabla)\Lambda \mathrm{rot} b)\Lambda \mathrm{rot} b dx\right| \\
\le& C\|\nabla u\|_{L^\infty}\|\Lambda^2 b\|_{L^2}.
\end{aligned}$$

Taking $\Lambda^{\frac{3}{2}+\varepsilon}$ to (10), testing by $\Lambda^{\frac{3}{2}+\varepsilon} u$, it yields that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon} u\|_{L^2}^2 \\
=&-\int_{\mathbb{R}^3} \Lambda^{\frac{3}{2}+\varepsilon}(u\cdot\nabla u)\Lambda^{\frac{3}{2}+\varepsilon} u dx + \int_{\mathbb{R}^3} \Lambda^{\frac{3}{2}+\varepsilon}(b\cdot\nabla b)\Lambda^{\frac{3}{2}+\varepsilon} u dx \\
\le& C\|u\|_{L^3}\|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^6}\|\Lambda^{\frac{5}{2}+\varepsilon} u\|_{L^2} + C\|b\|_{L^\infty}\|\Lambda^{\frac{3}{2}+\varepsilon} b\|_{L^2}\|\Lambda^{\frac{5}{2}+\varepsilon} u\|_{L^2} \\
\le& C\|\Lambda^{\frac{1}{2}} u\|_{L^2}\|\Lambda^{\frac{5}{2}+\varepsilon} u\|_{L^2}^2 + C\|b\|_{H^2}\|\nabla b\|_{H^1}\|\Lambda^{\frac{5}{2}+\varepsilon} u\|_{L^2} \\
\le& C\|\Lambda^{\frac{1}{2}} u\|_{L^2}\|\Lambda^{\frac{5}{2}+\varepsilon} u\|_{L^2}^2 + C\|b\|_{H^2}(\|\nabla b\|_{H^1}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon} u\|_{L^2}^2),
\end{aligned} \tag{21}$$

Combining (19)–(21) together gives

$$\frac{d}{dt}\left(\|u\|^2_{H^{\frac{3}{2}+\varepsilon}}+\|b\|^2_{H^2}\right)+\|\nabla u\|^2_{H^{\frac{3}{2}+\varepsilon}}+\|\nabla b\|^2_{H^1}\leq C(\|u\|_{H^{\frac{3}{2}+\varepsilon}}+\|b\|_{H^2})(\|\nabla b\|^2_{H^1}+\|\nabla u\|^2_{H^{\frac{3}{2}+\varepsilon}}).$$

Combining condition (13) with proof by contradiction, the global bound as follows can be obtained:

$$\|u\|^2_{H^{\frac{3}{2}+\varepsilon}}+\|b\|^2_{H^2}+\int_0^t(\|\nabla u\|^2_{H^{\frac{3}{2}+\varepsilon}}+\|\nabla b\|^2_{H^1})d\tau\leq C,\quad \forall t>0. \tag{22}$$

From the local well-posedness result (see [16]), (19) and (22), one easily proves that Theorem 1 holds for $p=0$, i.e.,

$$\|\Lambda^m u\|^2_{L^2}+\|\Lambda^m b\|^2_{L^2}+\|\Lambda^{m+1}b\|^2_{L^2}\leq C(\|u_0\|^2_{H^K}+\|b_0\|^2_{H^K}), \tag{23}$$

provided that $K\geq\max\{m,3\}$.

In the following, the time derivatives of the solution in terms of the space derivatives will be bounded. Let $\mathcal{A}=(I-\Delta)$, then $\mathcal{A}^{-1}=(I-\Delta)^{-1}$. Applying $\partial_t^p\Lambda^m$ to the solution of systems (9)–(12), it yields that

$$\|\partial_t^{p+1}\Lambda^m u\|^2_{L^2}\leq C(\|\partial_t^p\Lambda^{m+2}u\|^2_{L^2}+\|\partial_t^p\Lambda^m(u\cdot\nabla u)\|^2_{L^2}+\|\partial_t^p\Lambda^m(b\cdot\nabla b)\|^2_{L^2}), \tag{24}$$

$$\begin{aligned}&\|\partial_t^{p+1}\Lambda^m(b-\Delta b)\|^2_{L^2}\\ \leq&C(\|\partial_t^p\Lambda^{m+2}b\|^2_{L^2}+\|\partial_t^p\Lambda^m(u\cdot\nabla b)\|^2_{L^2}+\|\partial_t^p\Lambda^m(b\cdot\nabla u)\|^2_{L^2}\\ &+\|\partial_t^p\Lambda^m\nabla\times[b\times(\nabla\times b)]\|^2_{L^2}+\|\partial_t^p\Lambda^m\nabla\times[(u\cdot\nabla)(\nabla\times b)]\|^2_{L^2}).\end{aligned} \tag{25}$$

and

$$\begin{aligned}&\|\partial_t^{p+1}\Lambda^m b\|^2_{L^2}\\ \leq&C(\|\partial_t^p\mathcal{A}^{-1}\Lambda^{m+2}b\|^2_{L^2}+\|\partial_t^p\mathcal{A}^{-1}\Lambda^m(u\cdot\nabla b)\|^2_{L^2}+\|\partial_t^p\mathcal{A}^{-1}\Lambda^m(b\cdot\nabla u)\|^2_{L^2}\\ &+\|\partial_t^p\mathcal{A}^{-1}\Lambda^m\nabla\times[b\times(\nabla\times b)]\|^2_{L^2}+\|\partial_t^p\mathcal{A}^{-1}\Lambda^m\nabla\times[(u\cdot\nabla)(\nabla\times b)]\|^2_{L^2}).\end{aligned} \tag{26}$$

Using Gagliardo–Nirenberg inequality, the second term on the right hand side of (24) can be bounded as

$$\begin{aligned}\|\partial_t^p\Lambda^m(u\cdot\nabla u)\|^2_{L^2}\leq&\sum_{\mathfrak{p}=0}^{p}\sum_{\mathfrak{m}=0}^{m}C_p^{\mathfrak{p}}C_m^{\mathfrak{m}}\|\partial_t^{\mathfrak{p}}\Lambda^{\mathfrak{m}}u\|^2_{L^3}\|\partial_t^{p-\mathfrak{p}}\Lambda^{m-\mathfrak{m}+1}u\|^2_{L^6}\\ \leq&C\sum_{\mathfrak{p}=0}^{p}\sum_{\mathfrak{m}=0}^{m}\|\partial_t^{\mathfrak{p}}\Lambda^{\mathfrak{m}+\frac{1}{2}}u\|^2_{L^2}\|\partial_t^{p-\mathfrak{p}}\Lambda^{m+2-\mathfrak{m}}u\|^2_{L^2}.\end{aligned} \tag{27}$$

Similarly,

$$\|\partial_t^p\Lambda^m(b\cdot\nabla b)\|^2_{L^2}\leq C\sum_{\mathfrak{p}=0}^{P}\sum_{\mathfrak{m}=0}^{M}\|\partial_t^{\mathfrak{p}}\Lambda^{\mathfrak{m}+\frac{1}{2}}b\|^2_{L^2}\|\partial_t^{p-\mathfrak{p}}\Lambda^{m+2-\mathfrak{m}}b\|^2_{L^2}. \tag{28}$$

$$\begin{aligned}&\|\partial_t^p\mathcal{A}^{-1}\Lambda^m(u\cdot\nabla b)\|^2_{L^2}+\|\partial_t^p\mathcal{A}^{-1}\Lambda^m(b\cdot\nabla u)\|^2_{L^2}\\ \leq&C\|\partial_t^p\Lambda^m(u\cdot\nabla b)\|^2_{L^2}+\|\partial_t^p\Lambda^m(b\cdot\nabla u)\|^2_{L^2}\\ \leq&C\sum_{\mathfrak{p}=0}^{p}\sum_{\mathfrak{m}=0}^{m}\left(\|\partial_t^{\mathfrak{p}}\Lambda^{\mathfrak{m}+\frac{1}{2}}u\|^2_{L^2}\|\partial_t^{p-\mathfrak{p}}\Lambda^{m+2-\mathfrak{m}}u\|^2_{L^2}+\|\partial_t^{\mathfrak{p}}\Lambda^{\mathfrak{m}+\frac{1}{2}}b\|^2_{L^2}\|\partial_t^{p-\mathfrak{p}}\Lambda^{m+2-\mathfrak{m}}b\|^2_{L^2}\right).\end{aligned} \tag{29}$$

In addition,

$$
\begin{aligned}
&\|\partial_t^p \mathcal{A}^{-1}\Lambda^m \nabla \times [b \times (\nabla \times b)]\|_{L^2}^2 \\
\leq &C\|\partial_t^p \Lambda^m \nabla \times [b \times (\nabla \times b)]\|_{L^2}^2 \\
\leq &C \sum_{\mathfrak{p}=0}^{p} \sum_{\mathfrak{m}=0}^{m+1} C_p^{\mathfrak{p}} C_{m+1}^{\mathfrak{m}} \|\partial_t^{\mathfrak{p}} \Lambda^{\mathfrak{m}} b\|_{L^\infty}^2 \|\partial_t^{p-\mathfrak{p}} \Lambda^{m+2-\mathfrak{m}} b\|_{L^2}^2 \\
\leq &C \sum_{\mathfrak{p}=0}^{p} \sum_{\mathfrak{m}=0}^{m+1} \|\partial_t^{\mathfrak{p}} \Lambda^{\mathfrak{m}+1} b\|_{L^2} \|\partial_t^{\mathfrak{p}} \Lambda^{\mathfrak{m}+2} b\|_{L^2} \|\partial_t^{p-\mathfrak{p}} \Lambda^{m+2-\mathfrak{m}} b\|_{L^2}^2,
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
&\|\partial_t^p \mathcal{A}^{-1}\Lambda^m \nabla \times [(u \cdot \nabla)(\nabla \times b)]\|_{L^2}^2 \\
\leq &C\|\partial_t^p \Lambda^m \nabla \times [(u \cdot \nabla)(\nabla \times b)]\|_{L^2}^2 \\
\leq &C \sum_{\mathfrak{p}=0}^{p} \sum_{\mathfrak{m}=0}^{m+1} C_p^{\mathfrak{p}} C_{m+1}^{\mathfrak{m}} \|\partial_t^{\mathfrak{p}} \Lambda^{\mathfrak{m}} u\|_{L^\infty}^2 \|\partial_t^{p-\mathfrak{p}} \Lambda^{m+3-\mathfrak{m}} b\|_{L^2}^2 \\
\leq &C \sum_{\mathfrak{p}=0}^{p} \sum_{\mathfrak{m}=0}^{m+1} \|\partial_t^{\mathfrak{p}} \Lambda^{\mathfrak{m}+1} u\|_{L^2} \|\partial_t^{\mathfrak{p}} \Lambda^{\mathfrak{m}+2} u\|_{L^2} \|\partial_t^{p-\mathfrak{p}} \Lambda^{m+3-\mathfrak{m}} b\|_{L^2}^2.
\end{aligned}
\tag{31}
$$

Moreover,

$$
\begin{aligned}
\|\partial_t^{p+1} \Lambda^{m+1} b\|_{L^2}^2 \leq &C(\|\partial_t^{p+1} \Lambda^m b\|_{L^2}^2 + \|\partial_t^{p+1} \Lambda^{m+2} b\|_{L^2}^2) \\
\leq &C(\|\partial_t^{p+1} \Lambda^m b\|_{L^2}^2 + \|\partial_t^{p+1} \Lambda^m (b - \Delta b)\|_{L^2}^2).
\end{aligned}
\tag{32}
$$

Putting (24)–(32) together gives

$$
\begin{aligned}
&\|\partial_t^{p+1} \Lambda^m u\|_{L^2}^2 + \|\partial_t^{p+1} \Lambda^m b\|_{L^2}^2 + \|\partial_t^{p+1} \Lambda^{m+1} bu\|_{L^2}^2 \\
\leq &C(\|\partial_t^p u\|_{H^{m+2}}^2 + \|\partial_t^p b\|_{H^{m+2}}^2 + \|\partial_t^p \Lambda b\|_{H^{m+2}}^2),
\end{aligned}
\tag{33}
$$

which means, for all $m, p, K \in \mathbb{N}$ such that $K \geq \max\{m, 3\} + 2p$,

$$
\|\partial_t^p \Lambda^m u\|_{L^2}^2 + \|\partial_t^p \Lambda^m b\|_{L^2}^2 + \|\partial_t^p \Lambda^{m+1} b\|_{L^2}^2 \leq C(\|u_0\|_{H^K}^2 + \|b_0\|_{H^K}^2 + \|\Lambda b_0\|_{H^K}^2).
$$

This complete the proof.

