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Jump Models with Delay—Option Pricing and Logarithmic Euler–Maruyama Scheme

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Abstract: In this paper, we obtain the existence, uniqueness, and positivity of the solution to delayed stochastic differential equations with jumps. This equation is then applied to model the price movement of the risky asset in a financial market and the Black–Scholes formula for the price of European option is obtained together with the hedging portfolios. The option price is evaluated analytically at the last delayed period by using the Fourier transformation technique. However, in general, there is no analytical expression for the option price. To evaluate the price numerically, we then use the Monte-Carlo method. To this end, we need to simulate the delayed stochastic differential equations with jumps. We propose a logarithmic Euler–Maruyama scheme to approximate the equation and prove that all the approximations remain positive and the rate of convergence of the scheme is proved to be 0.5.

Keywords: Lévy process; hyper-exponential processes; Poisson random measure; stochastic delay differential equations; positivity; options pricing; Black–Scholes formula; logarithmic Euler–Maruyama scheme; convergence rate

MSC: 91B28; 91G20; 91G60; 91B25; 65C30; 34K50

1. Introduction

The risky asset in the classical Black–Scholes market is described by the geometric Brownian motion given by the stochastic differential equation driven by standard Brownian motion:

$$dS(t) = S(t)\left[rdt + \sigma dW(t)\right] , \tag{1}$$

where r and σ are two positive constants and $W(t)$ is the standard Brownian motion. Ever since the seminal work of Black, Scholes and Merton, there have been many research works to extend the Black–Scholes–Merton theory of option pricing from the original Black–Scholes market to more sophisticated models.

One of these extensions is the delayed stochastic differential equation (SDDE) driven by the standard Brownian motion (e.g., [1], see also [2,3]). In these works, the risky asset is described by the following stochastic delay differential equation

$$dS(t) = S(t)\left[f(t, S_t)dt + g(t, S_t)dW(t)\right] ,$$

where $S_t = \{S(s)\,, t-b \le s \le t\}$ or $S_t = S(t-b)$ for some constant $b > 0$.

On the other hand, there have been some recent discoveries (see, e.g., [4–7]) showing that, to better fit some risky assets, it is more desirable to use the hyper-exponential jump process along with the classical Brownian motion:

$$dS(t) = S(t)\,[rdt + \sigma dW(t) + \beta dZ(t)]\ ,$$

where $Z(t)$ is a hyper-exponential jump process (see the definition in the next section).

Let $N(dt, dz)$ be the Poisson random measure associated with a jump process which includes the hyper-exponential jump process as a special case and let $\tilde{N}(dt, dz)$ denote its compensated Poisson random measure. Then, the above equation with $\sigma = 0$ is a special case of the following equation

$$dS(t) = S(t)\left(rdt + \beta \int_{[0,T]\times\mathbb{R}_0} z\tilde{N}(dz, ds)\right) \tag{2}$$

and it has been argued (e.g., [8–10]) that Equation (2) is a better model for stock prices than (1).

In this paper, we propose a new model to describe the risky asset by combining the hyper-exponential process with delay. More precisely, we propose the following stochastic differential equation as a model for the price movement of the risky asset:

$$dS(t) = S(t)\,[f(t, S(t-b))dt + g(t, S(t-b))dZ(t)]\ , \tag{3}$$

where f and g are two given functions, and $Z(t)$ is a Lévy process which includes the hyper-exponential jump processes as a special case. The above model along with the Brownian motion component can be found in [11], where the coefficient of Brownian motion cannot be allowed to be zero. In this work, we let the coefficient of the Brownian motion be zero and we use the Girsanov formula for the jump process to address the issue of completeness of the market and hedging portfolio missed in [11].

