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Topologically Stable Chain Recurrence Classes for Diffeomorphisms

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Abstract: Let $f : M \to M$ be a diffeomorphism of a finite dimension, smooth compact Riemannian manifold M. In this paper, we demonstrate that if a diffeomorphism f lies within the C^1 interior of the set of all chain recurrence class-topologically stable diffeomorphisms, then the chain recurrence class is hyperbolic.

Keywords: topologically stable; chain recurrence class; generic; hyperbolic

MSC: 37C75; 37C15

1. Introduction

Through the paper we assume that M is a finite dimensional, smooth, compact, and boundaryless Riemannian manifold and $f : M \to M$ is a C^1 diffeomorphism. Let d be the distance in M induced from a Riemannian metric $\|\cdot\|$ in the tangent bundle TM. A closed subset Λ of M is *hyperbolic* if Λ is f-invariant and there is an invariant splitting $T_xM = E^s_x \oplus E^u_x$ for each $x \in \Lambda$, a constant $\lambda > 1$ such that:

(a) $\|Df_x(u)\| \le \lambda^{-1}\|u\|$ for $x \in \Lambda$ and $u \in E^s_x$, and
(b) $\|Df_x(v)\| \ge \lambda\|v\|$ for $x \in \Lambda$ and $v \in E^u_x$.

Notice that a diffeomorphism f of M is *Anosov* if M is hyperbolic for f. Let $Hom(M)$ be the set of all homeomorphisms of M. A diffeomorphism f is *topologically stable* if for any positive ϵ, there is a $\delta > 0$ such that any $g \in Hom(M)$ with $d_0(f,g) < \delta$, there is a continuous map $h : M \to M$ for which $h \circ g = f \circ h$ and $d_0(h, id) < \epsilon$, where $d_0(f,g) = \sup\{d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x)) : x \in M\}$,, and id is the identity map. Note that if $f, g : M \to M$ are $C^r (r \ge 1)$ diffeomorphisms then we define the C^r distance between f and g to be:

$$d_r(f,g) = \sup\{|f(x) - g(x)|, |Df(x) - Dg(x)|, \ldots, |D^r f(x) - D^r g(x)| : x \in M\},$$

where $|\cdot|$ is the operator norm.

Walters [1] proved that if a diffeomorphism f is Anosov, then f is topologically stable. A periodic point p with a period $\pi(p)$ is *hyperbolic* if $D_p f^{\pi(p)}$ has no eigenvalues with a norm of 1. We define C^1 immersed manifolds $W^s(p)$, which are called the *stable manifolds* of p, and $W^u(p)$, which are called the *unstable manifolds* of p, as follows: $W^s(p) = \{x \in M : f^{n\pi(p)}(x) \to p \text{ as } n \to \infty\}$ and $W^u(p) = \{x \in M : f^{n\pi(p)}(x) \to p \text{ as } n \to \infty\}$. $P(f)$ denotes the set of all periodic points of f. A diffeomorphism f satisfies *Axiom A* if the non-wandering set $\Omega(f)$ is hyperbolic and comprises the closure of $P(f)$. A diffeomorphism f satisfies the *strong transversality condition* if for any hyperbolic $p, q \in P(f)$, the stable and unstable manifolds $W^s(p)$ and $W^u(q)$ are transverse. A diffeomorphism f is *structurally stable* if there is a C^1 neighborhood $\mathcal{U}(f)$ such that there is a given $g \in \mathcal{U}(f)$, and there is a homeomorphism $h : M \to M$ such that $h \circ f = g \circ h$. For the sake of simplicity, we write that any g C^1 in the neighborhood of f, g is equivalent to f. Robinson proved in [2] that a diffeomorphism f is structurally stable if and only if it satisfies Axiom A and the strong transversality condition. Nitecki proved in [3] that if a diffeomorphism f is structurally stable, then f is topologically stable. Moriyasu proved in [4] that if a diffeomorphism f lies within the C^1 interior of the set of all topologically stable diffeomorphisms, then it is structurally stable. If a diffeomorphism f satisfies Axiom A, then $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_m$, where Λ_i are closed, disjoint, and invariant sets, and each Λ_i contains dense periodic orbits. The sets $\Lambda_1, \ldots, \Lambda_m$ are referred to as *basic sets*.