4. Proof of Theorem 2

4.1. Auxiliary Lemmas

In [29], the author established the following result:

Lemma 10 (see [29]). *Let $(u_0, b_0) \in L^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ with $divu_0 = divb_0 = 0$. Suppose that (u, b) is the solution of systems (9)–(12) with initial value (u_0, b_0). Then*

$$
|\widehat{u}(\xi, t)|^2 \leq C \left[e^{-2|\xi|^2 t} |\widehat{u_0}(\xi)|^2 + |\xi|^2 \left(\int_0^t (\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) ds \right)^2 \right],
\tag{34}
$$

and

$$
|\widehat{b}(\xi, t)|^2 \leq C \left[e^{-\frac{2|\xi|^2 t}{1+|\xi|^2}} |\widehat{b_0}(\xi)|^2 + (|\xi|^2 + |\xi|^4 + |\xi|^6) \left(\int_0^t (\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) ds \right)^2 \right].
\tag{35}
$$

In order to characterize the decay estimates of systems (9)–(12), the following lemma is introduced.

Lemma 11. *Suppose that the assumptions listed in Lemma 10 are satisfied. Then, for $p > 0$,*

$$\begin{aligned}|\partial_t^p \widehat{u}(\xi,t)|^2 \leq & C|\xi|^{4p}e^{-2|\xi|^2 t}|\hat{u}_0(\xi,t)|^2 + C|\xi|^{4p+2}\left[\int_0^t (\|u\|_{L^2}^2 + \|b\|_{L^2}^2)ds\right]^2 \\ & + C\sum_{\mathfrak{p}=0}^{p-1}\sum_{l=0}^{\mathfrak{p}} |\xi|^{4p-4\mathfrak{p}-2}(\|\partial_t^l u\|_{L^2}^2 \|\partial_t^{\mathfrak{p}-l} u\|_{L^2}^2,\end{aligned} \tag{36}$$

and

$$\begin{aligned}|\partial_t^p \widehat{b}(\xi,t)|^2 \leq & C|\xi|^{4p}e^{-2\frac{|\xi|^2}{1+|\xi|^2}t}|\hat{b}_0(\xi,t)|^2 + C|\xi|^{4p+2}\left[\int_0^t (\|u\|_{L^2}^2 + \|b\|_{L^2}^2)ds\right]^2 \\ & + C\sum_{\mathfrak{p}=0}^{p-1}\sum_{l=0}^{\mathfrak{p}} |\xi|^{4p-4\mathfrak{p}-2}(1+|\xi|^2+|\xi|^4) \\ & \times (\|\partial_t^l u\|_{L^2}^2 \|\partial_t^{\mathfrak{p}-l} b\|_{L^2}^2 + \|\partial_t^l b\|_{L^2}^2 \|\partial_t^{\mathfrak{p}-l} u\|_{L^2}^2 + \|\partial_t^l b\|_{L^2}^2 \|\partial_t^{\mathfrak{p}-l} b\|_{L^2}^2).\end{aligned} \tag{37}$$

Proof. Note that

$$\begin{aligned}\partial_t^p \hat{u}(\xi,t) = & (-1)^p|\xi|^{2p}e^{-|\xi|^2 t}\hat{u}_0(\xi) + \sum_{\mathfrak{p}=0}^{p-1}(-1)^{p-\mathfrak{p}-1}|\xi|^{2p-2\mathfrak{p}-2}\partial_t^{\mathfrak{p}}(\widehat{u\cdot\nabla u} - \widehat{b\cdot\nabla b})(\xi,t) \\ & + \int_0^t (-1)^p|\xi|^{2p}e^{-|\xi|^2(t-s)}(\widehat{u\cdot\nabla u} - \widehat{b\cdot\nabla b})(\xi,s)ds.\end{aligned} \tag{38}$$

In addition,

$$\begin{aligned}|\partial_t^{\mathfrak{p}}(\widehat{u\cdot\nabla u} - \widehat{b\cdot\nabla b})(\xi,t)| \leq & |\partial_t^{\mathfrak{p}}\sum_j \xi_j \widehat{u_j u}(\xi,t)| + |\partial_t^{\mathfrak{p}}\sum_j \xi_j \widehat{b_j b}(\xi,t)| \\ \leq & C\sum_{l=0}^{\mathfrak{p}} |\xi|(\|\partial_t^l u\|_{L^2}\|\partial_t^{\mathfrak{p}-l} u\|_{L^2} + \|\partial_t^l b\|_{L^2}\|\partial_t^{\mathfrak{p}-l} b\|_{L^2}).\end{aligned} \tag{39}$$

Adding (38) and (39) together, by using (34), it yields (36). On the other hand, the following equality holds:

$$\begin{aligned}\partial_t^p \hat{b}(\xi,t) = & (-1)^p|\xi|^{2p}e^{-\frac{|\xi|^2}{1+|\xi|^2}t}\hat{b}_0(\xi) + \sum_{\mathfrak{p}=0}^{p-1}(-1)^{p-\mathfrak{p}-1}|\xi|^{2(p-\mathfrak{p}-1)}\Big[\widehat{u\cdot\nabla b} - \widehat{b\cdot\nabla u} \\ & + \nabla\times[\widehat{(\nabla\times b)\times b}] + \nabla\times[\widehat{u\cdot\nabla(\nabla\times b)}]\Big](\xi,t) \\ & + \int_0^t (-1)^p|\xi|^{2p}e^{-\frac{|\xi|^2}{1+|\xi|^2}(t-s)}\Big[\widehat{u\cdot\nabla b} - \widehat{b\cdot\nabla u} + \nabla\times[\widehat{(\nabla\times b)\times b}] \\ & + \nabla\times[\widehat{u\cdot\nabla(\nabla\times b)}]\Big](\xi,s)ds.\end{aligned} \tag{40}$$

Moreover,

$$|\partial_t^{\mathfrak{p}}(\widehat{u\cdot\nabla b} - \widehat{b\cdot\nabla u})(\xi,t)| \leq C\sum_{l=0}^{\mathfrak{p}} |\xi|(\|\partial_t^l u\|_{L^2}\|\partial_t^{\mathfrak{p}-l} b\|_{L^2} + \|\partial_t^l b\|_{L^2}\|\partial_t^{\mathfrak{p}-l} u\|_{L^2}), \tag{41}$$

$$|\partial_t^{\mathfrak{p}}\nabla\times(\widehat{(\nabla\times b)\times b})(\xi,t)| \leq C\sum_{l=0}^{\mathfrak{p}} |\xi|^2\|\partial_t^l b\|_{L^2}\|\partial_t^{\mathfrak{p}-l} b\|_{L^2}, \tag{42}$$

and

$$|\partial_t^{\mathfrak{p}}\nabla\times(\widehat{u\cdot\nabla(\nabla\times b)})(\xi,t)| \leq C\sum_{l=0}^{\mathfrak{p}} |\xi|^3(\|\partial_t^l u\|_{L^2}\|\partial_t^{\mathfrak{p}-l} b\|_{L^2} + \|\partial_t^l b\|_{L^2}\|\partial_t^{\mathfrak{p}-l} u\|_{L^2}), \tag{43}$$

Combining (40)–(43) together, applying (35), the estimate (38) is obtained, and the proof is completed.

□

4.2. Proof of Theorem 2

Theorem 2 is proven using the mathematical induction in this subsection.

First of all, the fact that Theorem 2 holds for the case $p=1$ is proven:

Lemma 12. *Let $m \in \mathbb{N}$, $K \geq \max\{5, m+2\}$ and $r^* = r^*(u_0) = r^*(b_0) \in (-\frac{3}{2}, +\infty)$ be the decay character. Suppose that $(u_0, b_0) \in H^K(\mathbb{R}^3) \times H^{K+1}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then, there exists a positive constant $C = C(\|u_0\|_{H^K}, \|b_0\|_{H^{K+1}})$, such that*

$$\|\partial_t \Lambda^m u\|_{L^2}^2 + \|\partial_t \Lambda^m b\|_{L^2}^2 + \|\partial_t \Lambda^{m+1} b\|_{L^2}^2 \leq C(1+t)^{-\min\{\frac{7}{2}+r^*+m, \frac{9}{2}+m\}}, \ \text{for large } t. \tag{44}$$

Proof. In order to prove Lemma 12, one first proves the case $m=0$. Applying ∂_t to (10) and (11), multiplying both side by $\partial_t u$ and $\partial_t b$ respectively, integrating over $\mathbb{R}^3$, gives that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial_t u\|_{L^2}^2 + \|\partial_t b\|_{L^2}^2 + \|\partial_t \Lambda b\|_{L^2}^2) + \|\partial_t \Lambda u\|_{L^2}^2 + \|\partial_t \Lambda b\|_{L^2}^2 \\
\leq& |(u \cdot \partial_t \Lambda u, \partial_t u)| + |(b \cdot \partial_t \Lambda u, \partial_t b)| + |(u \cdot \partial_t \Lambda b, \partial_t b)| + |(b \cdot \partial_t \Lambda b, \partial_t u)| + |(b \cdot \partial_t \Lambda b, \partial_t \Lambda b)| \\
&+ |(\partial_t u \cdot \nabla(\nabla \times b), \nabla \times b)| + |(\partial_t u \cdot \partial_t \Lambda u, u)| + |(\partial_t b \cdot \partial_t \Lambda u, b)| + |(\partial_t u \cdot \partial_t \Lambda b, b)| \\
&+ |(\partial_t b \cdot \partial_t \Lambda b, u)| + |(\partial_t b \cdot \partial_t \Lambda b, \Lambda b)| + |(u \cdot \nabla(\nabla \times b_t), \nabla \times b_t)| \\
\leq& \|u\|_{L^3}\|\partial_t \Lambda u\|_{L^2}\|\partial_t u\|_{L^6} + \|b\|_{L^3}\|\partial_t \Lambda u\|_{L^2}\|\partial_t b\|_{L^6} + \|u\|_{L^3}\|\partial_t \Lambda b\|_{L^2}\|\partial_t b\|_{L^6} \\
&+ \|b\|_{L^3}\|\partial_t \Lambda b\|_{L^2}\|\partial_t u\|_{L^6} + \|b\|_{L^\infty}\|\partial_t \Lambda b\|_{L^2}^2 + \|\Lambda^2 b\|_{L^3}\|\partial_t \Lambda b\|_{L^2}\|\partial_t u\|_{L^6} \\
&+ \|u\|_{L^3}\|\partial_t u\|_{L^6}\|\partial_t \Lambda u\|_{L^2} + \|b\|_{L^3}\|\partial_t b\|_{L^6}\|\partial_t \Lambda u\|_{L^2} + \|u\|_{L^3}\|\partial_t b\|_{L^6}\|\partial_t \Lambda b\|_{L^2} \\
&+ \|b\|_{L^3}\|\partial_t u\|_{L^6}\|\partial_t \Lambda b\|_{L^2} + \|\Lambda b\|_{L^3}\|\partial_t b\|_{L^6}\|\partial_t \Lambda b\|_{L^2} + \|\Lambda u\|_{L^\infty}\|\partial_t \Lambda b\|_{L^2}^2 \\
\leq& (\|\Lambda^{\frac{1}{2}} u\|_{L^2} + \|\Lambda^{\frac{1}{2}} b\|_{L^2} + \|\Lambda^{\frac{1}{2}} u\|_{L^2} + \|\Lambda b\|_{L^2}^{\frac{1}{2}}\|\Lambda^2 b\|_{L^2}^{\frac{1}{2}} + \|\Lambda^{\frac{5}{2}} b\|_{L^2} + \|\Lambda^{\frac{3}{2}} b\|_{L^2} \\
&+ \|u\|_{H^K})(\|\partial_t \Lambda u\|_{L^2}^2 + \|\partial_t \Lambda b\|_{L^2}^2) \\
\leq& \frac{1}{2}(\|\partial_t \Lambda u\|_{L^2}^2 + \|\partial_t \Lambda b\|_{L^2}^2), \quad \text{for large } t,
\end{aligned}$$

where

$$(u \cdot \nabla(\nabla \times b_t), \nabla \times b_t) = 0.$$

has been used. Then,

$$\frac{d}{dt}(\|\partial_t u\|_{L^2}^2 + \|\partial_t b\|_{L^2}^2 + \|\partial_t \Lambda b\|_{L^2}^2) + \|\partial_t \Lambda u\|_{L^2}^2 + \|\partial_t \Lambda b\|_{L^2}^2 \leq 0, \quad \text{for large } t. \tag{45}$$

