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Every Planar Graph with the Distance of 5^--Cycles at Least 3 from Each Other Is DP-3-Colorable

Yueying Zhao [1,2] and Lianying Miao [1,*]

1 School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China; zhaoyueying@cumt.edu.cn
2 School of Mathematics and Statistics, Xuzhou University of Technology, Xuzhou 221000, China
* Correspondence: miaolianying@cumt.edu.cn; Tel.: +86-136-8514-9128

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Abstract: DP-coloring was introduced by Dvořák and Postle [J. Comb. Theory Ser. B 2018, 129, 38–54]. In this paper, we prove that every planar graph in which the 5^--cycles are at distance of at least 3 from each other is DP-3-colorable, which improves the result of Montassier et al. [Inform. Process. Lett. 2008, 107, 3–4] and Yin and Yu [Discret. Math. 2019, 342, 2333–2341].

Keywords: DP-coloring; planar graphs

1. Introduction

All graphs in this paper are finite, undirected, and simple. A planar graph is a graph that can be embedded into the plane. A plane graph is a particular embedding of a planar graph into the plane. We set a plane graph $G = (V, E, F)$, where V, E, and F are sets of vertices, edges, and faces of G, respectively. Two faces are intersecting if they have a common vertex, and are adjacent if they have a common edge. A vertex is incident to a face if it is on the face. A vertex is adjacent to a face if it is not incident to the face but adjacent to a vertex on the face. For a face $f \in F$, if the vertices on f in a cyclic order are $v_1, v_2, \cdots, v_k$, then we write $f = [v_1, v_2, \cdots, v_k]$. The degree $d(x)$ of $x \in V$ is the number of edges incident with x. The degree $d(x)$ of $x \in F$ is the number of vertices incident with x. Let a k-vertex (k^+-vertex, k^--vertex) be a vertex of degree k (at least k, at most k), and a k-face (k^+-face, k^--face) be a face of degree k (at least k, at most k). The same notation will be applied to cycles. A $(l_1, l_2, \cdots, l_k)$-face is a k-face $f = [v_1 v_2 \cdots v_k]$ with $d(v_i) = l_i$, respectively. A (l_1, l_2)-edge is an edge $e = v_1 v_2$ with $d(v_i) = l_i$. Let C be a cycle of a plane graph G, $|C|$ is the length of the cycle C. A triangle is a 3-cycle. An edge or a vertex of G is triangular if it is on a triangle. A chord in a cycle C is triangular if it splits the cycle C into a triangle and a cycle of length $d(C) - 1$. We use $Int(C)$ and $Ext(C)$ to denote the sets of vertices located inside and outside of C, respectively, and put $\overline{Int(C)} = G - Ext(C)$, $\overline{Ext(C)} = G - Int(C)$. The cycle C is called a separating cycle if $Int(C) \neq \emptyset$ and $Ext(C) \neq \emptyset$. A set of independent edges of G is called a matching. Identifying vertices means merging the vertices into a single vertex.

The distance between two vertices u and v in G, denoted by $d_G(u, v)$, is the length (number of edges) of the shortest path between u and v in G. The distance between two cycles C and C' of G, denoted by $d(C, C')$, is defined as follows:

$$d(C, C') = min\{d_G(u, v) : u \in V(C), v \in V(C')\}.$$

A proper k-coloring of G is a function $f : V(G) \to [k] := \{1, 2, \cdots, k\}$ such that for every edge $uv \in E(G)$, $f(u) \neq f(v)$. The smallest k such that G has a k-coloring is called the chromatic number of G and is denoted by $\chi(G)$. A list assignment of a graph G is a mapping L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of colors. An L-coloring of G is a function $\lambda : V \to \bigcup_{v \in V} L(v)$ and $\lambda(v) \in L(v)$ for every $v \in V$ such that $\lambda(u) \neq \lambda(v)$ for every edge $uv \in E$. A graph G is k-choosable if G is L-colorable for every assignment L with $|L(v)| \geq k$. The smallest k such that G is k-choosable is called the choice number of G and is denoted by $\chi_l(G)$.

It is well known that the problem of deciding whether a planar graph is 3-colorable is NP-complete. This provides motivation to look for sufficient conditions for planar graphs to be 3-colorable. Grötzsch [1] showed that every planar graph without triangles is 3-colorable. In 1976, Steinberg [2] conjectured that every planar graph without 4-cycle and 5-cycle is 3-colorable. Havel [3] proposed to make d^{Δ} large enough, where d^{Δ} is the smallest distance between triangles. Dvor̂ák, Kral, and Thomas [4] showed that $d^{\Delta} \geq 10^{100}$ suffices. Borodin and Glebov [5] showed that planar graphs without 5-cycles and $d^{\Delta} \geq 2$ are 3-colorable.

Vizing [6], and, independently Erdős, Rubin, and Taylor [7], introduced list coloring as a generalization of proper coloring. Thomassen [8,9] showed that every planar graph is 5-choosable and every planar graph without $\{3,4\}$-cycles is 3-choosable. Voigt [10] constructed a non-3-choosable planar graph without cycles of length 4 and 5. Montassier, Raspaud, Wang, and Wang [11] gave the following condition for a planar graph to be 3-choosable.

Theorem 1. *Every planar graph with the distance of 5^--cycles at least 4 from each other is 3-choosable.*

For ordinary coloring, the identification of vertices is involved in the reduction configurations. In list coloring, since different vertices may have different lists, it is not possible for one to use the identification of vertices. To overcome this difficulty, Dvořák and Postle [12] introduced DP-coloring (under the name correspondence coloring) as a generalization of list-coloring.

Definition 1. *Let G be a simple graph, and L be a list assignment of $V(G)$. For each vertex $v \in V(G)$, let $L_v = \{v\} \times L(v)$. For each edge uv in G, let M_{uv} be a partial matching between the sets L_u and L_v and let $\mathcal{M} = \{M_{uv} : uv \in E(G)\}$, called the matching assignment. The matching assignment is called a k-matching assignment if $L(v) = [k]$ for each $v \in V(G)$.*

Definition 2. *A cover of G is a graph $\mathcal{G}_{L,\mathcal{M}}$ (simply write $\mathcal{G}$) satisfying the following two conditions:*

(1) the vertex set of $\mathcal{G}$ is the disjoint union of L_v for all $v \in V(G)$;
(2) the edge set of $\mathcal{G}$ is the matching assignment $\mathcal{M}$.

Note that the induced subgraph $\mathcal{G}[L_v]$ is an independent set for each vertex $v \in V(G)$.