With the introduction of this new market model, the first question is whether the equation has a unique solution or not and if the unique solution exists whether the solution is positive or not (since the price of an asset is always positive). We first answer these questions in Section 2, where we prove the existence, uniqueness and positivity of the solutions to a larger class of equations than (3). To guarantee that the solution is positive, we need to assume that the jump part $g(t, S(t-b))dZ(t)$ of the equation is bounded from below by some constant (see Assumption (A3) in the next section for the precise meaning). The class of equations to which our results can be applied is larger in the following two aspects: (1) $Z(t)$ can be replaced by a more general Lévy process or more general Poisson random measure; and (2) the equation can be multi-dimensional.

Following the Black–Scholes–Merton principle, we then obtain a formula for the fair price for the European option and the corresponding replica hedging portfolio is also given. To evaluate this formula during the last delay period, we propose a Fourier transformation method. This method appears more explicit than the partial differential equation method in the literature and is closer to the original Black–Scholes formula in spirit. This is done in Section 4.

Due to the involvement of $f(S(t-b))$ and $g(S(t-b))$, the above analytical expression for the fair option price formula is only valid in the last delay period. Then, how do we perform the evaluation by using this option price formula? We propose to use Monte-Carlo method to get the numerical value approximately. For this reason, we need to simulate Equation (3) numerically. We observe that there have been a lot of works (e.g., [12–14]) on Euler–Maruyama convergence scheme for SDDE models. There has already been study on the Euler–Maruyama scheme for SDDE models with jumps (e.g., [15]). However, in general, the Euler–Maruyama scheme cannot preserve the positivity of the solution. Since the solution to Equation (3) is positive (when the initial condition is positive), we wish all of our approximations of the solution are also positive. To this end and motivated by the similar work in the Brownian motion case (see, e.g., [16]), we introduce a logarithmic Euler–Maruyama scheme, a variant of the Euler–Maruyama scheme for (3). With this scheme, all the approximate solutions are positive and the rate of the convergence of this scheme is also 0.5. This rate is optimal even in the

Brownian motion case (e.g., [17]). Let us point out that the 0.5 rate of the usual Euler–Maruyama scheme for SDDE with jumps studied in [15] is only obtained in the L^2 sense. Not only our logarithmic Euler–Maruyama scheme preserves the positivity, its rate is 0.5 in L^p for any $p \geq 2$. This is done in Section 3.

Finally, in Section 5, we present some numerical attempts and compare the classical Black–Scholes price formula against the market price for some famous call options in the real financial market.

2. Delayed Stochastic Differential Equations with Jumps

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{\{t\geq 0\}}$ satisfying the usual conditions. On $(\Omega, \mathcal{F}, \mathbb{P})$, let $Z(t)$ be a Lévy process adapted to the filtration $\mathcal{F}_t$. We consider the following delayed stochastic differential equation driven by the Lévy process $Z(t)$:

$$\begin{cases} dS(t) = f(S(t-b))S(t)dt + g(S(t-b))S(t-)dZ(t), & t \geq 0\,, \\ S(t) = \phi(t)\,, \qquad t \in [-b, 0]\,, \end{cases} \tag{4}$$

where

(i) $f, g : \mathbb{R} \to \mathbb{R}$ are some given bounded measurable functions;
(ii) $b > 0$ is a given number representing the delay of the equation; and
(iii) $\phi : [-b, 0] \to \mathbb{R}$ is a (deterministic) measurable function.

To study the above stochastic differential equation, it is common to introduce the Poisson random measure associated with this Lévy process $Z(t)$ (see, e.g., [9,18–20] and references therein). First, we write the jump of the process Z at time t by

$$\Delta Z(t) := Z(t) - Z(t-) \quad \text{if } \Delta Z(t) \neq 0\,.$$

Denote $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$ and let $\mathcal{B}(\mathbb{R}_0)$ be the Borel σ-algebra generated by the family of all Borel subsets $U \subset \mathbb{R}$, such that $\bar{U} \subset \mathbb{R}_0$. For any $t > 0$ and for any $U \in \mathcal{B}(\mathbb{R}_0)$, we define the *Poisson random measure*, $N : [0, T] \times \mathcal{B}(\mathbb{R}_0) \times \Omega \to \mathbb{R}$, associated with the Lévy process Z by