Applying Plancherel's theorem to (45) gives

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\left[|\widehat{\partial_t u}(\xi,t)|^2 + (1+|\xi|^2)|\widehat{\partial_t b}(\xi,t)|^2\right] d\xi \\
&+ \int_{\mathbb{R}^3}\frac{|\xi|^2}{1+|\xi|^2}\left[|\widehat{\partial_t u}(\xi,t)|^2 + (1+|\xi|^2)|\widehat{\partial_t b}(\xi,t)|^2\right] d\xi \leq 0, \quad \text{for large } t.
\end{aligned} \tag{46}$$

Set

$$B(t) := \left\{\xi \in \mathbb{R}^3 \big| |\xi|^2 \leq \frac{g'(t)}{g(t)-g'(t)}\right\}, \quad B^c(t) := \mathbb{R}^3 \setminus B(t),$$

where $g(t)$ is a differentiable function of t satisfying

$$g(0) = 1, \ \ g'(t) > 0 \ \text{ and } 2g(t) > g'(t), \ \ \forall t > 0.$$

Multiplying (46) by $g(t)$ gives

$$\frac{d}{dt}\left\{g(t)\int_{\mathbb{R}^3}\left[|\widehat{\partial_t u}(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t b}(\xi,t)|^2\right]d\xi\right\}$$
$$\leq g'(t)\int_{B(t)}\left[|\widehat{\partial_t u}(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t b}(\xi,t)|^2\right]d\xi, \quad \text{for large } t.$$

It then follows from Lemma 11 that

$$\begin{aligned}
&g(t)\int_{\mathbb{R}^3}\left[|\widehat{\partial_t u}(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t b}(\xi,t)|^2\right]d\xi\\
\leq& C+\int_0^t g'(s)\int_{B(s)}|\xi|^4\left[e^{-2|\xi|^2 t}|\hat{u}_0(\xi)|^2+(1+|\xi|^2)e^{-2\frac{|\xi|^2}{1+|\xi|^2}t}|\hat{b}_0(\xi)|^2\right]d\xi ds\\
&+C\int_0^t g'(s)\int_{B(s)}|\xi|^2\left(\|u(s)\|_{L^2}^4+\|b(s)\|_{L^2}^4+\|u(s)\|_{L^2}^2\|b(s)\|_{L^2}^2\right)d\xi ds\\
&+C\int_0^t g'(s)\int_{B(s)}|\xi|^6(1+|\xi|^2+|\xi|^4)\left[\int_0^t(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau\right]^2 d\xi ds, \text{ for large } t.
\end{aligned} \tag{47}$$

The right hand side of (47) is estimated in the following. For the first term, by using the estimates from Lemma 6, it yields that

$$\begin{aligned}
&\int_0^t g'(s)\int_{B(s)}|\xi|^4\left[e^{-2|\xi|^2 t}|\hat{u}_0(\xi)|^2+(1+|\xi|^2)e^{-2\frac{|\xi|^2}{1+|\xi|^2}t}|\hat{b}_0(\xi)|^2\right]d\xi ds\\
\leq& C\int_0^t g'(s)(\|\partial_t\bar{u}(s)\|_{L^2}^2+\|\partial_t\bar{b}(s)\|_{L^2}^2+\|\partial_t\Lambda\bar{b}(s)\|_{L^2}^2)ds\\
\leq& C\int_0^t g'(s)(1+s)^{-\min\{r^*+\frac{7}{2},\frac{9}{2}\}}ds,
\end{aligned} \tag{48}$$

where $(\bar{u},\bar{b})$ is the solution to the linear system (18). For the second term, after integrating in polar coordinates in $B(t)$, one can deduce that

$$\begin{aligned}
&C\int_0^t g'(s)\int_{B(s)}|\xi|^2\left(\|u(s)\|_{L^2}^4+\|b(s)\|_{L^2}^4+\|u(s)\|_{L^2}^2\|b(s)\|_{L^2}^2\right)d\xi ds\\
\leq& C\int_0^t g'(s)(1+s)^{-\frac{5}{2}}(1+s)^{-2\min\{r^*+\frac{3}{2},\frac{5}{2}\}}ds\\
\leq& C\int_0^t g'(s)(1+s)^{-\min\{2r^*+\frac{11}{2},\frac{15}{2}\}}ds, \quad \text{for large } t.
\end{aligned} \tag{49}$$

In addition, if $r^*+\frac{3}{2}<\frac{5}{2}$, the following estimate holds:

$$\left[\int_0^t(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau\right]^2\leq C(1+t)^{-2r^*-1}, \quad \text{for large } t,$$

then the third term of the right hand side of (47) can be estimated as

$$\begin{aligned}
&C\int_0^t g'(s)\int_{B(s)}|\xi|^6(1+|\xi|^2+|\xi|^4)\left[\int_0^t(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau\right]^2 d\xi ds\\
\leq& C\int_0^t g'(s)(1+s)^{-\frac{9}{2}}(1+s)^{-2r^*-1}ds\leq C\int_0^t g'(s)(1+s)^{-2r^*-\frac{11}{2}}ds, \quad \text{for large } t.
\end{aligned} \tag{50}$$

If $r^*+\frac{3}{2}\geq\frac{5}{2}$,

$$\left[\int_0^t(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau\right]^2\leq C, \quad \text{for large } t,$$

the third term of the right hand side of (47) satisfies

$$
\begin{aligned}
&C\int_0^t g'(s)\int_{B(s)}|\xi|^6(1+|\xi|^2+|\xi|^4)\left[\int_0^t(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau\right]^2 d\xi ds\\
&\leq C\int_0^t g'(s)(1+s)^{-\frac{9}{2}}ds, \quad \text{for large } t.
\end{aligned}
\tag{51}
$$

For a fixed r^*, choose $g(t)=(1+t)^m$, for some $m>\max\{r^*+\frac{7}{2},\frac{9}{2}\}$. Then $\rho(t)=C(1+t)^{-\frac{1}{2}}$. It follows from (47)–(51) that

$$
\begin{aligned}
&\|\partial_t u\|_{L^2}^2+\|\partial_t b\|_{L^2}^2+\|\partial_t\Lambda b\|_{L^2}^2\\
\leq& C(1+t)^{-m}+C(1+t)^{-\min\{r^*+\frac{7}{2},\frac{9}{2}\}}+C(1+t)^{-\min\{2r^*+\frac{11}{2},\frac{15}{2}\}}\\
&+C(1+t)^{-\min\{2r^*+\frac{11}{2},\frac{9}{2}\}}\\
\leq& C(1+t)^{-\min\{r^*+\frac{7}{2},\frac{9}{2}\}}, \quad \text{for large } t.
\end{aligned}
\tag{52}
$$

Suppose that Lemma 12 holds for $m\leq N\in\mathbb{N}^+$, then one can prove that it also holds for $m=N+1$. Applying $\partial_t\Lambda^{N+1}$ to (10) and (11), multiplying both side by $\partial_t\Lambda^{N+1}u$ and $\partial_t\Lambda^{N+1}b$ respectively, integrating over $\mathbb{R}^3$, gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\partial_t\Lambda^{N+1}u\|_{L^2}^2+\|\partial_t\Lambda^{N+1}b\|_{L^2}^2+\|\partial_t\Lambda^{N+2}b\|_{L^2}^2\right)\\
&+\|\partial_t\Lambda^{N+2}u\|_{L^2}^2+\|\partial_t\Lambda^{N+2}b\|_{L^2}^2\\
=&\sum_{p=0}^{1}\sum_{m=0}^{N+1}C_1^pC_{N+1}^m\Big[-(\partial_t^p\Lambda^m u\cdot\partial_t\Lambda^{N+2}u,\partial_t^{1-p}\Lambda^{N+1-m}u)\\
&+(\partial_t^p\Lambda^m b\cdot\partial_t\Lambda^{N+2}u,\partial_t^{1-p}\Lambda^{N+1-m}b)-(\partial_t^p\Lambda^m u\cdot\partial_t\Lambda^{N+2}b,\partial_t^{1-p}\Lambda^{N+1-m}b)\\
&+(\partial_t^p\Lambda^m b\cdot\partial_t\Lambda^{N+2}b,\partial_t^{1-p}\Lambda^{N+1-m}u)+(\partial_t^p\Lambda^m b\cdot\partial_t\Lambda^{N+2}b,\partial_t^{1-p}\Lambda^{N+2-m}b)\\
&+(\partial_t^p\Lambda^m u\cdot\partial_t\Lambda^{N+2}b,\partial_t^{1-p}\Lambda^{N+3-m}b)\Big]\\
=&I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}
$$

Note that

$$
\begin{aligned}
I_1\leq&\sum_{m=0}^{N+1}C_{N+1}^m\Big[|(\Lambda^m u\cdot\partial_t\Lambda^{N+2}u,\partial_t\Lambda^{N+1-m}u)|+|(\partial_t\Lambda^m u\cdot\partial_t\Lambda^{N+2}u,\Lambda^{N+1-m}u)|\Big]\\
\leq& C\|u\|_{L^3}\|\partial_t\Lambda^{N+2}u\|_{L^2}\|\partial_t\Lambda^{N+1}u\|_{L^6}+C\|\Lambda u\|_{L^3}\|\partial_t\Lambda^{N+2}u\|_{L^2}\|\partial_t\Lambda^{N}u\|_{L^6}\\
&+C\sum_{m=2}^{N+1}\|\Lambda^m u\|_{L^3}\|\partial_t\Lambda^{N+2}u\|_{L^2}\|\partial_t\Lambda^{N+1-m}u\|_{L^6}\\
&+C\sum_{m=0}^{N-1}\|\partial_t\Lambda^m u\|_{L^6}\|\partial_t\Lambda^{N+2}u\|_{L^2}\|\Lambda^{N+1-m}u\|_{L^3}\\
&+C\|\partial_t\Lambda^N u\|_{L^6}\|\partial_t\Lambda^{N+2}u\|_{L^2}\|\Lambda u\|_{L^3}+C\|\partial_t\Lambda^{N+1}u\|_{L^6}\|\partial_t\Lambda^{N+2}u\|_{L^2}\|u\|_{L^3}\\
\leq& C\|\Lambda^{\frac{1}{2}}u\|_{L^2}\|\partial_t\Lambda^{N+2}u\|_{L^2}^2+C\sum_{m=2}^{N+1}\|\Lambda^{m+\frac{1}{2}}u\|_{L^2}\|\partial_t\Lambda^{N+2}u\|_{L^2}\|\partial_t\Lambda^{N+2-m}u\|_{L^2}\\
&+C\|\Lambda^{\frac{3}{2}}u\|_{L^2}\|\partial_t\Lambda^{N+2}u\|_{L^2}^{\frac{3}{2}}\|\partial_t\Lambda^N u\|_{L^2}^{\frac{1}{2}}+C\sum_{m=0}^{N-1}\|\partial_t\Lambda^{m+1}u\|_{L^2}\|\partial_t\Lambda^{N+2}u\|_{L^2}\|\Lambda^{N+\frac{3}{2}-m}u\|_{L^2}\\
\leq&\frac{1}{8}\|\partial_t\Lambda^{N+2}u\|_{L^2}^2+C\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2\|\partial_t\Lambda^N u\|_{L^2}^2+C\sum_{m=2}^{N+1}\|\Lambda^{m+\frac{1}{2}}u\|_{L^2}^2\|\partial_t\Lambda^{N+2-m}u\|_{L^2}^2\\
&+C\sum_{m=0}^{N-1}\|\partial_t\Lambda^{m+1}u\|_{L^2}^2\|\Lambda^{N+\frac{3}{2}-m}u\|_{L^2}^2, \quad \text{for large } t.
\end{aligned}
$$