We say that a diffeomorphism f is Ω-*stable* if for every diffeomorphism g in the neighborhood of f, $g|_{\Omega(g)}$ is equivalent to $f|_{\Omega(f)}$, where $f|_{\Omega(f)} : \Omega(f) \to \Omega(f)$. Smale [5] proved that if a diffeomorphism f is Axiom A and has no-cycles, then it is Ω-stable. Conversely, Palis [6] proved that if a diffeomorphism f is Ω-stable, then f is Axiom A and has no-cycles.

Moriyasu [4] introduced the concept of Ω-topological stability. We say that a diffeomorphism f is Ω-*topologically stable* if for any positive ϵ, there is a positive δ such that given $g \in Hom(M)$ with $d_0(f, g) < \delta$, one can choose a continuous map $h : \Omega(g) \to \Omega(f)(h(\Omega(g)) \subset \Omega(f))$ such that $h \circ g = f \circ h$ on $\Omega(g)$ and $d_0(h, id) < \epsilon$.

Nitecki proved in [3] that if a diffeomorphism f is Axiom A and has no-cycles, then it is Ω-topologically stable. Conversely, Moryasu proved in [4] that if a diffeomorphism f lies within the C^1 interior of the set of all Ω-topologically stable diffeomorphisms, then it is Axiom A and has no-cycles.

For a given $\delta > 0$, a bi-sequence of points $\{x_i\}_{i\in\mathbb{Z}}$ of M is said to be a δ-*pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta \ \forall i \in \mathbb{Z}$. For a given $x, y \in M$, we denote $x \rightsquigarrow y$ if for any $\delta > 0$, there is a finite δ-pseudo orbit (or δ-chain from x to y) $\{x_i\}_{i=0}^{n} (n \geq 1)$ of f such that $x_0 = x$ and $x_n = y$. We denote $x \leftrightsquigarrow y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The set $\{x \in M : x \leftrightsquigarrow x\}$ is referred to as the *chain recurrent set* of f and is denoted as $\mathcal{R}(f)$. It is seen that $\overline{P(f)} \subset \Omega(f) \subset \mathcal{R}(f)$. The relationship $\leftrightsquigarrow$ induces an equivalence relationship on $\mathcal{R}(f)$, whose classes are called *chain recurrence classes* of f and are denoted as C_f. In general, a chain recurrent class C_f is a closed and f-invariant set. It is known that if $\mathcal{R}(f)$ is hyperbolic, then $\mathcal{R}(f) = \overline{P(f)}$. Therefore, if the chain recurrent set $\mathcal{R}(f)$ is hyperbolic, then it satisfies Axiom A.

Let $C_f(p) = \{x \in M : x \leftrightsquigarrow p\}$. Note that if a hyperbolic $p \in P(f)$, then there exist a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of p such that there is a given $g \in \mathcal{U}(f)$, the maximal invariant set $\bigcap_{n\in\mathbb{Z}} g^n(U)$ of f in U consists of a single hyperbolic $p_g \in P(g)$, which it has the same period of p and index$(p) =$ index(p_g), where $P(g)$ is the set of periodic points of g.

Wen and Wen [7] introduced a local version of structural stability. We say that a chain recurrence class $C_f(p)$ is C^1-*structurally stable* if there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that any $g \in \mathcal{U}(f)$, one can take a homeomorphism $h : C_f(p) \to C_g(p_g)$ such that $h \circ f = g \circ h$ on $C_f(p)$, where p_g is a continuation of p and $C_g(p_g)$ is the chain recurrence class of g associated with p_g. Furthermore, they proved that if the codimension one-chain recurrence class $C_f(p)$ is C^1-structurally stable, then it is hyperbolic. Wang [8] proved that if a chain recurrence class $C_f(p)$ is C^1-structurally stable, then it is also hyperbolic, which is a generalization of the result presented by Wen and Wen [7]. Based on the definition below, we consider a local version of the topological stability of the chain recurrence class $C_f(p)$.

For a hyperbolic $p \in P(f)$, and a closed f-invariant set $\Lambda \subset M$, we say that a diffeomorphism f is *chain recurrence class* $C_f(p)$-*topologically stable* if for a given positive ϵ, there is a positive δ such that there is a given $g \in Hom(M)$ with $d_0(f, g) < \delta$, there is a continuous surjective map $h : \Lambda \to C_f(p)$ for which $h \circ g = f \circ h$ on Λ, where $d(h(x), x) < \epsilon$ for all $x \in \Lambda$.