$$N(t, U) := \sum_{0 \leq s \leq t,\ \Delta Z_s \neq 0} \chi_U(\Delta Z(s))\,, \tag{5}$$

where χ_U is the indicator function of U. The associated Lévy measure ν of the Lévy process Z is given by

$$\nu(U) := \mathbb{E}[N(1, U)] \tag{6}$$

and the compensated Poisson random measure $\tilde{N}$ associated with the Lévy process $Z(t)$ is defined by

$$\tilde{N}(dt, dz) := N(dt, dz) - \mathbb{E}\left[N(dt, dz)\right] = N(dt, dz) - \nu(dz)dt\,. \tag{7}$$

For some technical reason, we assume that the process $Z(t)$ has only bounded negative jumps to guarantee that the solution $S(t)$ to (4) is positive. This means that there is an interval $\mathbb{J} = [-R, \infty)$ bounded from the left such that $\Delta Z(t) \in \mathbb{J}$ for all $t > 0$. With these notations, we can write

$$Z(t) = \int_{[0,t]\times\mathbb{J}} zN(ds, dz) \quad \text{or} \quad dZ(t) = \int_{\mathbb{J}} zN(dt, dz)$$

and Equation (4) becomes

$$\begin{aligned} dS(t) \quad &= \quad \left[f(S(t-b)) + g(S(t-b))\int_{\mathbb{J}} z\nu(dz)\right] S(t)dt \\ &\qquad + g(S(t-b))S(t-)\int_{\mathbb{J}} z\tilde{N}(dt, dz)\,. \end{aligned}$$

It is a special case of the following equation:

$$dS(t) = f(S(t-b))S(t)dt + \int_{\mathbb{J}} g(z, S(t-b))S(t-)\tilde{N}(dt, dz)\,. \tag{8}$$

Theorem 1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ are bounded measurable functions such that there is a constant $\alpha_0 > 1$ satisfying $g(z, x) \geq \alpha_0 > -1$ for all $z \in \mathbb{J}$ and for all $x \in \mathbb{R}$, where $\mathbb{J}$ is the supporting set of the Poisson measure $N(t, dz)$. Then, the stochastic differential delay Equation (8) admits a unique pathwise solution with the property that, if $\phi(0) > 0$, then, for all $t > 0$, the random variable $X(t) > 0$ almost surely.*

Proof. First, let us consider the interval $[0, b]$. When t is in this interval $f(X(t-b)) = f(\phi(t-b))$ and $g(z; X(t-b)) = g(z; \phi(t-b))$ are known given functions of t (and z). Thus, (8) is a linear equation driven by Poisson random measure. The standard theory (see, e.g., [18,20]) can be used to show that the equation has a unique solution. Moreover, it is also well-known (see the above-mentioned books or [21]) that by Itô's formula the solution to (8) can be written as

$$\begin{aligned} X(t) \;\; = \;\; & \phi(0) \exp\left\{ \int_0^t f(\phi(s-b))ds + \int_{[0,t]\times\mathbb{J}} \log\left[1 + g(z, \phi(s-b))\right] \tilde{N}(ds, dz) \right. \\ & \left. + \int_{[0,t]\times\mathbb{J}} \Big(\log\left[1 + g(z, \phi(s-b))\right] - g(z, \phi(s-b)) \Big) ds\nu(dz) \right\}. \end{aligned}$$

From this formula, we see that, if $\phi(0) > 0$, then the random variable $X(t) > 0$ almost surely for every $t \in [0, b]$.