Similarly,

$$I_2 \leq \frac{1}{8}\|\partial_t\Lambda^{N+2}u\|_{L^2}^2 + C\|\Lambda^{\frac{3}{2}}b\|_{L^2}^2\|\partial_t\Lambda^N b\|_{L^2}^2 + C\sum_{m=2}^{N+1}\|\Lambda^{m+\frac{1}{2}}b\|_{L^2}^2\|\partial_t\Lambda^{N+2-m}b\|_{L^2}^2$$
$$+ C\sum_{m=0}^{N-1}\|\partial_t\Lambda^{m+1}b\|_{L^2}^2\|\Lambda^{N+\frac{3}{2}-m}b\|_{L^2}^2, \quad \text{for large } t,$$

and

$$\begin{aligned}
&I_3 + I_4\\
\leq&\frac{1}{8}\|\partial_t\Lambda^{N+2}b\|_{L^2}^2 + C(\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}}b\|_{L^2}^2)(\|\partial_t\Lambda^N u\|_{L^2}^2 + \|\partial_t\Lambda^N b\|_{L^2}^2)\\
&+ C\sum_{m=2}^{N+1}(\|\Lambda^{m+\frac{1}{2}}u\|_{L^2}^2 + \|\Lambda^{m+\frac{1}{2}}b\|_{L^2}^2)(\|\partial_t\Lambda^{N+2-m}u\|_{L^2}^2 + \|\partial_t\Lambda^{N+2-m}b\|_{L^2}^2)\\
&+ C\sum_{m=0}^{N-1}(\|\partial_t\Lambda^{m+1}u\|_{L^2}^2 + \|\partial_t\Lambda^{m+1}b\|_{L^2}^2)(\|\Lambda^{N+\frac{3}{2}-m}u\|_{L^2}^2 + \|\Lambda^{N+\frac{3}{2}-m}b\|_{L^2}^2), \quad \text{for large } t,
\end{aligned}$$

The following estimate also holds:

$$\begin{aligned}
I_5 \leq& \sum_{m=0}^{N+1} C_{N+1}^m\left[|(\Lambda^m b\cdot\partial_t\Lambda^{N+2}b, \partial_t\Lambda^{N+2-m}b)| + |(\partial_t\Lambda^m b\cdot\partial_t\Lambda^{N+2}b, \Lambda^{N+2-m}b)|\right]\\
\leq&\|b\|_{L^\infty}\|\partial_t\Lambda^{N+2}b\|_{L^2}^2 + \|\Lambda b\|_{L^3}\|\partial_t\Lambda^{N+2}b\|_{L^2}\|\partial_t\Lambda^{N+1}b\|_{L^6} + \|\Lambda^2 b\|_{L^2}\|\partial_t\Lambda^{N+2}b\|_{L^2}\|\partial_t\Lambda^N b\|_{L^6}\\
&+ \sum_{m=3}^{N+1}\|\Lambda^m b\|_{L^3}\|\partial_t\Lambda^{N+2}b\|_{L^2}\|\partial_t\Lambda^{N+2-m}b\|_{L^6} + \sum_{m=0}^{N-1}\|\partial_t\Lambda^m b\|_{L^6}\|\partial_t\Lambda^{N+2}b\|_{L^2}\|\Lambda^{N+2-m}b\|_{L^3}\\
&+ \|\partial_t\Lambda^N b\|_{L^6}\|\partial_t\Lambda^{N+2}b\|_{L^2}\|\Lambda^2 b\|_{L^3} + \|\partial_t\Lambda^{N+1}b\|_{L^6}\|\partial_t\Lambda^{N+2}b\|_{L^2}\|\Lambda b\|_{L^3}\\
\leq&\frac{1}{16}\|\partial_t\Lambda^{N+2}b\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}}b\|_{L^2}^2\|\partial_t\Lambda^N b\|_{L^2}^2 + \sum_{m=3}^{N+1}\|\Lambda^{m+\frac{1}{2}}b\|_{L^2}^2\|\partial_t\Lambda^{N+3-m}b\|_{L^2}^2\\
&+ \sum_{m=0}^{N-1}\|\partial_t\Lambda^{m+1}b\|_{L^2}^2\|\Lambda^{N+\frac{5}{2}-m}b\|_{L^2}^2, \quad \text{for large } t,
\end{aligned}$$

Moreover, I_6 satisfies

$$\begin{aligned}
I_6 =& \sum_{m=0}^{N+1} C_{N+1}^m\left[|(\Lambda^m u\cdot\partial_t\Lambda^{N+2}b, \partial_t\Lambda^{N+3-m}b)| + |(\partial_t\Lambda^m u\cdot\partial_t\Lambda^{N+2}b, \Lambda^{N+3-m}b)|\right]\\
\leq&C[(u\cdot\nabla\Lambda^{N+2}b_t, \Lambda^{N+2}b_t) + \|\Lambda u\|_{L^\infty}\|\partial_t\Lambda^{N+2}b\|_{L^2}^2 + \|\Lambda^2 u\|_{L^3}\|\partial_t\Lambda^{N+2}b\|_{L^2}\|\partial_t\Lambda^{N+1}b\|_{L^6}\\
&+ \|\Lambda^3 u\|_{L^2}\|\partial_t\Lambda^{N+2}b\|_{L^2}\|\partial_t\Lambda^N b\|_{L^\infty} + \sum_{m=4}^{N+1}\|\Lambda^m u\|_{L^3}\|\partial_t\Lambda^{N+2}b\|_{L^2}\|\partial_t\Lambda^{N+3-m}b\|_{L^6}\\
&+ \|\partial_t\Lambda^{N+1}u\|_{L^6}\|\partial_t\Lambda^{N+2}b\|_{L^2}\|\Lambda^2 b\|_{L^3} + \|\partial_t\Lambda^N u\|_{L^6}\|\partial_t\Lambda^{N+2}b\|_{L^2}\|\Lambda^3 b\|_{L^3}\\
&+ \sum_{m=0}^{N-1}\|\partial_t\Lambda^m u\|_{L^6}\|\partial_t\Lambda^{N+2}b\|_{L^2}\|\Lambda^{N+2-m}b\|_{L^3}]\\
\leq&\frac{1}{16}\|\partial_t\Lambda^{N+2}b\|_{L^2}^2 + \|\Lambda^{\frac{7}{2}}b\|_{L^2}^2\|\partial_t\Lambda^N b\|_{L^2}^2 + \|\Lambda^3 u\|_{L^2}^2\|\partial_t\Lambda^{N+1}b\|_{L^2}^2\\
&+ \sum_{m=4}^{N+1}\|\Lambda^{m+\frac{1}{2}}u\|_{L^2}^2\|\partial_t\Lambda^{N+4-m}b\|_{L^2}^2 + \sum_{m=0}^{N-1}\|\partial_t\Lambda^{m+1}u\|_{L^2}^2\|\Lambda^{N+\frac{5}{2}-m}b\|_{L^2}^2, \quad \text{for large } t,
\end{aligned}$$

where

$$\|\partial_t\Lambda^{N+1}v\|_{L^2}^2 \leq \varepsilon\|\partial_t\Lambda^{N+2}v\|_{L^2}^2 + C_\varepsilon\|\partial_t\Lambda^N v\|_{L^2}^2.$$

has been used. Note that $\|v\|_{L^\infty} \le C\|\Lambda v\|_{L^2}^{\frac{1}{2}}\|\Lambda^2 v\|_{L^2}^{\frac{1}{2}}$. By using the previous decay results, it yields that

$$\begin{aligned}&\frac{d}{dt}(\|\partial_t\Lambda^{N+1}u\|_{L^2}^2+\|\partial_t\Lambda^{N+1}b\|_{L^2}^2+\|\partial_t\Lambda^{N+2}b\|_{L^2}^2)+\|\partial_t\Lambda^{N+2}u\|_{L^2}^2+\|\partial_t\Lambda^{N+2}b\|_{L^2}^2\\&\le C(1+t)^{-\min\{2r^*+\frac{15}{2}+N,\frac{19}{2}+N\}},\quad \text{for large } t.\end{aligned}\tag{53}$$

Applying the Plancherel's theorem to (53) gives

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3}\left[|\widehat{\partial_t\Lambda^{N+1}}u(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t\Lambda^{N+1}}b(\xi,t)|^2\right]d\xi\\&+\int_{\mathbb{R}^3}\frac{|\xi|^2}{1+|\xi|^2}\left[|\widehat{\partial_t\Lambda^{N+1}}u(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t\Lambda^{N+1}}b(\xi,t)|^2\right]d\xi\\&\le C(1+t)^{-\min\{2r^*+\frac{15}{2}+N,\frac{19}{2}+N\}},\quad \text{for large } t.\end{aligned}\tag{54}$$

Multiplying (54) by $g(t)$, it yields

$$\begin{aligned}&\frac{d}{dt}\left\{g(t)\int_{\mathbb{R}^3}\left[|\widehat{\partial_t\Lambda^{N+1}}u(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t\Lambda^{N+1}}b(\xi,t)|^2\right]d\xi\right\}\\&\le g'(t)\int_{B(t)}\left[|\widehat{\partial_t\Lambda^{N+1}}u(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t\Lambda^{N+1}}b(\xi,t)|^2\right]d\xi\\&\quad+Cg(t)(1+t)^{-\min\{2r^*+\frac{15}{2}+N,\frac{19}{2}+N\}},\quad \text{for large } t.\end{aligned}$$

Hence

$$\begin{aligned}&g(t)\int_{\mathbb{R}^3}\left[|\widehat{\partial_t\Lambda^{N+1}}u(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t\Lambda^{N+1}}b(\xi,t)|^2\right]d\xi\\&\le C+\int_0^t g'(s)\int_{B(s)}\left[|\widehat{\partial_t\Lambda^{N+1}}u(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t\Lambda^{N+1}}b(\xi,t)|^2\right]d\xi ds\\&\quad+C\int_0^t g(s)(1+s)^{-\min\{2r^*+\frac{15}{2}+N,\frac{19}{2}+N\}}ds\\&\le C+\int_0^t g'(s)\int_{B(s)}|\xi|^{2N+2}\left[|\widehat{\partial_t u}(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t b}(\xi,t)|^2\right]d\xi ds\\&\quad+C\int_0^t g(s)(1+s)^{-\min\{2r^*+\frac{15}{2}+N,\frac{19}{2}+N\}}ds,\quad \text{for large } t.\end{aligned}\tag{55}$$

It then follows from (36), (38) and (55) that

$$\begin{aligned}&g(t)\int_{\mathbb{R}^3}\left[|\widehat{\partial_t\Lambda^{N+1}}u(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t\Lambda^{N+1}}b(\xi,t)|^2\right]d\xi\\&\le C+\int_0^t g'(s)\int_{B(s)}|\xi|^{2N+6}\left[e^{-2|\xi|^2t}|\hat{u}_0(\xi)|^2+(1+|\xi|^2)e^{-2\frac{|\xi|^2}{1+|\xi|^2}t}|\hat{b}_0(\xi)|^2\right]d\xi ds\\&\quad+C\int_0^t g'(s)\int_{B(s)}|\xi|^{2N+4}\left(\|u(s)\|_{L^2}^4+\|b(s)\|_{L^2}^4+\|u(s)\|_{L^2}^2\|b(s)\|_{L^2}^2\right)d\xi ds\\&\quad+C\int_0^t g'(s)\int_{B(s)}|\xi|^{2N+8}(1+|\xi|^2+|\xi|^4)\left[\int_0^t(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau\right]^2 d\xi ds\\&\quad+C\int_0^t g(s)(1+s)^{-\min\{2r^*+\frac{15}{2}+N,\frac{19}{2}+N\}}ds,\quad \text{for large } t.\end{aligned}\tag{56}$$

Consider the first term of the right hand side of (56):

$$\begin{aligned}&\int_0^t g'(s)\int_{B(s)}|\xi|^{2N+6}\left[e^{-2|\xi|^2 t}|\hat{u}_0(\xi)|^2+(1+|\xi|^2)e^{-2\frac{|\xi|^2}{1+|\xi|^2}t}|\hat{b}_0(\xi)|^2\right]d\xi ds\\ \le & C\int_0^t g'(s)(1+s)^{-\min\{r^*+\frac{9}{2}+N,\frac{11}{2}+N\}}ds.\end{aligned} \tag{57}$$

