mathematics

Article

Coefficient Estimates for Bi-Univalent Functions in Connection with Symmetric Conjugate Points Related to Horadam Polynomial

S. Melike Aydoğan [1,*] and Zeliha Karahüseyin [2]

[1] Department of Mathematics, Istanbul Technical University, 34467 Istanbul, Turkey
[2] Department of Mathematics and Computer Science, Istanbul Kültür University, 34158 Istanbul, Turkey; karahuseyinzeliha@gmail.com
* Correspondence: aydogansm@itu.edu.tr

Received: 8 June 2020; Accepted: 30 September 2020; Published: 31 October 2020

Abstract: In the current study, we construct a new subclass of bi-univalent functions with respect to symmetric conjugate points in the open disc E, described by Horadam polynomials. For this subclass, initial Maclaurin coefficient bounds are acquired. The Fekete–Szegö problem of this subclass is also acquired. Further, some special cases of our results are designated.

Keywords: bi-univalent functions; symmetric conjugate points; horadam polynomial; Fekete–Szegö problem

MSC: 30C45

1. Introduction

Let $\mathcal{A}$ represent the class of all functions which are analytic and given by the following form

$$s(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $E = \{z : z \in \mathbb{C},\ |z| < 1\}$. Let S be class of all functions belonging to $\mathcal{A}$ which are univalent and hold the conditions of normalized $s(0) = s'(0) - 1 = 0$ in E.

For the functions s and r in E analytic, it is known that the function s is subordinate to r in E given by $s(z) \prec r(z)$, $(z \in E)$, if there is an analytic Schwarz function $w(z)$ given in E with the conditions

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad \text{for all} \quad z \in E,$$

such that $s(z) = r(w(z))$ for all $z \in E$.

Moreover, it is given by

$$s(z) \prec r(z) \quad (z \in E) \Leftrightarrow s(0) = r(0) \quad \text{and} \quad s(E) \subset r(E)$$

when r is univalent. By the Koebe one-quarter theorem, we know that the range of every function which belongs to $\mathcal{S}$ contains the disc $\{w : \ |w| < \frac{1}{4}\}$ [1]. Therefore, it is obvious that every univalent function s has an inverse s^{-1}, introduced by

$$s(s^{-1}(z)) = z \quad (z \in E),$$

and

$$s(s^{-1}(w)) = w \quad \left(|w| < r_0(s);\ r_0(s) \geq \frac{1}{4}\right),$$

where

$$s^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{2}$$

A function $s \in \mathcal{A}$ is said to be bi-univalent in E if both $s(z)$ and $s^{-1}(z)$ are univalent in E. The class of all functions $s \in \mathcal{A}$, such that s and $s^{-1} \in \mathcal{A}$ are both univalent in E, will be denoted by σ.

In 1967, the class σ of bi-univalent functions was first enquired by Lewin [2] and it was derived that $|a_2| < 1.51$. Brannan and Taha [3] also considered subclasses of bi-univalent functions, and acquired estimates of initial coefficients. In 2010, Srivastava et al. [4] investigated various classes of bi-univalent functions. Moreover, many authors (see [5–9]) have introduced subclasses for bi-univalent functions.

We define the class $S^*(\varphi)$ of starlike functions and the class $K(\varphi)$ of convex functions by

$$S^*(\varphi) = \left\{s : s \in \mathcal{A},\ \frac{zs'(z)}{s(z)} \prec \varphi(z)\right\},\ z \in E,$$

and

$$K(\varphi) = \left\{s : s \in \mathcal{A},\ 1 + \frac{zs''(z)}{s(z)} \prec \varphi(z)\right\},\ z \in E.$$

These classes were described and studied by Ma and Minda [10].

It is especially clear that $K = K(0)$ and $S^* = S^*(0)$.

It is also obvious that if $s(z) \in K$, then $zs'(z) \in S^*$.