In a similar way, we can consider Equation (8) on $t \in [kb, (k+1)b]$ recursively for $k = 1, 2, 3, \cdots$, and obtain the same statements on this interval from previous results on the interval $t \in [-b, kb]$. □

Since (4) is a special case of (8), we can write down a corresponding result of the above theorem for (4).

Corollary 1. *Let the Lévy process $Z(t)$ have bounded negative jumps (e.g., $\Delta Z(t) \in \mathbb{J} \subseteq [-R, \infty)$). Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are bounded measurable functions such that there is a constant $\alpha_0 > 1$ satisfying $g(x) \leq \frac{\alpha_0}{R}$ for all $x \in \mathbb{R}$. Then, the stochastic differential delay Equation (4) admits a unique pathwise solution with the property that, if $\phi(0) > 0$, then for all $t > 0$ the random variable $X(t) > 0$ almost surely.*

Proof. Equation (4) is a special case of (8) with $g(z, x) = zg(x)$. The condition $g(x) \leq \frac{\alpha_0}{R}$ implies $g(z, x) \geq \alpha_0 > -1$ for all $z \in \mathbb{J}$ and for all $x \in \mathbb{R}$. Thus, Theorem 1 can be applied. □

Example 1. *One example of the Lévy process $Z(t)$ we have in mind, which is used in finance, is the hyper-exponential jump process, which we explain below. Let $Y_i, i = 1, 2, \cdots$ be independent and identically distributed random variables with the probability distribution given by*

$$f_Y(x) = \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i x} I_{\{x \geq 0\}} + \sum_{j=1}^{n} q_j \theta_j e^{\theta_j x} I_{\{x<0\}}\,,$$

where

$$\eta_i > 0,\; p_i \geq 0, \quad \theta_j > 0,\; q_j \geq 0\,, \quad i = 1, \cdots, m,\; j = 1, \cdots, n$$

with $\sum_{i=1}^{m} p_i + \sum_{j=1}^{n} q_j = 1$. Let N_t be a Poisson process with intensity λ. Then,

$$Z(t) = \sum_{i=1}^{N_t} Y_i$$

is a Lévy process. If $m = 1, n = 1$, then $Z(t)$ is called a double exponential process. The assumption on the boundedness of the negative jumps can be made possible by requiring that $q_j = 0$ for all $j = 1, \cdots, n$ or by replacing the negative exponential distribution by truncated negative exponential distributions, namely,

$$f_Y(x) = \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i x} I_{\{x \geq 0\}} + \sum_{j=1}^{n} q_j \frac{\theta_j}{1 - e^{-\theta_j R_j}} e^{\theta_j x} I_{\{-R_j < x < 0\}},$$

where

$$\eta_i > 0, \; p_i \geq 0, \quad \theta_j > 0, R_j > 0, \; q_j \geq 0, \quad i = 1, \cdots, m, \; j = 1, \cdots, n$$

with $\sum_{i=1}^{m} p_i + \sum_{j=1}^{n} q_j = 1$. For this truncated hyper-exponential process, we can take $\mathbb{J} = [-R, \infty)$ with $R = \max\{R_1, \cdots, R_n\}$.

Although this paper mainly is concerned with the one-dimensional delayed stochastic differential Equation (8) or (4), it is interesting to extend Theorem 1 to more than one dimension.

Let $\tilde{N}_j(ds, dz), j = 1, \cdots, d$ be independent compensated Poisson random measures. Consider the following system of delayed stochastic differential equations driven by Poisson random measures:

$$\begin{aligned} dS_i(t) &= \sum_{j=1}^{d} f_{ij}(S(t-b)) S_j(t) dt \\ &\quad + S_i(t-) \sum_{j=1}^{d} \int_{\mathbb{J}} g_{ij}(z, S(t-b)) \tilde{N}_j(dt, dz), \quad i = 1, \cdots, d, \\ S_i(t) &= \phi_i(t), \quad t \in [-b, 0], \; i = 1, \cdots, d, \end{aligned} \tag{9}$$

where $S(t) = (S_1(t), \cdots, S_d(t))^T$.