Definition 3. *Let G be a simple graph, and $\mathcal{G}$ be a cover of G. An $\mathcal{M}$-coloring of G is an independent set $\mathcal{I}$ in $\mathcal{G}$ such that $|\mathcal{I} \bigcap L_v| = 1$ for each vertex $v \in V(G)$. The graph G is DP-k-colorable if, for each k-list assignment L and each matching assignment $\mathcal{M}$ over L, it has an $\mathcal{M}$-coloring. The minimum k such that G is DP-k-colorable is the DP-chromatic number of G, denoted by $\chi_{DP}(G)$.*

Let $\mathcal{G}_{L,\mathcal{M}}$ be a cover of G. For a vertex $v \in V(G)$, if $c_1 \in L(v)$ and $c_2 \notin L(v)$, then consider the cover $\mathcal{G}'_{L',\mathcal{M}'}$ of G such that $L'(v) = (L(v) \bigcup \{c_2\}) \backslash \{c_1\}$ and $L'(u) = L(u)$ for each $u \in V(G) \backslash \{v\}$. For each $e \in E(\mathcal{G}_{L,\mathcal{M}})$, $\mathcal{M}'$ is obtained form $\mathcal{M}$ by replacing the vertex (v, c_1) by (v, c_2). Thus, $\mathcal{G}'_{L',\mathcal{M}'}$ is obtained from $\mathcal{G}_{L,\mathcal{M}}$ by replacing the vertex (v, c_1) by (v, c_2). Then, G can be $\mathcal{M}'$-colorable when G is $\mathcal{M}$-colorable by changing the color of v to c_2 when $\phi(v) = c_1$, and vice versa. We say that $\mathcal{M}'$ is obtained from $\mathcal{M}$ by renaming at the vertex v.

An edge $uv \in E(G)$ is straight in a k-matching assignment $\mathcal{M}$ if every $(u, c_1)(v, c_2) \in E(M_{u,v})$ satisfies $c_1 = c_2$. One can construct a cover of any graph G based on a list assignment for G, thus showing that list coloring is a special case of DP-coloring and, in particular, $\chi_{DP}(G) \geq \chi_l(G)$ for all graphs G. DP-coloring is quite different from list coloring—for example, Bernshteyn [13] showed that the DP-chromatic number of every graph G with average degree d is $\Omega(d/\log d)$, while Alon [14] proved that $\chi_l(G) = \Omega(\log d)$ and the bound is sharp.

DP-coloring is a generalization of list-coloring. Dvořák and Postle [12] proved that every planar graph G without cycles of length from 4 to 8 is 3-choosable. They also noted that $\chi_{DP}(G) \leq 3$ if G is a planar graph without $\{3, 4\}$-cycles. Liu and Li [15] proved that every planar graph G without adjacent cycles of length at most 8 is 3-choosable.

Much attention was drawn to this new coloring; see, for example, [16–19]. Liu et al. [20,21] gave some sufficient conditions for a planar graph to be DP-3-colorable, and DP-4-colorable planar graphs can be found in [22–24]. Yin and Yu [25] gave the following condition for a planar graph to be DP-3-colorable:

Theorem 2. *Every planar graph without* $\{4,\ 5\}$*-cycles and the distance of triangles at least 3 is DP-3-colorable.*

In this paper, we improve the results in Theorems 1 and 2.

A 9-cycle C is bad if it is in a subgraph of G isomorphic to the graphs in Figure 1, and is the outer cycle of the subgraph. A 9-cycle is good if it is not bad.

(a) One vertex is in *Int(C)* (b) Three vertices are in *Int(C)*

Figure 1. Bad 9-cycles. (**a**) One vertex in Int(C); (**b**) Three vertices in Int(C).

Theorem 3. *Let G be a planar graph in which the 5^--cycles are at distance of at least 3 from each other. Let C_0 be a 8^--cycle or a good 9-cycle in G. Then, each DP-3-coloring of C_0 can be extended to G.*

Corollary 1. *Every planar graph in which the 5^--cycles are at a distance of at least 3 from each other is DP-3-colorable (thus also 3-choosable).*

Proof. Let G be a planar graph. Either G is 4^--cycles free or it is not 4^--cycles free. In the first case, as proved in Reference [12], G is DP-3-colorable. Thus, we only have to consider the case when G contains a 4^--cycle. As proved in Reference [12], the 4^--cycle can be precolored. Then, by Theorem 3, G is DP-3-colorable extended from the coloring of the 4^--cycle when the 5^--cycles are at distance of at least 3 from each other. □

2. Proof of Theorem 3

We will prove Theorem 3 by reductio ad absurdum. Let's start by a temporary assumption that the theorem is wrong. Then, there has to be a non-empty set of counterexamples to this theorem. Assume that G is a minimal (least number of vertices) counterexample to Theorem 3. Let C_0 be a 8^--cycle or a good 9-cycle in G.

Lemma 1. *For each $v \in V(G - C_0)$, $d(v) \geq 3$.*

Proof. Let v be a 2^--vertex in $V(G - C_0)$. By the minimality of G, each DP-3-coloring of C_0 can be extended to $G - v$. Then, the coloring of $G - v$ can be extended to G by selecting a color $\phi(v)$ for v such that, for each neighbor u of v, $((u, \phi(u)(v, \phi(v)) \notin E(M_{uv})$, a contradiction. □

Lemma 2. *There exist no separating 8^--cycles or separating good 9-cycles.*

Proof. First of all, we show that C_0 is not a separating cycle. Otherwise, if C_0 is a separating cycle, we may extend the coloring of C_0 to both $Int(C_0)$ and $Ext(C_0)$, respectively, and then combine them to get a coloring of G, a contradiction.

Let $C \neq C_0$ be a separating 8^--cycle or separating good 9-cycle in G. By the minimality of G, the coloring of C_0 can be extended to $\overline{Ext(C)}$. Now that C is colored, thus the coloring of C can be extended to $Int(C)$ by the minimality of G again. Combining the inside and outside of C, we have a coloring of G extended from the coloring of C_0, a contradiction. □

Lemma 3. *C_0 is the boundary of the out face of the embedding of G.*

Proof. C_0 is not a separating cycle by Lemma 2. Thus, either $Int(C_0)$ or $Ext(C_0)$ is empty. Without loss of generality, we assume that $Int(C_0)$ is empty, and we can redraw the graph to make $Ext(C_0)$ empty instead. □

Lemma 4. *If a 9^--cycle C in G has an internal chord e, then $|C| \in \{7,8,9\}$ and either e is triangular, or $|C| = 8$ and e splits C into a 4-cycle and a 6-cycle, or $|C| = 9$ and e splits C into a 4-cycle and a 7-cycle, or $|C| = 9$ and e splits C into a 5-cycle and a 6-cycle.*

Proof. Due to fact that the cycles of lengths 3, 4, and 5 in G are at a distance of at least 3 from each other, C cannot have a chord if $|C| \leq 6$ and can have only a triangular one when $|C| = 7$. If $|C| = 8$ and e is not triangular, then e splits C into a 4-cycle and a 6-cycle. If $|C| = 9$ and e is not triangular, then e splits C into a 4-cycle and a 7-cycle, or e splits C into a 5-cycle and a 6-cycle. □

By Lemmas 3 and 4, if a bad 9-cycle C (one type in Figure 1) is a subgraph in G, then C must be induced.

Lemma 5. *C_0 has no chord.*

Proof. If C_0 contains a chord e, then e is one of the types described in Lemma 4. By Lemma 2, G has no separating 8^--cycles. Thus, G contains no other vertices and the coloring on C_0 is also a coloring of G, a contradiction. □

The following lemma from [21] provides a powerful tool to prove the reducibility.