El-Ashwah and Thomas [11] presented the class S^*_{sc} of functions known as starlike with respect to symmetric conjugate points. This class consists of the functions $s \in S$, satisfying the inequality

$$Re\left\{\frac{zs'(z)}{s(z) - \overline{s(-\bar{z})}}\right\} > 0, \quad z \in E.$$

A function $s \in S$ is said to be convex with respect to symmetric conjugate points if

$$Re\left\{\frac{(zs'(z))'}{(s(z) - \overline{s(-\bar{z})})'}\right\} > 0, \quad z \in E.$$

The class of all convex functions with respect to symmetric conjugate points is denoted by C_{sc}.

The Horadam polynomials $h_n(x)$ are given by the iteration relation (see [12])

$$h_n(x) = kxh_{n-1}(x) + lh_{n-2}(x), \quad (n \in \mathbb{N} \geq 2), \tag{3}$$

with $h_1(x) = c$, $h_2(x) = dx$, and $h_3(x) = kdx^2 + cl$, where c, d, k, l are some real constants.

Some special cases regarding Horadam polynomials can be found in [12]. For further knowledge related to Horadam polynomials, see [13–16].

Remark 1. ([9,12]). *Let $\Omega(x,z)$ be the generating function of the Horadam polynomials $h_n(x)$. At that time*

$$\Omega(x,z) = \frac{c + (d - ck)xz}{1 - kxz - lz^2} = \sum_{n=1}^{\infty} h_n(x)z^{n-1}. \tag{4}$$

We took our motivation from the paper written by Wanas and Majeed [17]. They obtained coefficient estimates using Chebyshev polynomials, but in our study we used Horadam Polynomials instead.

In the present paper, we introduce a new subclass of bi-univalent functions with respect to symmetric conjugate points by handling the Horadam polynomials $h_n(x)$ and the generating function $\Omega(x,z)$. Moreover, we find the initial coefficients and the problem of Fekete–Szegö for functions in this new subclass. Some special cases related to our results were also acquired.

2. Main Results

Definition 1. *For $0 < \alpha \leq 1$, a function $s \in \sigma$ is belong to the class $\mathcal{F}_{\sigma}^{sc}(\alpha, x)$ if it satisfies the following conditions*

$$\begin{aligned} \frac{2zs'(z)}{s(z)-\overline{s(-\overline{z})}} &+ \frac{2(zs'(z))'}{(s(z)-\overline{s(-\overline{z})})'} \\ &- \frac{2\alpha z^2 s''(z) + 2zs'(z)}{\alpha z(s(z)-\overline{s(-\overline{z})})' + (1-\alpha)(s(z)-\overline{s(-\overline{z})})} \\ &\prec \Omega(x,z) + 1 - c \end{aligned} \tag{5}$$

and

$$\begin{aligned} \frac{2wr'(w)}{r(w)-\overline{r(-\overline{w})}} &+ \frac{2(wr'(w))'}{(r(w)-\overline{r(-\overline{w})})'} \\ &- \frac{2\alpha w^2 r''(w) + 2wr'(w)}{\alpha w(r(w)-\overline{r(-\overline{w})})' + (1-\alpha)(r(w)-\overline{r(-\overline{w})})} \\ &\prec \Omega(x,w) + 1 - c \end{aligned} \tag{6}$$

where c, d, and l are real constants as in (3), and r is the extension of s^{-1}, presented by (2).

In particular, if we set $\alpha = 0$, we obtain the class $\mathcal{F}_{\sigma}^{sc}(0,x) = F_{\sigma}^{sc}(x)$, which holds the following conditions:

$$\frac{2(zs'(z))'}{(s(z)-\overline{s(-\overline{z})})'} \prec \Omega(x,z) + 1 - c$$

and

$$\frac{2(wr'(w))'}{(r(w)-\overline{r(-\overline{w})})'} \prec \Omega(x,w) + 1 - c,$$

where the function $r = s^{-1}$ is presented by (2).