Theorem 2. *Suppose that $f_{ij} : \mathbb{R} \to \mathbb{R}$ and $g_{ij} : \mathbb{J} \times \mathbb{R} \to \mathbb{R}$, $1 \leq i, j \leq d$ are bounded measurable functions such that there is a constant $\alpha_0 > 1$ satisfying $g_{ij}(z, x) \geq \alpha_0 > -1$ for all $1 \leq i, j \leq d$, for all $z \in \mathbb{J}$ and for all $x \in \mathbb{R}$, where $\mathbb{J}$ is the common supporting set of the Poisson measures $\tilde{N}_j(t, dz), j = 1, \cdots, d$. If for all $i \neq j$, $f_{ij}(x) \geq 0$ for all $x \in \mathbb{R}$, and $\phi_i(0) \geq 0$, $i = 1, \cdots, d$, then the stochastic differential delay Equation (9) admits a unique pathwise solution with the property that for all $i = 1, \cdots, d$ and for all $t > 0$, the random variable $S_i(t) \geq 0$ almost surely.*

Proof. We can follow the argument as in the proof of Theorem 1 to show that the system of delayed stochastic differential Equation (9) has a unique solution $S(t) = (S_1(t), \cdots, S_d(t))^T$. We modify slightly the method of [22] to show the positivity of the solution. Denote $\tilde{g}_{ij}(t, z) = g_{ij}(z, S(t-b))$. Let $Y_i(t)$ be the solution to the stochastic differential equation

$$dY_i(t) = Y_i(t-) \sum_{j=1}^{d} \int_{\mathbb{J}} \tilde{g}_{ij}(t, z) \tilde{N}_j(dt, dz)$$

with initial conditions $Y_i(0) = \phi_i(0)$. Since this is a scalar equation for $Y_i(t)$, its explicit solution can be represented

$$\begin{aligned} Y_i(t) &= \phi_i(0) \exp \left\{ \sum_{j=1}^{d} \log \left[1 + \tilde{g}_{ij}(s, z) \right] \tilde{N}_j(ds, dz) \right. \\ &\quad \left. + \sum_{j=1}^{d} \int_{[0,t] \times \mathbb{J}} \Big(\log \left[1 + \tilde{g}_{ij}(s, z) \right] - \tilde{g}_{ij}(s, z) \Big) ds \nu_j(dz) \right\}, \end{aligned}$$

where ν_j is the associated Lévy measure for $\tilde{N}_j(ds, dz)$. Denote $\tilde{f}_{ij}(t) = f_{ij}(S(t-b))$ and let $p_i(t)$ be the solution to the following system of equations

$$dp_i(t) = \sum_{j=1}^{d} \tilde{f}_{ij}(t)p_j(t)dt\,, \quad p_i(0) = 1\,, \quad i = 1, \cdots, d\,.$$

By the assumption on f, we have that, when $i \neq j$, $\tilde{f}_{ij}(t) \geq 0$ almost surely. By a theorem in [23] (p. 173), we see that $p_i(t) \geq 0$ for all $t \geq 0$ almost surely. Now, it is easy to check by the Itô formula that $\tilde{S}_i(t) = p_i(t)Y_i(t)$ is the solution to (9), which satisfies that $\tilde{S}_i(t) \geq 0$ almost surely. By the uniqueness of the solution, we see that $S_i(t) = \tilde{S}_i(t)$ for $i = 1, \cdots, d$. The theorem is then proved. □

3. Logarithmic Euler–Maruyama Scheme

Equation (4) or (8) is used in Section 4 to model the price of a risky asset in a financial market and its the solution is proved to be positive as in Theorem 1. As is well-known, the usual Euler–Maruyama scheme cannot preserve the positivity of the solution (e.g., [16] and references therein). Motivated by the work [16], we propose in this section a variant of the Euler–Maruyama scheme (which we call logarithmic Euler–Maruyama scheme) to approximate the solution so that all approximations are always non-negative. For the convenience of the future simulation, we consider only Equation (4), which we rewrite here:

$$dS(t) \quad = \quad f(S(t-b))S(t)dt + g(S(t-b))S(t-)dZ(t)\,, \tag{10}$$

where $Z(t) = \sum_{i=1}^{N_t} Y_i$ is a Lévy process. Here, N_t is a Poisson process with intensity λ and $Y_1, Y_2, \cdots$, are iid random variables.