Lemma 6. *Let $k \geq 3$ and H be a subgraph of G. If the vertices of H can be ordered as $v_1, v_2, \cdots, v_l$ such that the following hold*

(1) $v_1 v_l \in E(G)$, and v_1 has no neighbor outside of H,
(2) $d(v_l) \leq k$ and v_l has at least one neighbor in $G - H$,
(3) for each $2 \leq i \leq l-1$, v_i has at most $k-1$ neighbors in $G[v_1, \cdots, v_{i-1}] \bigcup (G - H)$, then a DP-$k$-coloring of $G - H$ can be extended to a DP-k-coloring of G.

A face in G is internal if it contains no vertex of C_0 and a vertex in G is internal if it is not incident to C_0. A 6-face f in G is bad if it is adjacent to a 5^--face and a 6-face f in G is good if it is not bad.

Lemma 7. *Let f be an internal 6-face in G. If f is a (3, 3, 3, 3, 3, 3)-face, then f cannot be adjacent to an internal face f_1 with 5 or less vertices such that all vertices on f_1 are vertices with degree 3.*

Proof. Let $f = [v_1 v_2 w_1 w_2 w_3 w_4]$ be a (3, 3, 3, 3, 3, 3)-face and $f_1 = [v_1 v_2 \cdots v_i] (i \in \{3,4,5\})$, so that $v_1 v_2$ is the common edge of f and f_1, and all vertices on f_1 are 3-vertices. Order the vertices on f and f_1 as $v_1, w_4, w_3, w_2, w_1, v_2, v_3, \cdots, v_i$ $(i \in \{3,4,5\})$. Let H be the set of vertices in the list. Since all vertices in H are from the internal faces f and f_1, no vertex in C_0 is going to be removed by such subtraction.

Because G is a minimal counterexample, by Lemma 6, every DP-3-coloring of $G - H$ can be extended to G, a contradiction. □

Let f be a (3, 3, 3, 3, 3, 3)-face adjacent to a 3-face f'. We call the vertex v on f' but not on f the roof of f, and f the base of v.

Lemma 8. *Let f be an internal 6-face in G and f_1 be an internal face with 5 or less vertices which is adjacent to f. If f_1 has one 4-vertex, while the other vertices incident with f_1 are vertices with degree 3, then each of the following holds:*

(a) *f cannot be adjacent to another face with five or less vertices;*
(b) *If f is a (4, 3, 3, 3, 3, 3)-face such that f and f_1 have a common (3, 4)-edge, then the other (3, 4)-edge of f_1 cannot be incident with another internal (4, 3, 3, 3, 3, 3)-face;*
(c) *If f is a 6-face that all vertices on f are vertices with degree 3, then f_1 cannot be adjacent to an internal (4, 3, 3, 3, 3, 3)-face f_2 that f_1 and f_2 have a common (3, 4)-edge. This means that a 4-vertex incident with an internal (4, 3, 3, 3, 3, 3)-face is not a roof.*

Proof.

(a) follows from the condition on the distance of 5^--faces.
(b) Let $f_1 = [v_1 v_2 \cdots v_i] (i \in \{3, 4, 5\})$, $d(v_2) = 4$ and $f = [v_1 v_2 w_1 w_2 w_3 w_4]$, so that $v_1 v_2$ is the common (3, 4)-edge of f_1 and f, and all other vertices incident with f and f_1 are 3-vertices. Let $f_2 = [v_3 u_1 u_2 u_3 u_4 v_2]$ be the other (3, 3, 3, 3, 3, 4)-face adjacent to f_1. Order the vertices on f, f_1 and f_2 as $v_2, u_4, u_3, u_2, u_1, v_3, \cdots, v_i, v_1, w_4, w_3, w_2, w_1$ $(i \in \{3, 4, 5\})$. Let H be the set of vertices in the list. By Lemma 6, every DP-3-coloring of $G - H$ can be extended to G, a contradiction.
(c) Let $f_1 = [v_1 v_2 \cdots v_i] (i \in \{3, 4, 5\})$ and $f = [v_1 v_2 w_1 w_2 w_3 w_4]$, so that $v_1 v_2$ is the common (3,3)-edge of f_1 and f. Let $f_2 = [v_j u_1 u_2 u_3 u_4 v_{j+1}]$ $(j \in \{2, \cdots, i-1\})$ be the (3, 3, 3, 3, 3, 4)-face adjacent to f_1, $d(v_{j+1}) = 4$ and all other vertices on f, f_1 and f_2 are 3-vertices. If $j = 2$, then $u_1 = w_1$ and order the vertices on f, f_1 and f_2 as $v_1, v_i, \cdots, v_3, v_2, u_4, u_3, u_2, u_1(w_1), w_2, w_3, w_4$ $(i \in \{3, 4, 5\})$. Let H be the set of vertices in the list. By Lemma 6, every DP-3-coloring of $G - H$ can be extended to G, a contradiction. If $j > 2$, then order the vertices on f, f_1 and f_2 as $v_1, v_i, \cdots, v_{j+1}, u_4, u_3, u_2, u_1, v_j, \cdots, v_2, w_1, w_2, w_3, w_4$ $(i \in \{3, 4, 5\})$. Let H be the set of vertices in the list. By Lemma 6, every DP-3-coloring of $G - H$ can be extended to G, a contradiction.

□

Lemma 9. *Let $f = [v_1 v_2 v_3 v_4 v_5 v_6]$ be an internal 6-face that is adjacent to an internal face with five or less vertices $f_1 = [v_1 v_2 w_1 \cdots w_i]$ $(i \in \{1, 2, 3\})$. If all of the vertices on f_1 are vertices with degree 3, then $d(v_3) \geq 4$ or $d(v_6) \geq 4$.*

Proof. We assume that $d(v_3) = d(v_6) = 3$, and we use u to denote the neighbor of w_1 that is not on f_1. First, we may rename the lists of vertices in $\{w_1, v_2, v_1, v_3, v_4\}$ so that each edge in $\{uw_1, w_1 v_2, v_1 v_2, v_2 v_3, v_3 v_4\}$ is straight.

Consider the graph G' obtained from $G - \{w_1, \cdots, w_i, v_1, v_2, v_3, v_6\}$ $(i \in \{1, 2, 3\})$ by identifying v_4 and u. We claim that no new cycles of length from 3 to 5 multiple edges or loop are created. Otherwise, there is a path of length 1, 2, 3, 4 or 5 from u to v_4 in G, which together with w_1, v_2, v_3, v_4 forms a cycle C, $5 \leq d(C) \leq 9$. Because f_1 is a 5^--face, C cannot be a 5-cycle.

- If v_1 and v_6 are in $Int(C)$, see Figure 2a, then C is not a bad 9-cycle. Otherwise, since v_1 and v_6 are in $Int(C)$, C must be the type (b) in Figure 1. Thus, v_1 and v_6 must be in a 3-cycle which is adjacent to f, a contradiction to Lemma 8(a). Thus, C is a separating $\{6, 7, 8\}$-cycle or separating good 9-cycle, a contradiction to Lemma 2.