We prove that our first theorem includes initial coefficients of the class $\mathcal{F}_{\sigma}^{sc}(\alpha, x)$.

Theorem 1. *Let the function $s \in \sigma$ denoted by (1) belong to the class $\mathcal{F}_{\sigma}^{sc}(\alpha, x)$. Then*

$$|a_2| \leq \frac{|dx|\sqrt{|dx|}}{\sqrt{2\,|[(3-2\alpha)d - 2(2-\alpha)^2 k]dx^2 - 2(2-\alpha)^2 cl|}} \tag{7}$$

and

$$|a_3| \leq \frac{|dx|}{2(3-2\alpha)} + \frac{(dx)^2}{4(2-\alpha)^2} \tag{8}$$

Proof. Let $s \in \sigma$ be presented by Maclaurin expansion (1). Let us consider the functions Ψ and Φ, which are analytic, and satisfy $\Psi(0) = \Phi(0) = 0$, $|\Psi(w)| < 1$ and $|\Phi(z)| < 1$, $z, w \in E$. Note that if

$$|\Phi(z)| = |p_1 z + p_2 z^2 + p_3 z^3 + \ldots| < 1 \quad (z \in E)$$

and

$$|\Psi(w)| = |q_1 w + q_2 w^2 + q_3 w^3 + \ldots| < 1 \quad (w \in E),$$

then

$$|p_i| \leq 1 \quad \text{and} \quad |q_i| \leq 1 \quad (i \in \mathbb{N}).$$

In light of Definition 1, we have

$$\begin{aligned} \frac{2zs'(z)}{s(z) - \overline{s(-\overline{z})}} &+ \frac{2(zs'(z))'}{(s(z) - \overline{s(-\overline{z})})'} \\ &- \frac{2\alpha z^2 s''(z) + 2zs'(z)}{\alpha z(s(z) - \overline{s(-\overline{z})})' + (1-\alpha)(s(z) - \overline{s(-\overline{z})})} \\ &= \Omega(x, \Phi(z)) + 1 - c \end{aligned}$$

and

$$\begin{aligned} \frac{2wr'(w)}{r(w) - \overline{r(-\overline{w})}} &+ \frac{2(wr'(w))'}{(r(w) - \overline{r(-\overline{w})})'} \\ &- \frac{2\alpha w^2 r''(w) + 2wr'(w)}{\alpha w(r(w) - \overline{r(-\overline{w})})' + (1-\alpha)(r(w) - \overline{r(-\overline{w})})} \\ &= \Omega(x, \Psi(w)) + 1 - c \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{2zs'(z)}{s(z) - \overline{s(-\overline{z})}} &+ \frac{2(zs'(z))'}{(s(z) - \overline{s(-\overline{z})})'} \\ &- \frac{2\alpha z^2 s''(z) + 2zs'(z)}{\alpha z(s(z) - \overline{s(-\overline{z})})' + (1-\alpha)(s(z) - \overline{s(-\overline{z})})} \\ &= 1 + h_1(x) - c + h_2(x)\Phi(z) + h_3(x)[\Phi(z)]^3 + \cdots \end{aligned} \tag{9}$$

and

$$\begin{aligned} \frac{2wr'(w)}{r(w) - \overline{r(-\overline{w})}} &+ \frac{2(wr'(w))'}{(r(w) - \overline{r(-\overline{w})})'} \\ &- \frac{2\alpha w^2 r''(w) + 2wr'(w)}{\alpha w(r(w) - \overline{r(-\overline{w})})' + (1-\alpha)(r(w) - \overline{r(-\overline{w})})} \\ &= 1 + h_1(x) - c + h_2(x)\Psi(w) + h_3(x)[\Psi(w)]^3 + \cdots \end{aligned} \tag{10}$$