The solution to the above equation can be written as

$$S(t) = \phi(0) \exp\Big(\int_0^t f(X(u-b))du + \sum_{0 \leq u \leq t, \Delta Z(u) \neq 0} \ln(1 + g(X(u-b))Y_{N(u)}) \Big)\,. \tag{11}$$

We consider a finite time interval $[0, T]$ for some fixed $T > 0$. Let $\Delta = \frac{T}{n} > 0$ be a time step size for some positive integer $n \in \mathbb{N}$. For any nonnegative integer $k \geq 0$, denote $t_k = k\Delta$. We consider the partition π of the time interval $[0, T]$:

$$\pi : 0 = t_0 < t_1 < \cdots < t_n = T\,.$$

On the subinterval $[t_k, t_{k+1}]$, the solution (11) can also be written as

$$\begin{aligned} S(t) \quad &= \quad S(t_k) \exp\Big(\int_{t_k}^t f(X(u-b))du \\ &\qquad + \textstyle\sum_{t_k \leq u \leq t, \Delta Z(u) \neq 0} \ln(1 + g(X(u-b))Y_{N(u)}) \Big)\,, t \in [t_k, t_{k+1}]\,. \end{aligned} \tag{12}$$

Motivated by Formula (12), we propose a logarithmic Euler–Maruyama scheme to approximate (4) as follows.

$$\begin{aligned} S^\pi(t_{k+1}) \quad &= \quad S^\pi(t_k) \exp\Big(f(S^\pi(t_k - b))\Delta \Big) \\ &\qquad \cdot \exp\Big(\ln(1 + g(S^\pi(t_k - b))\Delta Z_k) \Big)\,, \quad k = 0, 1, 2, ..., n-1 \end{aligned} \tag{13}$$

with $S^\pi(t) = \phi(t)$ for all $t \in [-b, 0]$. It is clear that, if $\phi(0) > 0$, then $S^\pi(t_k) > 0$ almost surely for all $k = 0, 1, 2, ..., n$. Then, our approximations $S^\pi(t_k)$ are always positive. Notice that the approximations from usual Euler–Maruyama scheme is always not positive preserving (see, e.g., [16] and references therein).

We prove the convergence and find the rate of convergence for the above scheme. For the convergence of the usual Euler–Maruyama scheme of jump equation with delay, we refer to [15]. To study the convergence of the above logarithmic Euler–Maruyama scheme, we make the following assumptions.

(A1) The initial data $\phi(0) > 0$ and they are Hölder continuous, i.e. there exist constant $\rho > 0$ and $\gamma \in [1/2, 1])$ such that for $t, s \in [-b, 0]$

$$|\phi(t) - \phi(s)| \leq \rho |t-s|^{\gamma}. \tag{14}$$

(A2) f is bounded. f and g are global Lipschitz. This means that there exists a constant $\rho > 0$ such that

$$\begin{cases} \Big|g(x_1) - g(x_2)\Big| \leq \rho|x_1 - x_2|\,; \\ \Big|f(x_1) - f(x_2)\Big| \leq \rho|x_1 - x_2|\,, \quad \forall\, x, x_2 \in \mathbb{R}\,; \\ \big|f(x)\big| \leq \rho\,, \quad \forall x \in \mathbb{R} \end{cases}$$

(A3) The support $\mathbb{J}$ of the Poisson random measure N is contained in $[-R, \infty)$ for some $R > 0$ and there are constants $\alpha_0 > 1$ and $\rho > 0$ satisfying $-\rho \leq g(x) \leq \frac{\alpha_0}{R}$ for all $x \in \mathbb{R}$.