Remark 1. *In the above notion, Λ is a chain recurrence class for g. For any point $x \in C_f(p)$, let $\{x_i : x_0 = x, x_n = p, x_{2n} = x\} \subset C_f(p)$ be a δ-chain from x to x. Since $h(\Lambda) = C_f(p)$, for any $x_i \in C_f(p)$, there is a $y_i \in \Lambda$ such that $h(y_i) = x_i$ for all $i = 0, 1, \ldots, 2n$, where $y_0 = y$. Therefore, we know that:*

$$d(f(x_i), x_{i+1}) = d(f(h(y_i)), h(y_{i+1})) = d(h(g(y_i)), h(y_{i+1})),$$

for all $i = 0, 1, \ldots, 2n$. Since $d(h(x), x) < \epsilon$, one can see that $d(g(y_i), y_{i+1}) < 3\epsilon$ and $\{y_i\}_{n=0}^{2n} \subset \Lambda$, meaning $\{y_i\}_{n=0}^{2n}$ is an 3ϵ-chain from y to y. Therefore, Λ is a chain recurrence class for g.

Remark 2. *In the above notion, because $h : \Lambda \to C_f(p)$ is a continuous surjective map, $h(\Lambda) = C_f(p)$, $h \circ g = f \circ h$ on Λ, and $d(h(x), x) < \epsilon$ for all $x \in \Lambda$, meaning we have $h^{-1}(p) \neq \emptyset$. Therefore, one can take $q \in h^{-1}(p)$. However, $q \in h^{-1}(p)$ is not unique. It is clear that the point q is a periodic point of g. Therefore, we set $q = p_g$. Furthermore, we define $C_g(p_g)$, which is the chain recurrence class of g associated with p_g. Here p_g is not the continuation of p but an element of $h^{-1}(p)$.*

For the property of a continuation, we can see in [7,9]. According to Remark 2, the definition can be written as follows.

Definition 1. Let $p \in P(f)$ be hyperbolic. We say that a diffeomorphism f is *chain recurrence class $C_f(p)$-topologically stable* if for a given positive ϵ, there is a positive δ such that given $g \in Hom(M)$ with $d_0(f, g) < \delta$, there is a continuous subjective map $h : C_g(p_g) \to C_f(p)$ for which $h \circ g = f \circ h$ on $C_g(p_g)$, where $d(h(x), x) < \epsilon$ for all $x \in C_g(p_g)$.

$\mathcal{TS}(C_f(p))$ denotes the set of all chain recurrence class $C_f(p)$-topologically stable diffeomorphism. We say that a diffeomorphism f lies within the C^1 *interior of the set of all $C_f(p)$-topologically stable diffeomorphisms* if there exists a C^1 neighborhood $\mathcal{U}(f)$ of f such that given $g \in \mathcal{U}(f)$, g is $C_g(p_g)$-topologically stable, where $C_g(p_g)$ is the chain recurrence class of g and p_g is the continuation of p. Here, since g is a diffeomorphism, it guarantees that p_g is the continuation of p. Note that in the definition above, g is a homeomorphism, it does not belong to $int\mathcal{TS}(C_g(p_g))$.

It is known that if $C_f(p)$ is C^1-structurally stable then $C_f(p)$ ix topologically stable (see [8]). But, the converse is not true. So, we consider the C^1 interior elements of $C_f(p)$-topologically stable diffeomorphisms.

$int\mathcal{TS}(C_f(p))$ denotes the set of C^1 interior elements of $\mathcal{TS}(C_f(p))$. The following theorem is the main conclusion of our research.

Theorem A *Let $p \in P(f)$ be hyperbolic. If a diffeomorphism $f \in int\mathcal{TS}(C_f(p))$, then $C_f(p)$ is hyperbolic for f.*

2. Proof of Theorem A

Let M be defined as shown previously and let Diff(M) be the set of all diffeomorphisms of M. For a closed f-invariant set $A \subset M$, A is called *normally hyperbolic* for f if there is a Df-invariant splitting $T_\Lambda M = E^s \oplus E^u \oplus TA$ and $\lambda \in (0, 1)$ such that for all $x \in A$:

$$\|D_x f|_{E^s_x}\| < \lambda, \ \|D_x f^{-1}|_{E^u_x}\| < \lambda, \text{ and}$$

$$\|D_x f|_{E^s_x}\| \cdot \|D_{f(x)} f^{-1}|_{D_{f(x)}A}\| < \lambda, \ \|D_x f^{-1}|_{E^u_x}\| \cdot \|D_{f^{-1}(x)} f|_{D_{f^{-1}(x)}A}\| < \lambda.$$

It is known that if $x \in M \setminus A$ then x is hyperbolic point of f.