For the second term, after integrating in polar coordinates in $B(t)$,

$$\begin{aligned}&C\int_0^t g'(s)\int_{B(s)}|\xi|^{2N+4}\left(\|u(s)\|_{L^2}^4+\|b(s)\|_{L^2}^4+\|u(s)\|_{L^2}^2\|b(s)\|_{L^2}^2\right)d\xi ds\\ \le & C\int_0^t g'(s)(1+s)^{-\frac{7}{2}-N}(1+s)^{-2\min\{r^*+\frac{3}{2},\frac{5}{2}\}}ds\\ \le & C\int_0^t g'(s)(1+s)^{-\min\{2r^*+\frac{13}{2}+N,\frac{17}{2}+N\}}ds, \quad \text{for large } t.\end{aligned} \tag{58}$$

In addition, if $r^*+\frac{3}{2}<\frac{5}{2}$, the third term of the right hand side of (56) can be estimated as

$$\begin{aligned}&C\int_0^t g'(s)\int_{B(s)}|\xi|^{2N+8}(1+|\xi|^2+|\xi|^4)\left[\int_0^t(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau\right]^2 d\xi ds\\ \le & C\int_0^t g'(s)(1+s)^{-\frac{11}{2}-N}(1+s)^{-2r^*-1}ds\\ \le & C\int_0^t g'(s)(1+s)^{-(2r^*+\frac{13}{2}+N)}ds, \quad \text{for large } t.\end{aligned} \tag{59}$$

If $r^*+\frac{3}{2}\ge\frac{5}{2}$, then, the third term of the right hand side of (56) satisfies

$$\begin{aligned}&C\int_0^t g'(s)\int_{B(s)}|\xi|^{2N+8}(1+|\xi|^2+|\xi|^4)\left[\int_0^t(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau\right]^2 d\xi ds\\ \le & C\int_0^t g'(s)(1+s)^{-(N+\frac{11}{2})}ds, \text{ for large } t.\end{aligned} \tag{60}$$

The last term satisfies

$$\begin{aligned}&C\int_0^t g(s)(1+s)^{-\min\{2r^*+\frac{15}{2}+N,\frac{19}{2}+N\}}ds\\ \le & C\int_0^t g'(s)(1+s)^{-\min\{2r^*+\frac{13}{2}+N,\frac{17}{2}+N\}}ds, \quad \text{for large } t.\end{aligned} \tag{61}$$

For a fixed r^*, we choose $g(t)=(1+t)^m$, for some $m>\max\{r^*+\frac{9}{2}+N,\frac{11}{2}+N\}$. Then $\rho(t)=C(1+t)^{-\frac{1}{2}}$ and from (56)–(61), it yields that

$$\begin{aligned}&\|\partial_t\Lambda^{N+1}u\|_{L^2}^2+\|\partial_t\Lambda^{N+1}b\|_{L^2}^2+\|\partial_t\Lambda^{N+1}b\|_{L^2}^2\\ &\le C(1+t)^{-\min\{r^*+\frac{9}{2}+N,\frac{11}{2}+N\}}, \quad \text{for large } t.\end{aligned} \tag{62}$$

Through mathematical induction, one concludes that for $P=1$ and $0<M+\frac{5}{2}P<K$,

$$\|\partial_t\Lambda^M u\|_{L^2}^2+\|\partial_t\Lambda^M b\|_{L^2}^2+\|\partial_t\Lambda^M b\|_{L^2}^2\le C(1+t)^{-\min\{r^*+\frac{7}{2}+M,\frac{9}{2}+M\}}, \text{ for large } t. \tag{63}$$

Hence, the proof is complete. □

Now, suppose that Theorem 2 holds for $p\le P_0\in\mathbb{N}^+$, and prove it also holds for $p=P_0+1$.

Lemma 13. *Let $m \in \mathbb{N}$, $K = \max\{5 + 2P_0, m + 2P_0 + 2\}$ and $r^* = r^*(u_0) = r^*(b_0) \in (-\frac{3}{2}, +\infty)$ be the decay character. Suppose that $(u_0, b_0) \in H^K(\mathbb{R}^3) \times H^{K+1}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then, there exists a positive constant $C = C(, \|u_0\|_{H^K}, \|b_0\|_{H^{K+1}})$, such that*

$$\begin{aligned}
&\|\partial_t^{P_0+1}\Lambda^m u\|_{L^2}^2 + \|\partial_t^{P_0+1}\Lambda^m b\|_{L^2}^2 + \|\partial_t^{P_0+1}\Lambda^{m+1} b\|_{L^2}^2 \\
&\le C(1+t)^{-\min\{\frac{3}{2}+r^*+m+2(P_0+1), \frac{5}{2}+m+2(P_0+1)\}}, \quad \text{for large } t.
\end{aligned} \tag{64}$$

Proof. First, assume that $m = 0$. Applying $\partial_t^{P_0+1}$ to (10) and (11), multiplying both side by $\partial_t^{P_0+1}u$ and $\partial_t^{P_0+1}b$, respectively, integrating over $\mathbb{R}^3$, it yields that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial_t^{P_0+1}u\|_{L^2}^2 + \|\partial_t^{P_0+1}b\|_{L^2}^2 + \|\partial_t^{P_0+1}\Lambda b\|_{L^2}^2) + \|\partial_t^{P_0+1}\Lambda u\|_{L^2}^2 + \|\partial_t^{P_0+1}\Lambda b\|_{L^2}^2 \\
\le& \sum_{p=0}^{P_0+1}\Big[|(\partial_t^p u\cdot\partial_t^{P_0+1}\Lambda u, \partial_t^{P_0+1-p}u)| + |(\partial_t^p b\cdot\partial_t^{P_0+1}\Lambda u, \partial_t^{P_0+1-p}b)| \\
&+|(\partial_t^p u\cdot\partial_t^{P_0+1}\Lambda b, \partial_t^{P_0+1-p}b)| + |(\partial_t^p b\cdot\partial_t^{P_0+1}\Lambda b, \partial_t^{P_0+1-p}u)| + |(\partial_t^p b\cdot\partial_t^{P_0+1}\Lambda b, \partial_t^{P_0+1-p}\Lambda b)|\Big] \\
&+ |(u\cdot\partial_t^{P_0+1}\Lambda b, \partial_t^{P_0+1}\Lambda^2 b)| + \sum_{p=1}^{P_0+1}|(\partial_t^p u\cdot\partial_t^{P_0+1}\Lambda b, \partial_t^{P_0+1-p}\Lambda^2 b)|.
\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial_t^{P_0+1}u\|_{L^2}^2 + \|\partial_t^{P_0+1}b\|_{L^2}^2 + \|\partial_t^{P_0+1}\Lambda b\|_{L^2}^2) + \|\partial_t^{P_0+1}\Lambda u\|_{L^2}^2 + \|\partial_t^{P_0+1}\Lambda b\|_{L^2}^2 \\
\le& 2\|u\|_{L^3}\|\partial_t^{P_0+1}\Lambda u\|_{L^2}\|\partial_t^{P_0+1}u\|_{L^6} + \sum_{p=1}^{P_0}\|\partial_t^p u\|_{L^6}\|\partial_t^{P_0+1}\Lambda u\|_{L^2}\|\partial_t^{P_0+1-p}u\|_{L^3} \\
&+ 2\|b\|_{L^3}\|\partial_t^{P_0+1}\Lambda u\|_{L^2}\|\partial_t^{P_0+1}b\|_{L^6} + \sum_{p=1}^{P_0}\|\partial_t^p b\|_{L^6}\|\partial_t^{P_0+1}\Lambda u\|_{L^2}\|\partial_t^{P_0+1-p}b\|_{L^3} \\
&+ 2\|u\|_{L^3}\|\partial_t^{P_0+1}\Lambda b\|_{L^2}\|\partial_t^{P_0+1}b\|_{L^6} + \sum_{p=1}^{P_0}\|\partial_t^p u\|_{L^6}\|\partial_t^{P_0+1}\Lambda b\|_{L^2}\|\partial_t^{P_0+1-p}b\|_{L^3} \\
&+ 2\|b\|_{L^3}\|\partial_t^{P_0+1}\Lambda b\|_{L^2}\|\partial_t^{P_0+1}u\|_{L^6} + \sum_{p=1}^{P_0}\|\partial_t^p b\|_{L^6}\|\partial_t^{P_0+1}\Lambda b\|_{L^2}\|\partial_t^{P_0+1-p}u\|_{L^3} \\
&+ \|b\|_{L^\infty}\|\partial_t^{P_0+1}\Lambda b\|_{L^2}^2 + \|\partial_t^{P_0+1}b\|_{L^6}\|\partial_t^{P_0+1}\Lambda b\|_{L^2}\|\Lambda b\|_{L^3} \\
&+ \sum_{p=1}^{P_0}\|\partial_t^p b\|_{L^6}\|\partial_t^{P_0+1}b\|_{L^2}\|\partial_t^{P_0+1-p}\Lambda b\|_{L^3} + \|\Lambda u\|_{L^\infty}\|\partial_t^{P_0+1}\Lambda b\|_{L^2}^2 \\
&+ \|\Lambda^2 b\|_{L^3}\|\partial_t^{P_0+1}\Lambda b\|_{L^2}\|\partial_t^{P_0+1}u\|_{L^6} + \sum_{p=1}^{P_0}\|\partial_t^p u\|_{L^6}\|\partial_t^{P_0+1}\Lambda^b\|_{L^2}\|\partial_t^{P_0+1-p}\Lambda^2 b\|_{L^3} \\
\le& \frac{1}{2}(\|\partial_t^{P_0+1}\Lambda u\|_{L^2}^2 + \|\partial_t^{P_0+1}\Lambda b\|_{L^2}^2) + C\sum_{p=1}^{P_0}(\|\partial_t^p\Lambda u\|_{L^2}^2 + \|\partial_t^p\Lambda b\|_{L^2}^2)(\|\partial_t^{P_0+1-p}\Lambda^{\frac{1}{2}}u\|_{L^2}^2 \\
&+ \|\partial_t^{P_0+1-p}\Lambda^{\frac{1}{2}}b\|_{L^2}^2 + \|\partial_t^{P_0+1-p}\Lambda^{\frac{3}{2}}b\|_{L^2}^2 + \|\partial_t^{P_0+1-p}\Lambda^{\frac{5}{2}}b\|_{L^2}^2), \quad \text{for large } t.
\end{aligned}$$

Applying the previous decay estimates, it yields that

$$\begin{aligned}
&\frac{d}{dt}(\|\partial_t^{P_0+1}u\|_{L^2}^2+\|\partial_t^{P_0+1}b\|_{L^2}^2+\|\partial_t^{P_0+1}\Lambda b\|_{L^2}^2)+\|\partial_t^{P_0+1}\Lambda u\|_{L^2}^2+\|\partial_t^{P_0+1}\Lambda b\|_{L^2}^2\\
\le& C\sum_{p=1}^{P_0}(\|\partial_t^{p}\Lambda u\|_{L^2}^2+\|\partial_t^{p}\Lambda b\|_{L^2}^2)(\|\partial_t^{P_0+1-p}\Lambda^{\frac12}u\|_{L^2}^2+\|\partial_t^{P_0+1-p}\Lambda^{\frac12}b\|_{L^2}^2+\|\partial_t^{P_0+1-p}\Lambda^{\frac32}b\|_{L^2}^2\\
&+\|\partial_t^{P_0+1-p}\Lambda^{\frac52}b\|_{L^2}^2)\\
\le& C(1+t)^{-\min\{2r^*+\frac{13}{2}+2P_0,\frac{17}{2}+2P_0\}},\quad \text{for large } t.
\end{aligned} \tag{65}$$

Using Plancherel's theorem to (65) gives

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\left[|\widehat{\partial_t^{P_0+1}u}(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t^{P_0+1}b}(\xi,t)|^2\right]d\xi\\
&+\int_{\mathbb{R}^3}\frac{|\xi|^2}{1+|\xi|^2}\left[|\widehat{\partial_t^{P_0+1}u}(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t^{P_0+1}b}(\xi,t)|^2\right]d\xi\\
\le& C(1+t)^{-\min\{2r^*+\frac{13}{2}+2P_0,\frac{17}{2}+2P_0\}},\quad \text{for large } t.
\end{aligned} \tag{66}$$