If $\Phi(z) = p_1 z + p_2 z^2 + p_3 z^3 + \cdots \quad (z \in E)$ and $\Psi(w) = q_1 w + q_2 w^2 + q_3 w^3 + \cdots \quad (w \in E)$, from the equalities of (9) and (10), we obtain

$$\begin{aligned} \frac{2zs'(z)}{s(z)-\overline{s(-\bar{z})}} &+ \frac{2(zs'(z))'}{(s(z)-\overline{s(-\bar{z})})'} \\ &- \frac{2\alpha z^2 s''(z)+2zs'(z)}{\alpha z(s(z)-\overline{s(-\bar{z})})'+(1-\alpha)(s(z)-\overline{s(-\bar{z})})} \\ &= 1+h_2(x)p_1 z+\left[h_2(x)p_2+h_3(x)p_1^2\right]z^2+\cdots \end{aligned} \tag{11}$$

and

$$\begin{aligned} \frac{2wr'(w)}{r(w)-\overline{r(-\bar{w})}} &+ \frac{2(wr'(w))'}{(r(w)-\overline{r(-\bar{w})})'} \\ &- \frac{2\alpha w^2 r''(w)+2wr'(w)}{\alpha w(r(w)-\overline{r(-\bar{w})})'+(1-\alpha)(r(w)-\overline{r(-\bar{w})})} \\ &= 1+h_2(x)q_1 w+\left[h_2(x)q_2+h_3(x)q_1^2\right]w^2+\cdots \end{aligned} \tag{12}$$

Thus, upon equating the coincident coefficients in (11) and (12), after some basic calculations, we acquired

$$2(2-\alpha)a_2 = h_2(x)p_1 \tag{13}$$

$$2(3-2\alpha)a_3 = h_2(x)p_2 + h_3(x)p_1^2 \tag{14}$$

$$-2(2-\alpha)a_2 = h_2(x)q_1 \tag{15}$$

$$2(3-2\alpha)(2a_2^2-a_3) = h_2(x)q_2 + h_3(x)q_1^2 \tag{16}$$

From (13) and (15), we obtain that

$$p_1 = -q_1 \tag{17}$$

and

$$8(2-\alpha)^2 a_2^2 = h_2^2(x)(p_1^2+q_1^2). \tag{18}$$

Furthermore, by using (16) and (14), we obtain

$$4(3-2\alpha)a_2^2 = h_2(x)(p_2+q_2) + h_3(x)(p_1^2+q_1^2). \tag{19}$$

By using (18) in (19), we get

$$\left[4(3-2\alpha) - h_3(x)\frac{8(2-\alpha)^2}{h_2^2(x)}\right] a_2^2 = h_2(x)(p_2+q_2). \tag{20}$$

From (3) and (20), we acquired the result which is desired in (7).

Later, in order to derive the coefficient bound on $|a_3|$, by subtracting (16) from (14)

$$-4(3-2\alpha)(a_2^2-a_3) = h_2(x)(p_2-q_2) + h_3(x)(p_1^2-q_1^2)$$

and using (17) and (18), we have

$$\frac{-4(3-2\alpha)h_2^2(x)(p_1^2+q_1^2)}{8(2-\alpha)^2}+4(3-2\alpha)a_3=h_2(x)(p_2-q_2)$$

$$a_3=\frac{h_2(x)(p_2-q_2)}{4(3-2\alpha)}+\frac{h_2^2(x)(p_1^2+q_1^2)}{8(2-\alpha)^2}. \tag{21}$$

Hence, using (17) and applying (3), we obtain the desired result in (8). □

For $\alpha=0$ the class $\mathcal{F}_\sigma^{sc}(\alpha,x)$ reduced to the class $\mathcal{F}_\sigma^{sc}(x)$. The following corollary belongs to reduced class $\mathcal{F}_\sigma^{sc}(x)$.