- If v_1 and v_6 are not in $Int(C)$, see Figure 2b. Due to $d(v_3) = 3$ and f being a 6-cycle, by Lemma 2 and 4, v_3 is incident with an edge e in $Int(C)$. Either the other vertex incident to e is in the cycle C, or it is not. If it is in the cycle C, then edge e is its chord, $7 \leq d(C) \leq 9$ and e must be incident with a 5^--cycle C' by Lemma 4. The distance between f_1 and C' is at most 1, a contradiction. If the other vertex incident to e is not in cycle C, then C is a separating cycle. If C is a bad 9-cycle, recall that e is incident v_3 and in $Int(C)$, the v_3 must be incident with or adjacent to a 3-cycle. Thus, the distance between f_1 and the 3-cycle is at most 2, a contradiction. Thus, C must be a separating $\{6, 7, 8\}$-cycle or separating good 9-cycle, a contradiction to Lemma 2.

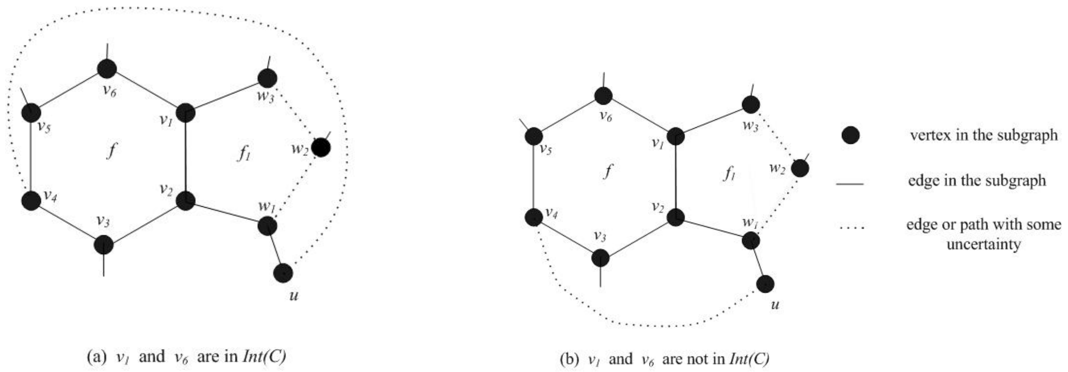

Figure 2. An internal 6-face f is adjacent to an internal 5^- face f_1 that all vertices on f_1 are 3-vertices. v_1 and v_6 are in Int(C) (**a**) and not in Int(C) (**b**).

Now, we claim that there is no new chord in C_0 of G'. Otherwise, u is on C_0 and v_4 is adjacent to a vertex v_4' which is on C_0. Then, there is a path between u and v_4' on C_0 with length at most four, which forms a $\{5, 6, 7, 8, 9\}$-cycle with w_1, v_2, v_3, v_4 in G. Similar to the proof process above, this does not occur.

Because f_1 is a 5^--cycle, the distance from v_4 to other 5^--cycles is at least one and the distance from u to other 5^--cycles is at least two. Thus, the 5^--cycles are at a distance of at least 3 from each other in G'. Since C_0 is not a bad 9-cycle in G and every bad 9-cycles in G is induced, C_0 is not a bad 9-cycle in G'. Because C_0 still is the boundary of the out face of the embedding of G' and no new chord in C_0 is formed in G' and G is a minimal counterexample, the DP-3-coloring of C_0 can be extended to G'. Now, we color v_4 and u with the color of the identified vertex and keep the colors of all vertices in G'. Now, we color v_3 first, and then color w_1 with the color of v_3. We can do this because the edges in $\{uw_1, w_1v_2, v_1v_2, v_2v_3, v_3v_4\}$ are straight and the color of v_3 is different from the color of v_4 and u. If $d(f_1) = 3$, then we color v_6, v_1, v_2 in the order. If $d(f_1) = 4$ or 5, then we color $w_2, (w_3), v_6, v_1, v_2$ in the order. Then, G has been colored, a contradiction. □

Lemma 10. *Let $P = w_1, u_1, u_2, w_2, v_1, v_2, w_3$ be a path in Int(C_0) and $f = [w_1', w_2', \cdots, w_i']$ ($i \in \{3, 4, 5\}$) be an internal face with five or less vertices such that all vertices of f are vertices with degree 3, so that $w_1w_1', w_2w_2', w_3w_3' \in E(G)$. If $d(w_1) = d(u_1) = d(u_2) = 3$, then $d(w_2) \geq 5$. (In addition, similarly, if $d(w_3) = d(v_1) = d(v_2) = 3$, then $d(w_2) \geq 5$.)*

Proof. Assume that $d(w_2) \leq 4$. Since there is no separating 6-cycle and every 6-cycle has no chords by Lemma 2 and Lemma 4, the 6-cycle $w_1u_1u_2w_2w_2'w_1'$ and $w_2v_1v_2w_3w_3'w_2'$ are both 6-faces. If $d(w_1) = 3$, then, by Lemma 9, $d(w_2) = 4$. Let w_2'' be the fourth neighbor of w_2. Since the cycles of length 3, 4, and 5 in G are at a distance of at least 3 from each other, w_2'' is not one vertex in $\{w_1, u_1, u_2, w_2, w_2', \cdots, w_i', w_1', w_3, v_2, v_1\}$. We may rename the list of vertices in $\{w_2, w_2', w_1', w_3', w_3\}$ so that the edges $\{w_2''w_2, w_2w_2', w_1'w_2', w_2'w_3', w_3'w_3\}$ are straight.

Consider the graph G' obtained from $G - \{w_1, u_1, u_2, w_2, w_2', \cdots, w_i', w_1'\}$ ($i \in \{3, 4, 5\}$) by identifying w_3 and w_2''. Since the cycles of length 3, 4, and 5 in G are a distance of at least 3 from each other, the distance between w_3 and w_2'' in G' is at least 2 and the distance between v_1 and w_2'' in G' is at least 4. We claim that no new cycles of length from 2 to 5 are created. Otherwise, there is a path of length 2 to 5 from w_3 and w_2'' in G, which together with w_2, w_2', w_3' forms a $\{6,7,8,9\}$-cycle C. If u_1 is in $Int(C)$, then C is not a bad 9-cycle. Otherwise, since u_1 is in $Int(C)$, u_1, u_2, w_1 and w_1' are in $Int(C)$. Then, C is not isomorphic to a bad 9-cycle in Figure 1. Thus, C is a separating $\{6,7,8\}$-cycle or separating good 9-cycle, a contradiction to Lemma 2. Let v_1 be in $Int(C)$ (Since the distance between v_1 and w_2'' in G' is at least 4, v_1 is not on C). Because the cycles of length 3,4, and 5 in G are at a distance of at least 3 from each other, v_1 cannot be on triangles. Then, C is not isomorphic to a bad 9-cycle see Figure 3b. Thus, C is a separating $\{6,7,8\}$-cycle or separating good 9-cycle, a contradiction to Lemma 2.