(A4) For any $q > 1$, there is a $\rho_q > 0$

$$\int_{\mathbb{J}} (1 + |z|)^q \nu(dz) \leq \rho_q\,, \quad \forall x \in \mathbb{R}\,. \tag{15}$$

For notational simplicity, we introduce two step processes

$$\begin{cases} v_1(t) = \sum_{k=0}^{\infty} \mathbb{I}_{[t_k, t_{k+1})}(t) S^{\pi}(t_k) \\ [1ex] v_2(t) = \sum_{k=0}^{\infty} \mathbb{I}_{[t_k, t_{k+1})}(t) S^{\pi}(t_k - b). \end{cases}$$

Define the continuous interpolation of the logarithmic Euler–Maruyama approximate solution on the whole interval $[-b, T]$ (not only on $t_k, k = 0, \cdots, n$) as follows:

$$S^{\pi}(t) = \begin{cases} \phi(t) & t \in [-b, 0] \\ \phi(0) \exp\Big(\int_0^t f(v_2(u)) du \\ \qquad + \sum_{0 \leq u \leq t, \Delta Z(u) \neq 0} \ln(1 + g(v_2(u)) Y_{N(u)}) \Big) & t \in [0, T]. \end{cases} \tag{16}$$

With this interpolation, we see that $S^{\pi}(t) > 0$ almost surely for all $t \geq 0$.

Lemma 1. *Let Assumptions (A1)–(A4) be satisfied. Then, for any $q \geq 1$, there exists K_q, independent of the partition π, such that*

$$\mathbb{E}\Big[\sup_{0 \leq t \leq T} |S(t)|^q\Big] \vee \mathbb{E}\Big[\sup_{0 \leq t \leq T} |S^{\pi}(t)|^q\Big] \leq K_q.$$

Proof. We can assume that $q > 2$. First, let us prove $\mathbb{E}\Big[\sup_{0 \leq t \leq T} |S^{\pi}(t)|^q\Big] \leq K_q$.

From (16), it follows

$$\begin{aligned} \mathbb{E}\Big[\sup_{0 \leq t \leq T} |S^{\pi}(t)|^q\Big] \quad &\leq \quad |\phi(0)|^q \mathbb{E}\Big[\sup_{0 \leq t \leq T} \exp\Big(q \int_0^t f(v_2(u)) du \\ &\qquad + q \sum_{0 \leq u \leq t, \Delta Z(u) \neq 0} \ln(1 + g(v_2(u)) Y_{N(u)})\Big)\Big]. \end{aligned}$$

Since $|f(t)| \leq \rho$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leq t\leq T}|S^{\pi}(t)|^{q}\Big] \\
&\leq \phi(0)^{q}e^{q\rho T}\mathbb{E}\Big[\sup_{0\leq t\leq T}\exp\Big(q\textstyle\sum_{0\leq u\leq t,\Delta Z(u)\neq 0}\ln(1+g(v_2(u))Y_{N(u)})\Big)\Big] \\
&= \phi(0)^{q}e^{q\rho T}\mathbb{E}\Big[\sup_{0\leq t\leq T}\exp\Big(q\int_{\mathbb{T}}\ln(1+zg(v_2(u)))N(du,dz)\Big)\Big],
\end{aligned} \tag{17}$$

where, and throughout the remaining part of this paper, we denote $\mathbb{T} = [0,t] \times \mathbb{J}$. Now, we handle the factor

$$I := \mathbb{E}\Big[\sup_{0\leq t\leq T}\exp\Big(q\int_{\mathbb{T}}\ln(1+zg(v_2(u)))N(du,dz)\Big)\Big].$$