Remark 3. *For a closed f-invariant set $L \subset M$, if the derivative map $D_x f$ has an eigenvalue λ ($x \in L$) such that $|\lambda| = 1$, then for some g C^1 close to f, we can construct a small closed curve $\mathcal{J}$ such that $g|_{\mathcal{J}} : \mathcal{J} \to \mathcal{J}$ is the identity map, meaning $\mathcal{J}$ is a normally hyperbolic set of g.*

Regarding Remark 3, we have the following.

Lemma 1. *For a diffeomorphism $f : M \to M$, if a closed f-invariant set $\mathcal{I} \subset C_f(p)$ is normally hyperbolic and $f|_{\mathcal{I}} : \mathcal{I} \to \mathcal{I}$ is the identity map, then f is not $C_f(p)$-topologically stable.*

Proof. To derive a contradiction, we assume that f is $C_f(p)$-topologically stable. Let $\mathrm{diam}\mathcal{I} = l$ and take $0 < \epsilon < l/8$. Since f is $C_f(p)$-topologically stable, there is a C^0 neighborhood $\mathcal{U}_0(f)$ of f such that given $g \in \mathcal{U}_0(f)$, there is a continuous surjective map $h : C_g(p_g) \to C_f(p)$ for which $h \circ f = g \circ h$ on $C_g(p_g)$ and $d(h(x), x) < \epsilon$ for all $x \in C_g(p_g)$. For any $x \in \mathcal{I}$, there is a $y \in C_g(p_g)$ such that $h(y) = x$. Since $f|_{\mathcal{I}} : \mathcal{I} \to \mathcal{I}$ is the identity map, for any $x \in \mathcal{I}$, one can see that $f^i(x) = x$ for all $i \in \mathbb{Z}$.

We take $c, d \in C_g(p_g)$ such that (i) $d(c, d) < \epsilon/4$, (ii) $h(c) \in \mathcal{I}$, and $h(d) \in C_f(p) \setminus \mathcal{I}$, and (iii) $d(g^k(c), g^k(d)) < \epsilon$ for some $k \in \mathbb{Z}$.

Let $h(c) = a$ and $h(d) = b$. Since $\mathcal{I} \subset C_f(p)$ is normally hyperbolic and $b \in C_f(p) \setminus \mathcal{I}$, by hyperbolicity of b there is a $j \in \mathbb{Z}$ such that $d(f^j(b), a) = 8\epsilon$. Since $h \circ f = g \circ h$ on $C_g(p_g)$ and $d(h(x), x) < \epsilon$ for all $x \in C_g(p_g)$, we have:

$$\begin{aligned} 8\epsilon = d(f^j(b), a) &= d(f^j(b), f^j(a)) = d(f^j(h(d)), f^j(h(c))) = d(h(g^j(d)), h(g^j(c))) \\ &\le d(h(g^j(d)), g^j(d)) + d(g^j(d), g^j(c)) + d(g^j(c), h(g^j(c))) \\ &< \epsilon + \epsilon + \epsilon = 3\epsilon. \end{aligned}$$

This creates a contradiction. Therefore, f is not $C_f(p)$-topologically stable if $\mathcal{I} \subset C_f(p)$ is normally hyperbolic and $f|_{\mathcal{I}} : \mathcal{I} \to \mathcal{I}$ is the identity map. □

The following lemma is called the Franks' lemma [10]. It plays an essential role in our proofs.

Lemma 2. *For any C^1 neighborhood $\mathcal{U}(f)$ of f, one can take a positive ϵ and C^1 neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that given $g \in \mathcal{U}_0(f)$, there exists a finite set $S = \{x_1, x_2, \cdots, x_N\}$, neighborhood U of S, and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ for which $\|L_i - D_{x_i}g\| \le \epsilon$ ($1 \le i \le N$), one can find a diffeomorphism $g_1 \in \mathcal{U}(f)$ such that:*

(a) $g_1(x) = g(x)$ *if* $x \in S \cup (M \setminus U)$ *and*
(b) $D_{x_i}g_1 = L_i$ *for all* $1 \le i \le N$.