Multiplying (66) by $g(t)$ gives

$$\begin{aligned}
&\frac{d}{dt}\left\{g(t)\int_{\mathbb{R}^3}\left[|\widehat{\partial_t^{P_0+1}u}(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t^{P_0+1}b}(\xi,t)|^2\right]d\xi\right\}\\
\le& g'(t)\int_{B(t)}\left[|\widehat{\partial_t^{P_0+1}u}(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t^{P_0+1}b}(\xi,t)|^2\right]d\xi\\
&+Cg(t)(1+t)^{-\min\{2r^*+\frac{13}{2}+2P_0,\frac{17}{2}+2P_0\}},\quad \text{for large } t.
\end{aligned}$$

Hence

$$\begin{aligned}
&g(t)\int_{\mathbb{R}^3}\left[|\widehat{\partial_t^{P_0+1}u}(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t^{P_0+1}b}(\xi,t)|^2\right]d\xi\\
\le& C+\int_0^t g'(s)\int_{B(s)}\left[|\widehat{\partial_t^{P_0+1}u}(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t^{P_0+1}b}(\xi,t)|^2\right]d\xi ds\\
&+C\int_0^t g(s)(1+s)^{-\min\{2r^*+\frac{13}{2}+2P_0,\frac{17}{2}+2P_0\}}ds,\quad \text{for large } t.
\end{aligned} \tag{67}$$

It then follows from (36), (38) and (67) that

$$\begin{aligned}
&g(t)\int_{\mathbb{R}^3}\left[|\widehat{\partial_t^{P_0+1}u}(\xi,t)|^2+(1+|\xi|^2)|\widehat{\partial_t^{P_0+1}b}(\xi,t)|^2\right]d\xi\\
\le& C+\int_0^t g'(s)\int_{B(s)}|\xi|^{4P_0+4}\left[e^{-2|\xi|^2t}|\hat{u}_0(\xi)|^2+(1+|\xi|^2)e^{-2\frac{|\xi|^2}{1+|\xi|^2}t}|\hat{b}_0(\xi)|^2\right]d\xi ds\\
&+C\int_0^t g'(s)\int_{B(s)}\sum_{p=0}^{P_0}\sum_{l=0}^{p}|\xi|^{4P_0+2-4p}(\|\partial_t^l u\|_{L^2}^2+\|\partial_t^l b\|_{L^2}^2)(\|\partial_t^{p-l}u\|_{L^2}^2+\|\partial_t^{p-l}b\|_{L^2}^2)d\xi ds\\
&+C\int_0^t g'(s)\int_{B(s)}|\xi|^{4P_0+6}(1+|\xi|^2+|\xi|^4)\left[\int_0^t(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau\right]^2 d\xi ds\\
&+C\int_0^t g(s)(1+s)^{-\min\{2r^*+\frac{13}{2}+2P_0,\frac{17}{2}+2P_0\}}ds,\quad \text{for large } t.
\end{aligned} \tag{68}$$

For the first term of the right hand side of (68),

$$\begin{aligned}
&\int_0^t g'(s)\int_{B(s)}|\xi|^{4P_0+4}\left[e^{-2|\xi|^2t}|\hat{u}_0(\xi)|^2+(1+|\xi|^2)e^{-2\frac{|\xi|^2}{1+|\xi|^2}t}|\hat{b}_0(\xi)|^2\right]d\xi d\xi ds\\
\le& C\int_0^t g'(s)(1+s)^{-(r^*+\frac{7}{2}+2P_0)}ds.
\end{aligned} \tag{69}$$

For the second term, after integrating in polar coordinates in $B(t)$,

$$
\begin{aligned}
&C\int_0^t g'(s)\int_{B(s)}\sum_{p=0}^{P_0}\sum_{l=0}^{p}|\xi|^{4P_0+2-4p}(\|\partial_t^l u\|_{L^2}^2+\|\partial_t^l b\|_{L^2}^2)(\|\partial_t^{p-l}u\|_{L^2}^2+\|\partial_t^{p-l}b\|_{L^2}^2)d\xi ds\\
\le &C\int_0^t g'(s)(1+s)^{-(2P_0-2p+\frac{5}{2})}(1+s)^{-\min\{r^*+\frac{3}{2}+2l,\frac{5}{2}+2l\}}(1+s)^{-\min\{r^*+\frac{3}{2}+2p-2l,\frac{5}{2}+2p-2l\}}ds\\
\le &C\int_0^t g'(s)(1+s)^{-\min\{2r^*+\frac{11}{2}+2P_0,\frac{15}{2}+2P_0\}}ds, \quad \text{for large } t.
\end{aligned} \tag{70}
$$

In addition, if $r^*+\frac{3}{2}<\frac{5}{2}$, the third term of the right hand side of (68) can be estimated as

$$
\begin{aligned}
&C\int_0^t g'(s)\int_{B(s)}|\xi|^{4P_0+6}(1+|\xi|^2+|\xi|^4)\left[\int_0^t(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau\right]^2 d\xi ds\\
\le &C\int_0^t g'(s)(1+s)^{-\frac{9}{2}-2P_0}(1+s)^{-2r^*-1}ds\\
\le &C\int_0^t g'(s)(1+s)^{-(2r^*+\frac{11}{2}+2P_0)}ds, \quad \text{for large } t.
\end{aligned} \tag{71}
$$

If $r^*+\frac{3}{2}\ge\frac{5}{2}$,

$$
\begin{aligned}
&C\int_0^t g'(s)\int_{B(s)}|\xi|^{4P_0+6}(1+|\xi|^2+|\xi|^4)\left[\int_0^t(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau\right]^2 d\xi ds\\
\le\ &C\int_0^t g'(s)(1+s)^{-(2P_0+\frac{9}{2})}ds, \quad \text{for large } t.
\end{aligned} \tag{72}
$$

The last term satisfies

$$
\begin{aligned}
&C\int_0^t g(s)(1+s)^{-\min\{2r^*+\frac{13}{2}+2P_0,\frac{17}{2}+2P_0\}}ds\\
\le &C\int_0^t g'(s)(1+s)^{-\min\{2r^*+\frac{11}{2}+2P_0,\frac{15}{2}+2P_0\}}ds, \quad \text{for large } t.
\end{aligned} \tag{73}
$$

For a fixed r^*, choose $g(t)=(1+t)^m$, for some $m>\max\{r^*+\frac{7}{2}+2P_0,\frac{9}{2}+2P_0\}$. Then $\rho(t)=C(1+t)^{-\frac{1}{2}}$ and from (68)–(73),

$$
\|\partial_t^{P_0+1}u\|_{L^2}^2+\|\partial_t^{P_0+1}b\|_{L^2}^2+\|\partial_t^{P_0+1}b\|_{L^2}^2\le C(1+t)^{-\min\{r^*+\frac{7}{2}+2P_0,\frac{9}{2}+2P_0\}}, \text{ for large } t. \tag{74}
$$

Suppose that Lemma 13 holds for $m\le N\in\mathbb{N}^+$, then one can prove it also holds for $m=N+1$. Applying $\partial_t^{P_0+1}\Lambda^{N+1}$ to (10) and (11), multiplying both sides by $\partial_t^{P_0+1}\Lambda^{N+1}u$ and $\partial_t^{P_0+1}\Lambda^{N+1}b$, respectively, integrating over $\mathbb{R}^3$, it yields that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial_t^{P_0+1}\Lambda^{N+1}u\|_{L^2}^2+\|\partial_t^{P_0+1}\Lambda^{N+1}b\|_{L^2}^2+\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}^2)\\
&+\|\partial_t^{P_0+1}\Lambda^{N+2}u\|_{L^2}^2+\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}^2\\
\le &\sum_{p=0}^{P_0+1}\sum_{m=0}^{N+1}C_{P_0+1}^{p}C_{N+1}^{m}\Big[|(\partial_t^p\Lambda^m u\cdot\partial_t^{P_0+1}\Lambda^{N+2}u,\partial_t^{P_0+1-p}\Lambda^{N+1-m}u)|\\
&+|(\partial_t^p\Lambda^m b\cdot\partial_t^{P_0+1}\Lambda^{N+2}u,\partial_t^{P_0+1-p}\Lambda^{N+1-m}b)|+|(\partial_t^p\Lambda^m u\cdot\partial_t^{P_0+1}\Lambda^{N+2}b,\partial_t^{P_0+1-p}\Lambda^{N+1-m}b)|\\
&+|(\partial_t^p\Lambda^m b\cdot\partial_t^{P_0+1}\Lambda^{N+2}u,\partial_t^{P_0+1-p}\Lambda^{N+1-m}b)|+|(\partial_t^p\Lambda^m b\cdot\partial_t^{P_0+1}\Lambda^{N+2}b,\partial_t^{P_0+1-p}\Lambda^{N+2-m}b)|\\
&+|(\partial_t^p\Lambda^m u\cdot\partial_t^{P_0+1}\Lambda^{N+2}b,\partial_t^{P_0+1-p}\Lambda^{N+3-m}b)|\Big]\\
:=&I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}
$$

First of all, I_1 satisfies

$$
\begin{aligned}
I_1 \leq & \sum_{m=0}^{N+1} C_{N+1}^{m} \|\Lambda^m u\|_{L^3} \|\partial_t^{P_0+1}\Lambda^{N+2}u\|_{L^2} \|\partial_t^{P_0+1}\Lambda^{N+1-m}\|_{L^6} \\
& + \sum_{m=0}^{N+1} C_{N+1}^{m} \|\partial_t^{P_0+1}\Lambda^m u\|_{L^3} \|\partial_t^{P_0+1}\Lambda^{N+2}u\|_{L^2} \|\Lambda^{N+1-m}\|_{L^6} \\
& + \sum_{p=1}^{P_0}\sum_{m=0}^{N+1} C_{P_0+1}^{p} C_{N+1}^{m} \|\partial_t^{p}\Lambda^m u\|_{L^6} \|\partial_t^{P_0+1}\Lambda^{N+2}u\|_{L^2} \|\partial_t^{P_0+1-p}\Lambda^{N+1-m}\|_{L^3} \\
\leq & \frac{1}{16}\|\partial_t^{P_0+1}\Lambda^{N+2}u\|_{L^2}^2 \\
& + C\left[\sum_{m=0}^{N+1}\left(\|\Lambda^{m+\frac{1}{2}}u\|_{L^2}^2\|\partial_t^{P_0+1}\Lambda^{N+2-m}u\|_{L^2}^2 + \|\partial_t^{P_0+1}\Lambda^{m+1}u\|_{L^2}^2\|\Lambda^{N+\frac{3}{2}-m}u\|_{L^2}^2\right)\right] \\
& + C\left[\sum_{p=1}^{P_0}\sum_{m=0}^{N+1}\|\partial_t^{p}\Lambda^{m+\frac{1}{2}}u\|_{L^2}^2\|\partial_t^{P_0+1-p}\Lambda^{N+2-m}u\|_{L^2}^2\right] \\
\leq & \frac{1}{8}\|\partial_t^{P_0+1}\Lambda^{N+2}u\|_{L^2}^2 + C\sum_{m=1}^{N+1}\|\Lambda^{m+\frac{1}{2}}u\|_{L^2}^2\|\partial_t^{P_0+1}\Lambda^{N+2-m}u\|_{L^2}^2 \\
& + C\|\partial_t^{P_0+1}\Lambda^{N}u\|_{L^2}^2\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2 + C\sum_{m=0}^{N-1}\|\partial_t^{P_0+1}\Lambda^{m+1}u\|_{L^2}^2\|\Lambda^{N+\frac{3}{2}-m}u\|_{L^2}^2 \\
& + C\left[\sum_{p=1}^{P_0}\sum_{m=0}^{N+1}\|\partial_t^{p}\Lambda^{m+\frac{1}{2}}u\|_{L^2}^2\|\partial_t^{P_0+1-p}\Lambda^{N+2-m}u\|_{L^2}^2\right], \quad \text{for large } t,
\end{aligned}
$$

where the following fact

$$
\|\partial_t^{P_0+1}\Lambda^{N+1}u\|_{L^2}^2 \leq \frac{1}{2}(\|\partial_t^{P_0+1}\Lambda^{N+2}u\|_{L^2}^2 + \|\partial_t^{P_0+1}\Lambda^{N}u\|_{L^2}^2).
$$