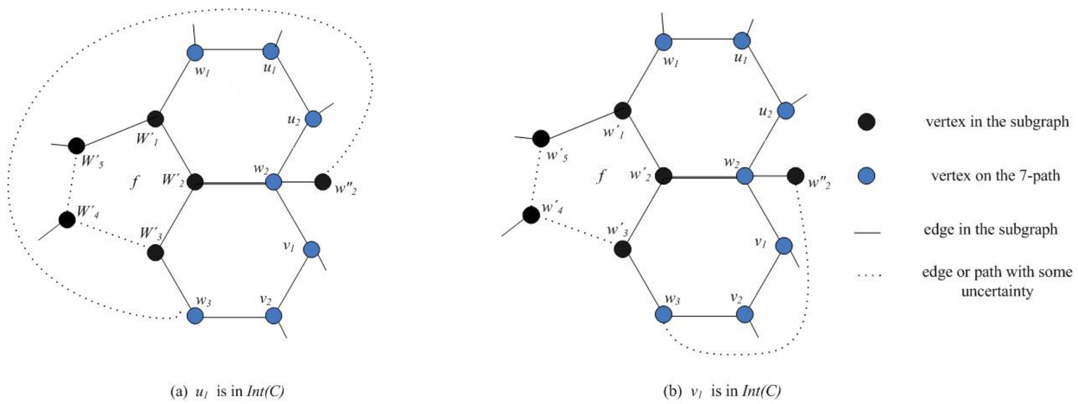

Figure 3. All the vertices on the 5^{-}-cycle f are 3-vertices. (**a**) u_1 in Int(C); (**b**) v_1 in Int(C).

Now, we claim that there is no new chord in C_0 of G'. Otherwise, w_2'' is on C_0 and w_3 is adjacent to a vertex w_3'' which is on C_0; then, there is a path between w_2'' and w_3'' on C_0 with length at most four, which forms a $\{5,6,7,8,9\}$-cycle with w_3, w_2, w_2', w_3' in G. Similar to the proof process above, this does not occur.

Because f is a 5^{-}-cycle, the distance from w_3 to other 5^{-}-cycles is at least two and the distance from w_2'' to other 5^{-}-cycles is at least one. Thus, the 5^{-}-cycles are at a distance of at least 3 from each other in G'. Since C_0 is not a bad 9-cycle in G and every bad 9-cycles in G is induced, C_0 is not a bad 9-cycle in G'. Because C_0 still is the boundary of the out face of the embedding of G' and no new chord in C_0 is formed in G' and G is a minimal counterexample, the DP-3-coloring of C_0 can be extended to G'. Now, color w_2'' and w_3 with the color of the identified vertex and keep the colors of all vertices in G'. Now, we color w_2 first, and then color w_3' with the color of w_2. We can do this because the edges in $\{w_2''w_2, w_2w_2', w_2'w_3', w_3'w_3\}$ are straight and the color of w_2 is different from the color of w_2'' and w_3. If $d(f_1) = 3$, then we color $u_2, u_1, w_1, w_1', w_2'$ in the order. If $d(f_1) = 4$ or 5, then we color $u_2, u_1, w_1, w_4', (w_5'), w_1', w_2'$ in the order. Then, G has been colored, a contradiction. □

We are now ready to present a discharging procedure that will complete the proof of the Theorem 3. Let each vertex $v \in V(G)$ have an initial charge of $\mu(v) = 2d(v) - 6$, and each face $f \neq f_0$ in our fixed plane drawing of G have an initial charge of $\mu(f) = d(f) - 6$. Let $\mu(f_0) = d(f_0) + 6$. By Euler's Formula, $\sum_{x \in V \bigcup F} \mu(x) = 0$.

Let $\mu^*(x)$ be the charge of $x \in V \bigcup F$ after the discharge procedure. To lead to a contradiction, we shall prove that $\mu^*(x) \geq 0$ for all $x \in V \bigcup F \setminus \{f_0\}$ and $\mu^*(f_0) > 0$.

For shortness, let $F_k = \{f : f$ be a k-face and $V(f) \bigcap C_0 \neq \emptyset\}$.

The discharging rules:

(R1): If v is an internal 4^{+}-vertex.

① If v is an internal 4-vertex and incident with a 3-face f, then v gives $\frac{3}{2}$ to its incident 3-face, $\frac{1}{2}$ to its base when v has a base and $\frac{1}{2}$ to its incident $(3,3,3,3,3,4^{+})$-face that is adjacent to f.

② If v is an internal 5^+-vertex and incident with a 3-face f, then v gives $\frac{3}{2}$ to its incident 3-face, $\frac{1}{2}$ to its base when v has a base and $\frac{1}{2}$ to its incident 6-face that is adjacent to f.
③ If v is incident with a 4-face, then v gives 2 to its incident 4-face.
④ If v is incident with a 5-face, then v gives 1 to its incident 5-face.
⑤ If v is an internal 4-vertex. If v is adjacent to a 5^--face f such that all vertices on f are 3-vertices, then v gives 1 to its adjacent 5^--face f and v gives $\frac{1}{2}$ to its incident 6-faces if any that are not adjacent to f.
⑥ If v is an internal 5^+-vertex. If v is adjacent to a 5^--face f such that all vertices on f are 3-vertices, then v gives 2 to its adjacent 5^--face f and $\frac{1}{2}$ to its incident 6-faces if any that are not adjacent to f.
⑦ If v is an internal 4^+-vertex. If v is not on a 5^--face nor adjacent to a 5^--face f such that all vertices on f are 3-vertices, then v gives $\frac{1}{2}$ to its incident 6-faces.

(R2): Each internal 6-face gives $\frac{1}{2}$ to its adjacent internal (3, 3, 4)-face.

Each internal 6-face f gives $\frac{1}{2}$ to its adjacent internal (3, 3, 3)-face f_1, if any, when f contains a 4^+ vertex v and v is not adjacent to f_1.

Each non-internal 6-face or 7^+-face other than f_0 gives 1 to each of its adjacent 5^--face, if any, and gives the rest to the outer face f_0.

(R3): The outer face f_0 get $\mu(v)$ from each $v \in C_0$, gives 3 to each face in F_3, 2 to each face in F_4, 1 to each face in F_5, and 1 to each face in F_6 adjacent to an internal face with 5 or less vertices.

Lemma 11. *Every vertex v in G has nonnegative final charge.*

Proof. By (R3), the outer face f_0 get $\mu(v)$ from each $v \in C_0$ whether $\mu(v)$ is positive or negative, each vertex on C_0 has final charge 0. Thus, we assume that v is an internal vertex of G, then $d(v) \geq 3$ by Lemma 1. If $d(v) = 3$, then $\mu^*(v) = 0$.