Lemma 3. *Suppose that a diffeomorphism $f \in int\mathcal{TS}(C_f(p))$. Then, every periodic point $q \in C_f(p)$ is hyperbolic.*

Proof. Let $f \in int\mathcal{TS}(C_f(p))$ and let $\mathcal{U}(f)$ be a C^1 neighborhood of f. Suppose that there are $g \in \mathcal{U}(f)$ and a periodic point $q \in C_g(p_g)$ such that q is not hyperbolic. For simplicity, we can assume that $g(q) = q$ (other cases are similar). Since q is not hyperbolic, there is an eigenvalue λ of D_qg such that $|\lambda| = 1$. Note that if all eigenvlaues of D_qg are one then we can get a similar result as this proof. Let E^c_q be the eigenspace corresponding to λ. Then we have a splitting $T_dM = E^s_q \oplus E^c_q \oplus E^u_q$, where E^s_q associated to all eigenvlaues that are less than one and E^u_q associated to all eigenvalues that are greater than one.

First, we consider $\dim E^c_q = 1$. The case means that the eignevlaue λ is real. For simplicity, we assume that $\lambda = 1$ (other cases are similar). Then, by Lemma 2, there is a $\alpha > 0$ and $g_1 \in \mathcal{U}_0(f) \subset \mathcal{U}(f)$ with the following properties:

(i) $g_1(q) = g(q) = q$,
(ii) $g_1(x) = \exp_q \circ D_qg \circ \exp_q^{-1}(x)$ for $x \in B_\alpha(q)$, and
(iii) $g_1(x) = g(x)$ for $x \notin B_{4\alpha}(q)$, where $B_\alpha(q)$ is an α neighborhood of q.

Consider $\alpha_1 < \alpha$ and define $E^c_q(p, \alpha_1) = E^c_q \cap T_qM(\alpha_1)$. Here, $T_qM(\alpha_1) = \{v \in T_qM : \|v\| \le \alpha_1\}$. Therefore, it is clear that $g_1|_{\exp_q(E^c(q,\alpha_1))}$ is the identity map of $\exp_q(E^c(q, \alpha_1))$.

Since g_1 is a diffeomorphism and p_g is a hyperbolic periodic point of g, we have $p_{g_1} \in P(g_1)$ and we can define the chain recurrence class $C_{g_1}(p_{g_1})$ associated with p_{g_1}. Since $g_1 \in \mathcal{TS}(C_{g_1}(p_{g_1}))$, according to Definition 1.4, for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $g_2 \in Hom(M)$ with $d_0(g_1, g_2) < \delta$, there is a continuous map $h : C_{g_2}(p_{g_2}) \to C_{g_1}(p_{g_1})$ such that $d(h(x), x) < \epsilon$, and $g_2 \circ h = h \circ g_1$ for all $x \in C_{g_2}(p_{g_2})$, where $p_{g_2} \in h^{-1}(p_{g_1})$ and $C_{g_2}(p_{g_2})$ is the chain recurrence class associated with p_{g_2}.

Let $\mathcal{J}_q = \exp_q(E^c(q, \delta_1))$. Since $g_1|_{\mathcal{J}_q}$ is the identity map, every point in $\mathcal{J}_q$ is chain transitive, meaning its points are mutually chain equivalent. Therefore, we know that $\mathcal{J}_q \subset C_{g_1}(p_{g_1})$. Furthermore, by Remark 3, we can see that $\mathcal{J}_q$ is a normally hyperbolic set.

Therefore, according to Lemma 1, g_1 is not $C_{g_1}(p_{g_1})$-topologically stable. This creates a contradiction.

Finally, we consider $\dim E^c_q = 2$. The case means that the eigenvalue λ is complex. In this case, to avoid notational complexity, we can assume that $g(q) = q$ for some $g \in \mathcal{U}(f)$. Then, similar to the proof of the first case, we can take $\epsilon_0 > 0$ and g_1 C^1 close to g with the following properties: (i) $g_1(q) = g(q) = q$ and (ii) g_1 has a small arc $\mathcal{L}_q$, where $\mathcal{L}_q = \exp_q(\{t \cdot v_0 : 1 \le t \le 1 + \epsilon_0/4\})$ for some $\epsilon_0 > 0$. The small arc $\mathcal{L}_q$ has the following properties for g_1:

(a) $g_1^i(\mathcal{L}_q) \cap g_1^j(\mathcal{L}_q) = \emptyset$ for $0 \le i \ne j < l - 1$,
(b) $g_1^l(\mathcal{L}_q) = \mathcal{L}_q$ for some $l > 0$, and
(c) $g_1^l|_{\mathcal{L}_q}$ is the identity map.