has been used. Similarly,

$$
\begin{aligned}
I_2 \leq & \frac{1}{8}\|\partial_t^{P_0+1}\Lambda^{N+2}u\|_{L^2}^2 + C\sum_{m=1}^{N+1}\|\Lambda^{m+\frac{1}{2}}b\|_{L^2}^2\|\partial_t^{P_0+1}\Lambda^{N+2-m}b\|_{L^2}^2 \\
& + C\|\partial_t^{P_0+1}\Lambda^{N}b\|_{L^2}^2\|\Lambda^{\frac{3}{2}}b\|_{L^2}^2 + C\sum_{m=0}^{N-1}\|\partial_t^{P_0+1}\Lambda^{m+1}b\|_{L^2}^2\|\Lambda^{N+\frac{3}{2}-m}b\|_{L^2}^2 \\
& + C\left[\sum_{p=1}^{P_0}\sum_{m=0}^{N+1}\|\partial_t^{p}\Lambda^{m+\frac{1}{2}}b\|_{L^2}^2\|\partial_t^{P_0+1-p}\Lambda^{N+2-m}b\|_{L^2}^2\right], \quad \text{for large } t,
\end{aligned}
$$

$$
\begin{aligned}
I_3 \leq & \frac{1}{8}\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}^2 + C\sum_{m=1}^{N+1}\|\Lambda^{m+\frac{1}{2}}u\|_{L^2}^2\|\partial_t^{P_0+1}\Lambda^{N+2-m}b\|_{L^2}^2 \\
& + C\|\partial_t^{P_0+1}\Lambda^{N}u\|_{L^2}^2\|\Lambda^{\frac{3}{2}}b\|_{L^2}^2 + C\sum_{m=0}^{N-1}\|\partial_t^{P_0+1}\Lambda^{m+1}u\|_{L^2}^2\|\Lambda^{N+\frac{3}{2}-m}b\|_{L^2}^2 \\
& + C\left[\sum_{p=1}^{P_0}\sum_{m=0}^{N+1}\|\partial_t^{p}\Lambda^{m+\frac{1}{2}}u\|_{L^2}^2\|\partial_t^{P_0+1-p}\Lambda^{N+2-m}b\|_{L^2}^2\right], \quad \text{for large } t,
\end{aligned}
$$

$$
\begin{aligned}
I_4 \leq & \frac{1}{8}\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}^2 + C\sum_{m=1}^{N+1}\|\Lambda^{m+\frac{1}{2}}b\|_{L^2}^2\|\partial_t^{P_0+1}\Lambda^{N+2-m}u\|_{L^2}^2 \\
& + C\|\partial_t^{P_0+1}\Lambda^{N}b\|_{L^2}^2\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2 + C\sum_{m=0}^{N-1}\|\partial_t^{P_0+1}\Lambda^{m+1}b\|_{L^2}^2\|\Lambda^{N+\frac{3}{2}-m}u\|_{L^2}^2 \\
& + C\left[\sum_{p=1}^{P_0}\sum_{m=0}^{N+1}\|\partial_t^{p}\Lambda^{m+\frac{1}{2}}b\|_{L^2}^2\|\partial_t^{P_0+1-p}\Lambda^{N+2-m}u\|_{L^2}^2\right], \quad \text{for large } t.
\end{aligned}
$$

For I_5,

$$
\begin{aligned}
I_5 \leq & \|b\|_{L^\infty}\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}^2 + \sum_{m=1}^{N+1}C_{N+1}^m\|\Lambda^m b\|_{L^3}\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}\|\partial_t^{P_0+1}\Lambda^{N+1-m}b\|_{L^6} \\
& + \|\partial_t^{P_0+1}b\|_{L^6}\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}\|\Lambda b\|_{L^3} + \sum_{m=0}^{N}\|\partial_t^{P_0+1}\Lambda^m b\|_{L^6}\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}\|\Lambda^{N+2-m}b\|_{L^3} \\
& + \sum_{p=1}^{P_0}\sum_{m=0}^{N+1}\|\partial_t^{p}\Lambda^m b\|_{L^6}\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}\|\partial_t^{P_0+1-p}\Lambda^{N+2-m}b\|_{L^3} \\
\leq & \frac{1}{8}\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}^2 + C\sum_{m=2}^{N+1}\|\Lambda^{m+\frac{1}{2}}b\|_{L^2}^2\|\partial_t^{P_0+1}\Lambda^{N+1}b\|_{L^2}^2 \\
& + \sum_{m=0}^{N-1}\|\Lambda^{N+\frac{5}{2}-m}b\|_{L^2}^2\|\partial_t^{P_0+1}\Lambda^{m+1}b\|_{L^2}^2 + (\|\Lambda^{\frac{3}{2}}b\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}}b\|_{L^2}^2)\|\partial_t^{P_0+1}\Lambda^N b\|_{L^2}^2 \\
& + \sum_{p=1}^{P_0}\sum_{m=0}^{N+1}\|\partial_t^{p}\Lambda^{m+1}b\|_{L^2}^2\|\partial_t^{P_0+1-p}\Lambda^{N+\frac{5}{2}-m}b\|_{L^2}^2, \quad \text{for large } t,
\end{aligned}
$$

where

$$
\|\partial_t^{P_0+1}\Lambda^{N+1}b\|_{L^2}^2 \leq \frac{1}{2}(\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}^2 + \|\partial_t^{P_0+1}\Lambda^{N}b\|_{L^2}^2).
$$

has been used. In addition,

$$
\begin{aligned}
I_6 \leq & C[\|\Lambda u\|_{L^\infty}\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}^2 + \sum_{m=2}^{N+1}\|\Lambda^m u\|_{L^3}\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}\|\partial_t^{P_0+1}\Lambda^{N+3-m}b\|_{L^6} \\
& + \|\partial_t^{P_0+1}\Lambda^{N+1}u\|_{L^6}\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}\|\Lambda^2 b\|_{L^3} + \sum_{m=0}^{N}\|\partial_t^{P_0+1}\Lambda^m u\|_{L^6}\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}\|\Lambda^{N+3-m}b\|_{L^2} \\
& + \sum_{p=1}^{P_0}\sum_{m=0}^{N+1}\|\partial_t\Lambda^m u\|_{L^6}\|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}\|\partial_t^{P_0+1-p}\Lambda^{N+3-m}b\|_{L^3}] \\
\leq & \frac{1}{8}(\|\partial_t^{P_0+1}\Lambda^{N+2}u\|_{L^2}^2 + \|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}^2) + C[\|\Lambda^{\frac{7}{2}}u\|_{L^2}^2\|\partial_t^{P_0+1}\Lambda^N b\|_{L^2}^2 \\
& + \sum_{m=4}^{N}\|\Lambda^{\frac{1}{2}+m}u\|_{L^2}^2\|\partial_t^{P_0+1}\Lambda^{N+4-m}b\|_{L^2}^2 + \sum_{m=0}^{N-1}\|\partial_t^{P_0+1}\Lambda^{m+1}u\|_{L^2}^2\|\Lambda^{N+\frac{7}{2}-m}b\|_{L^2}^2 \\
& + \|\Lambda^{\frac{7}{2}}b\|_{L^2}^2\|\partial_t^{P_0+1}\Lambda^N u\|_{L^2}^2 + \sum_{p=1}^{P_0}\sum_{m=0}^{N+1}\|\partial_t^{p}\Lambda^{m+1}u\|_{L^2}^2\|\partial_t^{P_0+1-p}\Lambda^{N+\frac{7}{2}-m}b\|_{L^2}^2], \quad \text{for large } t.
\end{aligned}
$$

Summing up, using the previous decay results gives

$$
\begin{aligned}
& \frac{d}{dt}(\|\partial_t^{P_0+1}\Lambda^{N+1}u\|_{L^2}^2 + \|\partial_t^{P_0+1}\Lambda^{N+1}b\|_{L^2}^2 + \|\partial_t^{P_0+1}\Lambda^{N+2}b\|_{L^2}^2) \\
\leq & C(1+t)^{-\min\{2r^*+\frac{15}{2}+N+2P_0, \frac{19}{2}+N+2P_0\}}, \quad \text{for large } t.
\end{aligned} \tag{75}
$$

Applying Plancherel's theorem to (75), it yields that

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3}\left[|\partial_t^{P_0+1}\widehat{\Lambda^{N+1}u}(\xi,t)|^2+(1+|\xi|^2)|\partial_t^{P_0+1}\widehat{\Lambda^{N+1}b}(\xi,t)|^2\right]d\xi \\ &+\int_{\mathbb{R}^3}\frac{|\xi|^2}{1+|\xi|^2}\left[|\partial_t^{\widehat{P_0+1}}u(\xi,t)|^2+(1+|\xi|^2)|\partial_t^{\widehat{P_0+1}}b(\xi,t)|^2\right]d\xi \\ \leq& C(1+t)^{-\min\{2r^*+\frac{15}{2}+N+2P_0,\frac{19}{2}+N+2P_0\}},\ \text{ for large } t. \end{aligned} \tag{76}$$

Multiplying (76) by $g(t)$ gives

$$\begin{aligned} &\frac{d}{dt}\left\{g(t)\int_{\mathbb{R}^3}\left[|\partial_t^{P_0+1}\widehat{\Lambda^{N+1}u}(\xi,t)|^2+(1+|\xi|^2)|\partial_t^{P_0+1}\widehat{\Lambda^{N+1}b}(\xi,t)|^2\right]d\xi\right\} \\ \leq& g'(t)\int_{B(t)}\left[|\partial_t^{P_0+1}\widehat{\Lambda^{N+1}u}(\xi,t)|^2+(1+|\xi|^2)|\partial_t^{P_0+1}\widehat{\Lambda^{N+1}b}(\xi,t)|^2\right]d\xi \\ &+Cg(t)(1+t)^{-\min\{2r^*+\frac{15}{2}+N+2P_0,\frac{19}{2}+N+2P_0\}},\quad \text{for large } t. \end{aligned}$$

Hence

$$\begin{aligned} &g(t)\int_{\mathbb{R}^3}\left[|\partial_t^{P_0+1}\widehat{\Lambda^{N+1}u}(\xi,t)|^2+(1+|\xi|^2)|\partial_t^{P_0+1}\widehat{\Lambda^{N+1}b}(\xi,t)|^2\right]d\xi \\ \leq& C+\int_0^t g'(s)\int_{B(s)}\left[|\partial_t^{P_0+1}\widehat{\Lambda^{N+1}u}(\xi,t)|^2+(1+|\xi|^2)|\partial_t^{P_0+1}\widehat{\Lambda^{N+1}b}(\xi,t)|^2\right]d\xi ds \\ &+C\int_0^t g(s)(1+s)^{-\min\{2r^*+\frac{15}{2}+N+2P_0,\frac{19}{2}+N+2P_0\}}ds,\quad \text{for large } t. \end{aligned} \tag{77}$$

It then follows from (36), (38) and (67) that

$$\begin{aligned} &g(t)\int_{\mathbb{R}^3}\left[|\partial_t^{P_0+1}\widehat{\Lambda^{N+1}u}(\xi,t)|^2+(1+|\xi|^2)|\partial_t^{P_0+1}\widehat{\Lambda^{N+1}b}(\xi,t)|^2\right]d\xi \\ \leq& C+\int_0^t g'(s)\int_{B(s)}|\xi|^{4P_0+2N+6}\left[e^{-2|\xi|^2t}|\hat{u}_0(\xi)|^2+(1+|\xi|^2)e^{-2\frac{|\xi|^2}{1+|\xi|^2}t}|\hat{b}_0(\xi)|^2\right]d\xi ds \\ &+C\int_0^t g'(s)\int_{B(s)}\sum_{p=0}^{P_0}\sum_{l=0}^{p}|\xi|^{4P_0+4+2N-4p}(\|\partial_t^l u\|_{L^2}^2+\|\partial_t^l b\|_{L^2}^2)(\|\partial_t^{p-l}u\|_{L^2}^2+\|\partial_t^{p-l}b\|_{L^2}^2)d\xi ds \\ &+C\int_0^t g'(s)\int_{B(s)}|\xi|^{4P_0+2N+8}(1+|\xi|^2+|\xi|^4)\left[\int_0^t(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau\right]^2 d\xi ds \\ &+C\int_0^t g(s)(1+s)^{-\min\{2r^*+\frac{13}{2}+2P_0,\frac{17}{2}+2P_0\}}ds,\quad \text{for large } t. \end{aligned} \tag{78}$$