If $d(v) = 4$. If v is on a 5^--face f, then it is not on or adjacent to other 5^--faces. If $d(f) = 3$, by Lemma 8 (b) and (c), v cannot be on two (3, 3, 3, 3, 3, 4)-faces which are adjacent to f at the same time, and v cannot be a roof and on a (3, 3, 3, 3, 3, 4)-face at the same time, then v gives $\frac{3}{2}$ to the 3-face, and at most $\frac{1}{2}$ to 6-faces by (R1) ①. Thus, $\mu^*(v) \geq 2d(v) - 6 - \frac{3}{2} - \frac{1}{2} \geq 0$. If $d(f) = 4$, then v gives 2 to the 4-face by (R1) ③. Thus, $\mu^*(v) \geq 2d(v) - 6 - 2 \geq 0$. If $d(f) = 5$, then v gives 1 to the 5-face by (R1) ④. Thus, $\mu^*(v) \geq 2d(v) - 6 - 1 \geq 0$. Now, assume that v is adjacent to a 5^--face f that all vertices on f are 3-vertices, then it is not on or adjacent to other 5^--faces. Thus, by (R1) ⑤, v gives 1 to the 5^--face, and $\frac{1}{2}$ to each other incident 6-faces that are not adjacent to the 5^--face. Thus, $\mu^*(v) \geq 2d(v) - 6 - 1 - \frac{1}{2} \times 2 = 0$. Finally, assume that v is not on a 5^--face or adjacent to a 5^--face f that all vertices on f are 3-vertices, then, by (R1) ⑦, its final charge is $\mu^*(v) \geq 2d(v) - 6 - \frac{1}{2} \times d(v) = \frac{3}{2} \times (d(v) - 4) = 0$.

If $d(v) = k \geq 5$. Because of this, the cycle of lengths 3, 4, and 5 are at a distance of at least 3 from each other. If v is on a 5^--face f. If $d(f) = 3$, then, by (R1) ②, v gives $\frac{3}{2}$ to the 3-face, and $\frac{1}{2}$ to its base or incident $(3, 3, 3, 3, 3, 4^+)$-faces that is adjacent to f. Thus, $\mu^*(v) \geq 2d(v) - 6 - \frac{3}{2} - \frac{1}{2} \times 3 > 0$. If $d(f) = 4$, then, by (R1) ③ v gives 2 to the 4-face. Thus, $\mu^*(v) \geq 2d(v) - 6 - 2 > 0$. If $d(f) = 5$, then, by (R1) ④ v gives 1 to the 5-face. Thus, $\mu^*(v) \geq 2d(v) - 6 - 1 > 0$. If v is adjacent to a 5^--face f that all vertices on f are 3-vertices. Thus, by (R1) ⑥, v gives at most 2 to the 5^--face, and $\frac{1}{2}$ to each other incident 6-faces that are not adjacent to the 5^--face. Hence, $\mu^*(v) \geq 2d(v) - 6 - 2 - \frac{1}{2} \times (d(v) - 2) = \frac{3}{2} \times (d(v) - \frac{14}{3}) > 0$. Finally, assume that v is not on a 5^--face or adjacent to a 5^--face f that all vertices on f are 3-vertices, then, by (R1) ⑦, its final charge is $\mu^*(v) \geq 2d(v) - 6 - \frac{1}{2} \times d(v) = \frac{3}{2} \times (d(v) - 4) > 0$. □

Lemma 12. *Every face other than f_0 in G has a nonnegative final charge.*

Proof. Let $d(f) = 3$. If f contains some vertices of C_0, then f gets 3 from f_0 by (R3), so $\mu^*(f) = 0$. Let f be an internal face. If f contains at least two 4^+-vertices, then, by (R1) ① and ②, f gets $\frac{3}{2}$ from each of the incident 4^+-vertices, thus $\mu^*(f) \geq d(f) - 6 + \frac{3}{2} \times 2 = 0$. If f is incident with exactly one

4^+-vertex, then f gets $\frac{3}{2}$ from the incident 4^+-vertex by (R1) ① and ②, and gets $\frac{1}{2}$ from each of the adjacent 6^+-face by (R2), thus $\mu^*(f) \geq d(f) - 6 + \frac{3}{2} \times 2 = 0$. Now, we assume that $f = [v_1'v_2'v_3']$ is an internal (3, 3, 3)-face. Let v_1v_1', v_2v_2', $v_3v_3' \in E(G)$ and let f_1, f_2, f_3 be the three adjacent faces of f so that f_1 contains v_1v_1', $v_1'v_2'$, $v_2'v_2$ and f_2 contains v_2v_2', $v_2'v_3'$, $v_3'v_3$. If f_1, f_2, f_3 are three 7^+ or non-internal 6-faces, then they sent 1 to f by (R2) and $\mu^*(f) \geq d(f) - 6 + 3 = 0$. Now, let f be adjacent to an internal 6-face, say f_1, then f is adjacent to at least one internal 4^+-vertex (say v_2), which is incident with f_1 by Lemma 9.

- If both f_2 and f_3 are internal 6-faces, then one of $\{v_1v_3\}$ is a 4^+-vertex by Lemma 9. By Lemma 10, either all of $\{v_1, v_2, v_3\}$ are 4-vertices, or one of them is a 5^+-vertex, or one of them (say v_1) is a 3-vertex and the other two are 4-vertices, in which case both f_1 and f_3 contain 4^+-vertices which are not adjacent to f. Thus, $\mu^*(f) \geq d(f) - 6 + min\{3 \times 1, 2 + 1, 2 + \frac{1}{2} \times 2\} = 0$ by (R1) ⑤, ⑥ and (R2).
- If both f_2 and f_3 are 7^+- or non-internal 6-faces, then, by (R1) ⑤, ⑥ and (R2), $\mu^*(f) \geq d(f) - 6 + 1 + 1 \times 2 = 0$.
- Thus, we may assume that one of $\{f_2, f_3\}$ is an internal 6-face and the other is a non-internal 6-face or a 7^+-face. If f_3 is an internal 6-face, then one of $\{v_1, v_3\}$ is a 4^+-vertex by Lemma 9. Thus, f gets 2 from the two adjacent 4^+-vertices by (R1) ⑤ and ⑥, and f gets 1 from f_2 by (R2). Thus, $\mu^*(f) \geq d(f) - 6 + 1 + 1 + 1 = 0$. Thus, we may assume that f_2 is an internal 6-face and f_3 is a 7^+-or non-internal 6-face. If both v_1 and v_3 are 3-vertices and $d(v_2) \geq 4$, by Lemma 10, both f_1 and f_2 contain at least one 4^+-vertex that is not adjacent to f. Thus, by (R2), f gets $\frac{1}{2} \times 2$ from f_1 and f_2, gets 1 from f_3, and gets 1 from v_2. Thus, $\mu^*(f) \geq d(f) - 6 + 1 + 1 + 1 = 0$. If $d(v_2) = 4$ and one vertex of $\{v_1, v_3\}$ is a 4^+-vertex, then, by (R1) ⑤, f gets 1 from v_2 and the 4^+-vertex of $\{v_1, v_3\}$, and by (R2) f gets 1 from f_3. Thus, $\mu^*(f) \geq d(f) - 6 + 1 + 1 + 1 = 0$. If $d(v_2) \geq 5$, then by (R1) ⑥ f gets 2 from v_2 and by (R2) f gets 1 from f_3. Thus, $\mu^*(f) \geq d(f) - 6 + 2 + 1 = 0$.