Corollary 1. *Let the function $s\in\sigma$, presented by (1), belong to the class $\mathcal{F}_\sigma^{sc}(x)$. Then*

$$|a_2|\leq\frac{|dx|\sqrt{|dx|}}{\sqrt{2|(3d-8k)dx^2-8cl|}} \tag{22}$$

$$|a_3|\leq\frac{|dx|}{6}+\frac{(dx)^2}{16}. \tag{23}$$

3. Fekete–Szegö Problem

For $s\in S$, $\left|a_3-\xi a_2^2\right|$ is the Fekete–Szegö functional, well-known for its productive history in the area of GFT. It started from the disproof by Fekete and Szegö [18] conjecture of Littlewood and Paley, suggesting that the coefficients of odd univalent functions are restricted by unity.

Theorem 2. *For $0<\alpha\leq 1$ and $\xi\in\mathbb{R}$, let s, given by (1), be in the class $\mathcal{F}_\sigma^{sc}(\alpha,x)$. Then*

$$\left|a_3-\xi a_2^2\right|\leq\begin{cases}\frac{|dx|}{2(3-2\alpha)} & ;\text{for } |\xi-1|\leq 1-\frac{2(2-\alpha)^2(kdx^2+cl)}{(3-2\alpha)(dx)^2}\\[2ex] \frac{|dx|^3|1-\xi|}{|2(3-2\alpha)(dx)^2-4(2-\alpha)^2(kdx^2+cl)|} & ;\text{for } |\xi-1|\geq 1-\frac{2(2-\alpha)^2(kdx^2+cl)}{(3-2\alpha)(dx)^2}.\end{cases}$$

Proof. It follows from (20) and (21) that

$$\begin{aligned}a_3-\xi a_2^2 &= \frac{[h_2(x)]^3(1-\xi)(p_2+q_2)}{4(3-2\alpha)h_2^2(x)-8(2-\alpha)^2h_3(x)}+\frac{h_2(x)(p_2-q_2)}{4(3-2\alpha)}\\ &= h_2(x)\left[\left(\Theta(\xi,x)+\frac{1}{4(3-2\alpha)}\right)p_2+\left(\Theta(\xi,x)-\frac{1}{4(3-2\alpha)}\right)q_2\right],\end{aligned}$$

where

$$\Theta(\xi,x)=\frac{[h_2(x)]^2(1-\xi)}{4(3-2\alpha)h_2^2(x)-8(2-\alpha)^2h_3(x)}.$$

Thus, we conclude that

$$\left|a_3-\xi a_2^2\right|\leq\begin{cases}\frac{|h_2(x)|}{2(3-2\alpha)} & ,|\Theta(\xi,x)|\leq\frac{1}{4(3-2\alpha)}\\[2ex] 2|h_2(x)||\Theta(\xi,x)| & ,|\Theta(\xi,x)|\geq\frac{1}{4(3-2\alpha)}.\end{cases}$$

In this way, the proof of Theorem 2 is completed. □

For $\alpha = 0$ the class $\mathcal{F}_{\sigma}^{sc}(\alpha, x)$ reduced to the class $\mathcal{F}_{\sigma}^{sc}(x)$. The following corollary belongs to reduced class $\mathcal{F}_{\sigma}^{sc}(x)$.

Corollary 2. *For $\xi \in \mathbb{R}$, let s, presented by (1), belong to the class $\mathcal{F}_{\sigma}^{sc}(x)$. Then*

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \frac{|dx|}{6} & ; \text{for } |\xi - 1| \leq 1 - \frac{8(kdx^2+cl)}{3(dx)^2} \\ \frac{|dx|^3|1-\xi|}{|6(dx)^2-16(kdx^2+cl)|} & ; \text{for } |\xi - 1| \geq 1 - \frac{8(kdx^2+cl)}{3(dx)^2}. \end{cases}$$

Upon taking $\xi = 1$ in Theorem 2, we easily acquire the corollary given below

Corollary 3. *For $0 < \alpha \leq 1$, let s, presented by (1), belong to the class $\mathcal{F}_{\sigma}^{sc}(\alpha, x)$. Then*

$$\left|a_3 - a_2^2\right| \leq \frac{|dx|}{2(3 - 2\alpha)}.$$

Remark 2. *Different subclasses and results were obtained for some special cases of parameters in our results, such as corollaries. Furthermore, when we take $d = 2, k = 2, c = -1, l = 1$, in our results, it can be seen that these results enhance the study by Wanas and Majeed [17].*