Consider the first term of the right hand side of (78),

$$\begin{aligned} &\int_0^t g'(s)\int_{B(s)}|\xi|^{4P_0+4}\left[e^{-2|\xi|^2t}|\hat{u}_0(\xi)|^2+(1+|\xi|^2)e^{-2\frac{|\xi|^2}{1+|\xi|^2}t}|\hat{b}_0(\xi)|^2\right]d\xi d\xi ds \\ \leq& C\int_0^t g'(s)(1+s)^{-(r^*+\frac{9}{2}+2P_0+N)}ds. \end{aligned} \tag{79}$$

For the second term, after integrating in polar coordinates in $B(t)$,

$$\begin{aligned} &C\int_0^t g'(s)\int_{B(s)}\sum_{p=0}^{P_0}\sum_{l=0}^{p}|\xi|^{4P_0+4+2N-4p}(\|\partial_t^l u\|_{L^2}^2+\|\partial_t^l b\|_{L^2}^2)(\|\partial_t^{p-l}u\|_{L^2}^2+\|\partial_t^{p-l}b\|_{L^2}^2)d\xi ds \\ \leq& C\int_0^t g'(s)(1+s)^{-\min\{2r^*+N+\frac{13}{2}+2P_0,\frac{17}{2}+N+2P_0\}}ds,\quad \text{for large } t. \end{aligned} \tag{80}$$

Moreover, if $r^* + \frac{3}{2} < \frac{5}{2}$, the third term of the right hand side of (78) can be estimated as

$$\begin{aligned} &C\int_0^t g'(s)\int_{B(s)}|\xi|^{4P_0+2N+8}(1+|\xi|^2+|\xi|^4)\left[\int_0^t(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau\right]^2 d\xi ds \\ \le &C\int_0^t g'(s)(1+s)^{-(2r^*+N+\frac{13}{2}+2P_0)}ds, \quad \text{for large } t. \end{aligned} \tag{81}$$

If $r^* + \frac{3}{2} \geq \frac{5}{2}$,

$$\begin{aligned} &C\int_0^t g'(s)\int_{B(s)}|\xi|^{4P_0+2N+8}(1+|\xi|^2+|\xi|^4)\left[\int_0^t(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2)d\tau\right]^2 d\xi ds \\ \le &C\int_0^t g'(s)(1+s)^{-(2P_0+\frac{11}{2}+N)}ds, \quad \text{for large } t. \end{aligned} \tag{82}$$

The last term can be estimated as

$$\begin{aligned} &C\int_0^t g(s)(1+s)^{-\min\{2r^*+\frac{15}{2}+N+2P_0,\frac{19}{2}+N+2P_0\}}ds \\ \le &C\int_0^t g'(s)(1+s)^{-\min\{2r^*+\frac{13}{2}+N+2P_0,\frac{17}{2}+N+2P_0\}}ds, \quad \text{for large } t. \end{aligned} \tag{83}$$

For a fixed r^*, choose $g(t) = (1+t)^m$, for some $m > \max\{r^* + \frac{9}{2} + N + 2P_0, \frac{11}{2} + N + 2P_0\}$. Then $\rho(t) = C(1+t)^{-\frac{1}{2}}$ and from (78)–(83), one can obtain

$$\begin{aligned} &\|\partial_t^{P_0+1}\Lambda^{N+1}u\|_{L^2}^2 + \|\partial_t^{P_0+1}\Lambda^{N+1}b\|_{L^2}^2 + \|\partial_t^{P_0+1}\Lambda^{N+1}b\|_{L^2}^2 \\ \le &C(1+t)^{-\min\{r^*+\frac{9}{2}+N+2P_0,\frac{11}{2}+N+2P_0\}}, \text{ for large } t. \end{aligned} \tag{84}$$

This completes the proof of Lemma 13. □

The proof of Theorem 2 is given in the following.

Proof of Theorem 2. Lemma 12 implies Theorem 2 holds for the case $p = 1$. Then, supposing that Theorem 2 holds for $p \leq P_0 \in \mathbb{N}^+$, one can also obtain that it holds for $p = P_0 + 1$ (Lemma 13). Hence, through mathematical induction, the proof of Theorem 2 is complete. □

5. Conclusions

The magnetic induction equation with Hall effect is a typical Hall–MHD equation. This model can be used to describe the reconnection phenomenon by simulating flows with differential typical scales L_0. From a mathematical point of view, the local well-posedness, global well-posedness and large time behavior of solutions are very interesting. In the previous works of Fan et al. [11], the authors studied the local well-posedness of strong solutions and gave the preliminary result on the small initial data global well-posedness; Zhao [29,30] considered the large time behavior of solutions, established the decay estimates for the weak solution (see also Lemma 1) and the strong solution (see Lemma 2). In this paper, one only assumes that $\|u_0\|_{H^{\frac{3}{2}+\varepsilon}} + \|b_0\|_{H^2}$ is sufficiently small, obtains the global well-posedness of strong solution and establishes the a priori estimates on higher order time and spatial derivatives of solutions. Moreover, by using the properties of decay character and the Fourier splitting method, one also shows the optimal decay rates for higher order time and spatial derivatives of solutions. In a sense, the results of this paper can be seen as an improvement of the previous results in [11,29,30].

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Nomenclature

∇	∇f is the gradient of f
Λ^l	the usual spatial derivatives of order l
$\subset$	the symbol of embedding
$\mathbb{R}^3$	3-dimensional Euclidean space
ρ	the density
u	the velocity field of the fluid
b	the magnetic field
π	the pressure
L_0	the normalizing length limit
δ_e	electron inertia
δ_i	ion inertia
P_r	the decay indicator
r^*	the decay character

References

1. Corti, P. Stable numerical scheme for the magnetic induction equation with Hall effect. In *Hyperbolic Problems: Theory, Numerics and Applications (In 2 Volumes)*; Li, T., Jiang, S., Eds.; Higher Education Press: Beijing, China, 2012; pp. 374–381.
2. Qian, X.; Balbás, J.; Bhattacharjee, A.; Yang, H. A numerical study of magnetic reconnection: A central scheme for Hall MHD. Hyperbolic Partial Differential Equations, Theory, Numerics and Applications. In Proceedings of the 12th International Conference Held in University of Maryland, College Park, MD, USA, 5–10 July 1998.
3. Acheritogaray, M.; Degond, P.; Frouvelle, A.; Liu, J. Kinetic formulation and global existence for the Hall-Magneto-hydrodynamics system. *Kinet. Relat. Models* **2011**, *4*, 901–918. [CrossRef]
4. Deng, X.; Matsumoto, H. Rapid magnetic reconnection in the Earth's magnetosphere mediated by whistler waves. *Nature* **2001**, *410*, 557–560. [CrossRef]
5. Ma, Z.; Bhattacharjee, A. Hall magnetohydrodynamic reconnection: The geospace environment modeling challenge. *J. Geophys. Res.* **2001**, *106*, 3773–3782. [CrossRef]
6. Chae, D.; Degond, P.; Liu, J. Well-posedness for Hall-magnetohydrodynamics. *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **2014**, *31*, 555–565. [CrossRef]
7. Chae, D.; Lee, J. On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics. *J. Differ. Equ.* **2014**, *256*, 3835–3858. [CrossRef]
8. Chae, D.; Schonbek, M. On the temporal decay for the Hall-magnetohydrodynamic equations. *J. Differ. Equ.* **2013**, *255*, 3971–3982. [CrossRef]
9. Chae, D.; Wolf, J. Regularity of the 3D stationary hall magnetohydrodynamic equations on the plane. *Comm. Math. Phys.* **2017**, *354*, 213–230. [CrossRef]
10. Dai, M. Regularity criterion for the 3D Hall-magneto-hydrodynamics. *J. Differ. Equ.* **2016**, *261*, 573–591. [CrossRef]
11. Fan, J.; Jia, X.; Nakamura, G.; Zhou, Y. On well-posedness and blowup criteria for the magnetohydrodynamics with the Hall and ion-slip effects. *Z. Angew. Math. Phys.* **2015**, *66*, 1695–1706. [CrossRef]
12. Wan, R.; Zhou, Y. On global existence, energy decay and blow-up criteria for the Hall-MHD system. *J. Differ. Equ.* **2015**, *259*, 5982–6008. [CrossRef]
13. Weng, S. Space-time decay estimates for the incompressible viscous resistive MHD and Hall-MHD equations. *J. Funct. Anal.* **2016**, *270*, 2168–2187. [CrossRef]
14. Weng, S. On analyticity and temporal decay rates of solutions to the viscous resistive Hall-MHD system. *J. Differ. Equ.* **2016**, *260*, 6504–6524. [CrossRef]
15. Wan, R.; Zhou, Y. Global well-posedness, BKM blow-up criteria and zero h limit for the 3D incompressible Hall-MHD equations. *J. Differ. Equ.* **2019**, *267*, 3724–2747. [CrossRef]

16. Fan, J.; Ahmad, B.; Hayat, T.; Zhou, Y. On blow-up criteria for a new Hall-MHD system. *Appl. Math. Comput.* **2016**, *274*, 20–24. [CrossRef]
17. Ma, C.; Jiang, Z.; Zhu, M. Global existence and zero α limit of the 3D incompressible two-fluid MHD equations. *Math. Methods Appl. Sci.* **2017**, *40*, 5760–5767. [CrossRef]
18. Schonbek, M. L^2 decay for weak solutions of the Navier-Stokes equations. *Arch. Ration. Mech. Anal.* **1985**, *88*, 209–222. [CrossRef]
19. Schonbek, M. Large time behaviour of solutions to the Navier-Stokes equations. *Commun. Partial Differ. Equ.* **1986**, *11*, 733–763. [CrossRef]
20. Schonbek, M.; Schonbek, T.; Süli, E.; Large-time behaviour of solutions to the magneto-hydrodynamics equations. *Math. Ann.* **1996**, *304*, 717–756. [CrossRef]
21. Brandolese, L.; Schonbek, M. Large time decay and growth for solutions of a viscous Boussinesq system. *Trans. Am. Math. Soc.* **2012**, *364*, 5057–5090. [CrossRef]
22. Dai, M.; Schonbek, M. Asymptotic behavior of solutions to Liquid crystal systems in $\mathbb{R}^3$. *Commun. Partial Differ. Equ.* **2012**, *37*, 2138–2164. [CrossRef]
23. Niche, C. Decay characterization of solutions to Navier-Stokes-Voigt equations in terms of the initial datum. *J. Differ. Equ.* **2016**, *260*, 4440–4453. [CrossRef]
24. Ferreira, L.; Niche, C.; Planas, G. Decay of solutions to dissipative modified quasi-geostrophic equations. *Proc. Am. Math. Soc.* **2017**, *145*, 287–301. [CrossRef]
25. Zhao, C.; Liang, Y.; Zhao, M. Upper and lower bounds of time decay rate of solutions for a class of third grade fluids in $\mathbb{R}^3$. *Nonlinear Anal.-RWA* **2014**, *15*, 229–238. [CrossRef]
26. Bjorland, C.; Schonbek, M. Poincaré's inequality and diffusive evolution equations. *Adv. Differ. Equ.* **2009**, *14*, 241–260.
27. Niche, C.; Schonbek, M. Decay characterization of solutions to dissipative equations. *J. Lond. Math. Soc.* **2015**, *91*, 573–595. [CrossRef]
28. Brandolese, L. Characterization of solutions to dissipative systems with sharp algebraic decay. *SIAM J. Math. Anal.* **2016**, *48*, 1616–1633. [CrossRef]
29. Zhao, X. Decay of solutions to a new Hall-MHD system in $\mathbb{R}^3$. *C. R. Acad. Sci. Paris Ser. I* **2017**, *355*, 310–317. [CrossRef]
30. Zhao, X. Asymptotic behavior of solutions to a new Hall-MHD system. *Acta Appl. Math.* **2018**, *157*, 205–216. [CrossRef]
31. Zhao, X. Global well-posedness and decay characterization of solutions to 3D MHD equations with Hall and ion-slip effects. *Z. Angew. Math. Phys.* **2020**, *71*, 89. [CrossRef]
32. Zhao, X. On the well-posdeness and decay characterization of solutions for incompressible electron inertial Hall-MHD equations. *Adv. Differ. Equ.* **2020**, *25*, 373–422.
33. Anh, C.; Trang, P. Decay characterization of solutions to the viscous Camassa-Holm equations. *Nonlineary* **2018**, *31*, 621–650. [CrossRef]

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