Let $d(f) = 4$. If f contains some vertices of C_0, then f gets 2 from f_0 by (R3), so $\mu^*(f) = 0$. Let f be an internal face. If f is contains a 4^+-vertex, then, by (R1) ③ f gets 2 from each of the incident 4^+-vertices, thus $\mu^*(f) \geq d(f) - 6 + 2 = 0$. Now, we assume that $f = [v_1'v_2'v_3'v_4']$ is an internal (3,3,3,3)-face. Let v_1v_1', v_2v_2', v_3v_3', $v_4v_4' \in E(G)$ and let f_1, f_2, f_3, f_4 be the four adjacent faces of f. If two faces of $\{f_1, f_2, f_3, f_4\}$ are 7^+- or non-internal 6-faces, then they sent 1 to f by (R2) and $\mu^*(f) \geq d(f) - 6 + 2 = 0$. Thus, we may assume that it is adjacent to three or four internal 6-faces. By Lemma 9, f is adjacent to at least two internal 4^+-vertices, so, by (R1) ⑤ and ⑥, f gets 1 + 1 from the two 4^+-vertices. Thus, $\mu^*(f) \geq d(f) - 6 + 2 = 0$.

Let $d(f) = 5$. If f contains some vertices of C_0, then f gets 1 from f_0 by (R3), so $\mu^*(f) = 0$. Let f be an internal face. If f contains a 4^+-vertex, then, by (R1) ④ f gets 1 from each of the incident 4^+-vertices, thus $\mu^*(f) \geq d(f) - 6 + 2 = 0$. Now, we assume that $f = [v_1'v_2'v_3'v_4'v_5']$ is an internal (3, 3, 3, 3, 3)-face. Let v_1v_1', v_2v_2', v_3v_3', v_4v_4',$v_5v_5' \in E(G)$ and let f_1, f_2, f_3, f_4, f_5 be the five adjacent faces of f. If one face of $\{f_1, f_2, f_3, f_4\}$ are 7^+- or non-internal 6-faces, then it sent 1 to f by (R2) and $\mu^*(f) \geq d(f) - 6 + 1 = 0$. Thus, we may assume that it is adjacent to five internal 6-faces. By Lemma 9, f is adjacent to at least three internal 4^+-vertices, so, by (R1) ⑤ and ⑥, f gets 1 + 1 from the three 4^+-vertices. Thus, $\mu^*(f) \geq d(f) - 6 + 3 > 0$.

Let $d(f) = 6$. If f contains vertices of C_0 or f is not adjacent to an internal 3-face, then, by (R2) and (R3), $\mu^*(f) = 0$. Now, we assume that f is an internal 6-face that is adjacent to an internal 3-face $f' = [v_1v_2v_3]$ with the common edge v_1v_2 and $d(v_1) \leq d(v_2)$.

- If $d(v_1), d(v_2) \geq 4$, then f gives nothing to f', so $\mu^*(f) = 0$.
- If $d(v_1) = 3$ and $d(v_2) \geq 5$. then, by (R1) ② and (R2), f gets $\frac{1}{2}$ from v_2 and gives $\frac{1}{2}$ to f'. Thus, $\mu^*(f) \geq d(f) - 6 + \frac{1}{2} - \frac{1}{2} = 0$.
- If $d(v_1) = d(v_2) = 3$. Now, we assume that $d(v_3) = 4$. If f has a 4^+-vertex, then, by (R1) ⑦ and (R2) f gets $\frac{1}{2}$ from the 4^+-vertex and gives $\frac{1}{2}$ to f'. Thus, $\mu^*(f) \geq d(f) - 6 + \frac{1}{2} - \frac{1}{2} = 0$. If f is

a (3, 3, 3, 3, 3, 3)-face, then, by (R1) ①, ② and (R2) f gets $\frac{1}{2}$ from v_3, and gives $\frac{1}{2}$ to f'. Thus, $\mu^*(f) \geq d(f) - 6 + \frac{1}{2} - \frac{1}{2} = 0$. If $d(v_3) = 3$, by Lemma 7 f has a 4^+-vertex. By (R2), f' gets $\frac{1}{2}$ from f when f contains a 4^+-vertex that is not adjacent to the 3-face, in which case, by (R1) ⑦ f gets $\frac{1}{2}$ from the 4^+-vertex, so $\mu^*(f) \geq d(f) - 6 + \frac{1}{2} - \frac{1}{2} = 0$.

- If $d(v_1) = 3$ and $d(v_2) = 4$. If f is an internal (4, 3, 3, 3, 3, 3)-face, then f gets $\frac{1}{2}$ from v_2 by (R1) ①, or else f contains another 4^+-vertex, then, by (R1) ⑦ f gets $\frac{1}{2}$ from the 4^+-vertex. Thus, $\mu^*(f) \geq d(f) - 6 + \frac{1}{2} - \frac{1}{2} = 0$.

If $d(f) \geq 7$. Since the cycles of lengths 3, 4, and 5 in G are a distance of at least 3 from each other, f is adjacent to at most $\lfloor \frac{d(f)}{4} \rfloor$ 3-faces. Thus, $\mu^*(f) \geq d(f) - 6 - \lfloor \frac{d(f)}{4} \rfloor \geq 0$ by (R2). □

We call a bad 6-face f in F_6 Special if f is adjacent to one internal 5^--face.

Lemma 13. *The final charge of f_0 is positive.*

Proof. Assume that $\mu^*(f_0) \leq 0$. Let $E(C_0, G - C_0)$ be the set of edges between C_0 and $G - C_0$. Let e' be the number of edges in $E(C_0, G - C_0)$ that is not on a 5^--face and x be the charges f_0 receives by (R2). Let $\ell_3 = |F_3|$, $\ell_4 = |F_4|$, $\ell_5 = |F_5|$ and ℓ_6 be the number of special 6-faces. By Lemma 5, C_0 has no chord, each 5^--face in F_3, F_4 and F_5 contains at least two edges in $E(C_0, G - C_0)$. By (R2) and (R3), the final charge of f_0 is

$$
\begin{aligned}
\mu^*(f_0) &= d(f_0) + 6 + \sum_{v \in C_0} (2d(v) - 6) - 3\ell_3 - 2\ell_4 - \ell_5 - \ell_6 + x \\
&= d(f_0) + 6 + \sum_{v \in C_0} 2(d(v) - 2) - 2d(C_0) - 3\ell_3 - 2\ell_4 - \ell_5 - \ell_6 + x \\
&= 6 - d(f_0) + 2|E(C_0, G - C_0)| - 3\ell_3 - 2\ell_4 - \ell_5 - \ell_6 + x \\
&\geq 6 - d(f_0) + 4\ell_3 + 4\ell_4 + 4\ell_5 + 2e' - 3\ell_3 - 2\ell_4 - \ell_5 - \ell_6 + x \\
&= 6 - d(f_0) + \ell_3 + 2\ell_4 + 3\ell_5 + 2e' - \ell_6 + x
\end{aligned}
$$

where the equality follows from that each 5^--face in F_3, F_4 and F_5 contains two edges in $E(C_0, G - C_0)$.