Since $g_1^l|_{\mathcal{L}_q}$ is the identity map, we can easily show that $\mathcal{L}_q \subset C_{g_1}(p_{g_1})$. Let $g_1^l = g_1$. Then, just as in the argument for the first case, we can derive a contradiction. Therefore, Lemma 3 is proved. □

Let p and q be hyperbolic periodic points. We say that q is *homoclinically related to* p if $W^s(p) \pitchfork W^u(q) \neq \emptyset$ and $W^u(p) \pitchfork W^s(q) \neq \emptyset$. Then, $W^s(p) \pitchfork W^u(q) \neq \emptyset$ and $W^u(p) \pitchfork W^s(q) \neq \emptyset$ are denoted by $p \sim q$. Next, we define $H_f(p) = \overline{\{q \in P_h(f) : q \sim p\}}$, where $P_h(f)$ is the set of all hyperbolic periodic points of f.

Let p be a periodic point with a period $\pi(p)$ of a diffeomorphism f. If $\lambda_1, \lambda_2, \ldots,$ and λ_d are the eigenvalues of $D_p f^{\pi(p)}$, then the numbers $\chi_i = \frac{1}{\pi(p)} \log|\lambda_i| (i = 1, \ldots, d)$ are called the *Lyapunov exponents of* p.

Lemma 4. *A diffeomorphism f in a dense $\mathcal{G}_\delta$ subset $\mathcal{G}$ in* Diff(M) *has the following properties:*

(a) A chain recurrence class $C_f(p)$ is a homoclinical class $H_f(p)$ for some hyperbolic periodic point p (see [11]);
(b) If a homoclinical class $H_f(p)$ is not hyperbolic, then one can find a periodic point q that is homoclinically related to p and has a Lyapunov exponent arbitrarily close to 0 *(see [8]).*

Lemma 5. *Let p be a hyperbolic periodic point of f. If the point q is homoclinically related to p and has a Lyapunov exponent arbitrarily close to* 0*, then there is a g C^1 close to f such that $D_{q_g} g^{\pi(q)}$ has an eigenvalue λ such that $|\lambda| = 1$.*

Proof. Suppose that there is a periodic point q that is homoclinically related to p and has a Lyapunov exponent arbitrarily close to 0. Then, we know that there is an eigenvalue λ_i of $D_q f^{\pi(q)}$ such that λ_i is close to 1 for some $i = 1, \ldots, d$. From Lemma 2, there is a g C^1 close to f such that $D_{q_g} g^{\pi(q_g)}$ has an eigenvalue μ_i such that $|\mu_i| = 1$. □

Proof of Theorem A. By contradiction, suppose that $C_f(p)$ is not hyperbolic. Let $\mathcal{U}(f)$ be a C^1 neighborhood of $f \in$ Diff(M). Since $f \in int\mathcal{TS}(C_f(p))$, there is a $g \in \mathcal{U}(f) \cap \mathcal{G}$ with the following properties:

(i) $g \in \mathcal{TS}(C_g(p_g))$,
(ii) $C_g(p_g) = H_g(p_g)$, and
(iii) There is a hyperbolic periodic point $q \in C_g(p_g)$ with a Lyapunov exponent arbitrarily close to 0 such that $q \sim p_g$.

Since q has a Lyapunov exponent arbitrarily close to 0, from Lemma 5, there is a $g_1 \in \mathcal{U}(g) \subset \mathcal{U}(f) \cap \mathcal{G}$ such that $D_{q_{g_1}} g_1^{\pi(q_{g_1})}$ has an eigenvalue λ such that $|\lambda| = 1$. Just as in the proof of Lemma 3, there is a $g_2 \in \mathcal{U}(g_1) \subset \mathcal{U}(f) \cap \mathcal{G}$ such that g_2 has a small arc $\mathcal{J}_{q_{g_2}}$ centered at q_{g_2}. Then, we have that $g_2^{\pi(q_{g_2})}$ is $\pm id$ on $\mathcal{J}_{q_{g_2}}$ and $\mathcal{J}_{q_{g_2}} \subset C_{g_2}(p_{g_2})$, where id is the identity map. Additionally, $\mathcal{J}_{q_{g_2}}$ is normally hyperbolic. From Lemma 1, this creates a contradiction. Therefore, if $f \in int\mathcal{TS}(C_f(p))$, then $C_f(p)$ is hyperbolic. □

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