Author Contributions: Data curation, S.M.A. and Z.K.; Funding acquisition, S.M.A.; Methodology, Z.K.; Resources, S.M.A.; Software, Z.K; Supervision, S.M.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Duren, P.L. *Univalent Functions*; Springer: New York, NY, USA, 1983.
2. Lewin, M. On a coefficient problem for bi-univalent functions. *Proc. Am. Math. Soc.* **1967**, *18*, 63–68. [CrossRef]
3. Brannan, D.A.; Taha, T.S. On some classes of bi-univalent functions. In *Mathematical Analysis and Its Applications*; Mazhar, S.M., Hamoul, A., Faour, N.S., Eds.; Elsevier Science Limited: Oxford, UK, 1988; pp. 53–60.
4. Srivastava, H.M.; Mishra, A.K.; Gochhayat, P. Certain subclasses of analytic and bi-univalent function. *Appl. Math. Lett.* **2010**, *23*, 1188–1192. [CrossRef]
5. Alamoush, A.G.; Darus, M. Coefficient bounds for new subclasses of bi-univalent functions using Hadamard products. *Acta Univ. Apulensis* **2014**, *38*, 153–161.
6. Akgül, A.; Sakar, F.M. A certain subclass of bi-univalent analytic functions introduced by means of the q-analogue of Noor integral operator and Horadam polynomials. *Turk. J. Math.* **2019**, *43*, 2275–2286. [CrossRef]
7. Altınkaya, Ş.; Yalçın, S. Coefficient estimates for two new subclasses of bi-univalent functions with respect to symmetric points. *J. Funct. Spaces* **2015**, *2015*, 145242.
8. Sakar, F.M. Estimating Coefficients for Certain Subclasses of Meromorphic and Bi-univalent Functions. *J. Inequal. Appl.* **2018**, *2018*, 283. [CrossRef] [PubMed]
9. Sakar, F.M.; Aydoğan, S.M. Initial Bounds for Certain Subclasses of Generalized Salagean Type Bi-univalent Functions Associated with the Horadam Polynomials. *Qual. Meas. Anal.* **2019**, *15*, 89–100.
10. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In Proceedings of the Conference on Complex Analysis (Nankai Institute of Mathematics), Tianjin, China, 19–23 June 1992; pp. 157–169.

11. El-Ashwah, R.M.; Thomas, D.K. Some subclasses of close-to-convex functions. *J. Ramanujan Math. Soc.* **1987**, *2*, 85–100.
12. Horcum, T.; Kocer, E.G. On some properties of Horadam Polynomials. *Internat. Math. Forum* **2009**, *4*, 1243–1252.
13. Horadam, A.F. Jacobsthal Representation Polynomials. *Fibonacci Q.* **1997**, *35*, 137–148.
14. Horadam, A.F.; Mahon, J.M. Pell and Pell-Lucas Polynomials. *Fibonacci Q.* **1985**, *23*, 7–20.
15. Koshy, T. *Fibonacci and Lucas Numbers with Applications*; A Wiley-Interscience Publication: New York, NY, USA, 2001.
16. Lupas, A. A Guide of Fibonacci and Lucas Polynomials. *Octagon Math. Mag.* **1999**, *7*, 2–12.
17. Wanas, A.K.; Majeed, A.H. Chebyshev polynomial bounded for analytic and bi-univalent functions with respect to symmetric conjugate points. *Appl. Math. E-Notes* **2019**, *19*, 14–21.
18. Fekete, M.; Szegö, G. Eine bemerkung uber ungerade schlichte funktionen. *J. Lond. Math. Soc.* **1933**, *8*, 85–89. [CrossRef]