Note that, for each special 6-face f, no edge in $E(C_0, G - C_0) \bigcap E(f)$ is on a 5^--face, then $e' \geq \ell_6$. When $e' = \ell_6 \neq 0$, $\mu^*(f_0) \geq 6 - d(f_0) + \ell_3 + 2\ell_4 + 3\ell_5 + \ell_6 + x$, then $\ell_3 = \ell_4 = \ell_5 = 0$ and $9 \geq d(f_0) \geq 6 + \ell_6 + x$. If $e' = \ell_6 = 1$, then $9 \geq d(f_0) \geq 7$. Thus, there is only a 7^+-face adjacent to C_0, a contradiction to $\ell_6 = 1$. If $e' = \ell_6 = 2$, then $9 \geq d(f_0) \geq 8 + x$ and $x \leq 1$. Since a special 6-face shares at most four vertices with C_0, so C_0 is adjacent to a 9^+-face f that contains at least four consecutive 2-vertices on C_0. Thus, there is only a 6-face adjacent to C_0, a contradiction to $\ell_6 = 2$. If $e' = \ell_6 \geq 3$, then $d(f_0) = 9$ and $x = 0$ and $e' = \ell_6 = 3$, in which case, we have a bad 9-cycle as in Figure 1. Thus, we may assume that $e' \geq \ell_6 + 1$. Thus,

$$
\begin{aligned}
\mu^*(f_0) &\geq 6 - d(f_0) + \ell_3 + 2\ell_4 + 3\ell_5 + 2e' - \ell_6 + x \\
&\geq 6 - d(f_0) + \ell_3 + 2\ell_4 + 3\ell_5 + \ell_6 + 2 + x \\
&= 8 - d(f_0) + \ell_3 + 2\ell_4 + 3\ell_5 + \ell_6 + x
\end{aligned}
$$

Since $\mu^*(f_0) \leq 0$, $d(f_0) \geq 8$. Thus, if $\ell_6 = 1$, then $d(f_0) = 9$, $x = \ell_3 = \ell_4 = \ell_5 = 0$ and $e' = 2$. Since the 6-face shares at most four vertices with C_0, C_0 is adjacent to a 10^+-face f that contains at least five consecutive 2-vertices on C_0. Thus, by (R2), $x \geq d(f) - 6 - \lceil \frac{d(f)-8}{4} \rceil > 0$, a contradiction.

Therefore, we may assume that $\ell_6 = 0$, and $9 \geq d(f_0) \geq 6 + \ell_3 + 2\ell_4 + 3\ell_5 + 2e' + x$, so $e' \leq 1$. Let $e' = 1$, it follows that $\ell_3 \leq 1$, $\ell_4 = 0$ and $\ell_5 = 0$.

- If $\ell_3 = 1$, then $d(f_0) = 9$ and $x = 0$. Since C_0 is not a bad 9-cycle, C_0 is adjacent to a 7^+-face f and f is adjacent to the 3-face. Thus, by (R2), f gives at least 1 to f_0, that is, $x \geq 1$, a contradiction.

- If $\ell_3 = 0$, then $d(f_0) \geq 8$ and $x \leq 1$. Note that C_0 is adjacent to a 8^+-face f that contains $d(C_0) - 1$ consecutive 2-vertices on C_0, thus, by (R2), f gives at least $x \geq d(f) - 6 - \lceil \frac{d(f)-d(f_0)}{4} \rceil \geq 2$ to f_0, a contradiction to $x \leq 1$.

  Finally, let $e' = 0$, then, $\ell_3 + 2\ell_4 + 3\ell_5 + x \leq d(f_0) - 6$, and each edge in $E(C_0, G - C_0)$ is on a 5^--face. Note that we may assume that $\ell_3 + \ell_4 + \ell_5 \geq 1$; otherwise, $G = C_0$, so $d(f_0) \geq 7$. Because of that, the cycle of lengths 3, 4, and 5 are a distance of at least 3 from each other, and, by Lemma 1, $d(v) \geq 3$ for each $v \in G - C_0$, so there must be a 8^+-face f adjacent to the 5^--face and C_0. The 8^+-face do not give charge to at least one 5^--face, so $x \geq d(f) - 6 - [\lceil \frac{d(f)}{4} \rceil - 1] \geq 1$. It follows that $\ell_3 \leq 2$, $\ell_4 \leq 1$ and $\ell_5 = 0$.
- If $\ell_3 = 2$, then $\ell_4 = 0$, $x = 1$ and $d(f_0) = 9$. There must be a 8^+-face f adjacent to the two 3-faces and C_0. The 8^+-face does not give charge to the two 3-faces, so $x \geq d(f) - 6 - [\lceil \frac{d(f)}{4} \rceil - 2] \geq 2$, a contradiction to $x = 1$.
- If $\ell_3 = 1$, then $\ell_4 = 0$, $1 \leq x \leq 2$ and $8 \leq d(f_0) \leq 9$. If $d(f_0) = 8$, then $x = 1$. In this case, C_0 is adjacent to a 9^+-face f that contains at least six consecutive 2-vertices on C_0; thus, by (R3), f gives at least $x \geq d(f) - 6 - \lceil \frac{d(f)-7}{4} \rceil \geq 2$ to f_0, a contradiction to $x = 1$. If $d(f_0) = 9$, then $x = 2$. In this case, C_0 is adjacent to a 10^+-face f that contains at least seven consecutive 2-vertices on C_0; thus, by (R3), f gives at least $x \geq d(f) - 6 - \lceil \frac{d(f)-8}{4} \rceil \geq 3$ to f_0, a contradiction to $x = 2$.
- If $\ell_3 = 0$, then $\ell_4 = 1$, $x = 1$ and $d(f_0) = 9$. C_0 is adjacent to a 9^+-face f that contains at least six consecutive 2-vertices on C_0; thus, by (R3), f gives at least $x \geq d(f) - 6 - \lceil \frac{d(f)-7}{4} \rceil \geq 2$ to f_0, a contradiction to $x = 1$.

□

Proof of Theorem 3. By Lemmas 11–13, $\sum_{x \in V \bigcup F} \mu^*(x) > 0$, a contradiction to $\sum_{x \in V \bigcup F} \mu(x) = 0$. Thus, the counterexample can't exist, which, in turn, would show that the theorem is true for all cases. □

3. Conclusions

The coloring theory of graphs plays a very important role in combinatorial optimization, computer theory, allocation of wireless communication channels, network data transmission, and so on. For example, the efficient design of airline schedules, the design of computer coding programs, etc.

It is well known that the problem of deciding whether a planar graph is 3-colorable is NP-complete. The Three Color Problem is very much alive, replete with an assortment of established results and an abundance of open problems. In addition, 3-coloring is the first significant graph coloring problem and, on the plane, the only unqualified graph coloring problem remaining. DP-coloring is one generalization of list coloring, which is a stronger version of proper coloring. It is very difficult to determine whether a graph is DP-colorable, or even whether a planar graph is DP-3-colorable.

It is unknown if Theorem 3 is most possible in the sense that there exists a planar graph with 5^--cycles that are a distance of at least 2 from each other is not DP-3-colorable.

Let d_1 denote the least integer k such that every planar graph with 5^--cycles are at distance from each other of at least k is DP-3-colorable.

Problem 1. *What is the exact value of d_1?*

Let d_2 denote the least integer k such that every planar graph with 5^--cycles are at distance from each other of at least k is 3-choosable.

Problem 2. *What is the exact value of d_2?*

It follows from Theorem 3 that $d_1 \leq 3$ and $d_2 \leq 3$.

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