Let $h = ((1+zg(v_2(u))^{2q}-1))/z$. Then,

$$\begin{aligned}
I &= \mathbb{E}\Big[\sup_{0\leq t\leq T}\exp\Big(\frac{1}{2}\int_{\mathbb{T}}\ln(1+zh)N(du,dz)\Big)\Big] \\
&= \mathbb{E}\Big[\sup_{0\leq t\leq T}\exp\Big(\frac{1}{2}\int_{\mathbb{T}}\ln(1+zh)\tilde{N}(du,dz)+\frac{1}{2}\int_{\mathbb{T}}\ln(1+zh)\nu(dz)du\Big)\Big] \\
&= \mathbb{E}\Big[\sup_{0\leq t\leq T}\exp\Big(\frac{1}{2}\int_{\mathbb{T}}\ln(1+zh)\tilde{N}(du,dz)+\frac{1}{2}\int_{\mathbb{T}}[\ln(1+zh)-zh]\,\nu(dz)du\Big)\Big] \\
&\qquad \sup_{0\leq t\leq T}\exp\Big(-\frac{1}{2}\int_{\mathbb{T}}(1+zg(v_2(u))^{2q}-1)\,\nu(dz)du\Big)\Big] \\
&\leq C_q\mathbb{E}\Big[\sup_{0\leq t\leq T}\exp\Big(\frac{1}{2}\int_{\mathbb{T}}\ln(1+zh)\tilde{N}(du,dz)+\frac{1}{2}\int_{\mathbb{T}}[\ln(1+zh)-zh]\,\nu(dz)du\Big)\Big],
\end{aligned}$$

where we use boundedness of g and Assumption (A4). Now, an application of the Cauchy–Schwartz inequality yields

$$I \leq C_q\Big\{\mathbb{E}\Big[\sup_{0\leq t\leq T}M_t\Big]\Big\}^{1/2},$$

where

$$M_t := \exp\Big(\int_{\mathbb{T}}\ln(1+zh)\tilde{N}(du,dz)+\int_{\mathbb{T}}[\ln(1+zh)-zh]\,\nu(dz)du\Big).$$

However, $(M_t, 0 \leq t \leq T)$ is an exponential martingale. Thus,

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}M_t\Big] \leq 2\mathbb{E}\Big[M_T\Big] = 2.$$

Inserting this estimate of I into (17) proves $\mathbb{E}\Big[\sup_{0\leq t\leq T}|S^{\pi}(t)|^{q}\Big] \leq K_q < \infty$. In the same way, we can show $\mathbb{E}\Big[\sup_{0\leq t\leq T}|S(t)|^{q}\Big] \leq K_q < \infty$. This completes the proof of the lemma. □

Lemma 2. *Assume Assumptions (A1)–(A4) are true. Then, there is a constant $K > 0$, independent of π, such that*

$$\mathbb{E}_{\mathbb{Q}}\Big|S^{\pi}(t)-v_1(t)\Big|^{p} \leq K\Delta^{p/2}, \quad \forall\, t\in[0,T].$$

Proof. Let $t \in [t_j, t_{j+1})$ for some j. Using $|e^x - e^y| \le (e^x + e^y)|x - y|$ we can write

$$\begin{aligned} \left|S^\pi(t) - v_1(t)\right| &= \left|S^\pi(t) - S^\pi(t_j)\right| \\ &\le \left|S^\pi(t) + S^\pi(t_j)\right| \cdot \left| \int_{t_j}^t f(v_2(s))ds + \sum_{t_j \le s \le t} \ln(1 + g(v_2(s))Y_{N(s)})\right|. \end{aligned}$$

An application of the Hölder inequality yields that, for any $p > 1$,

$$\begin{aligned} \mathbb{E}\left[\left|S^\pi(t) - v_1(t)\right|^p\right] &\le \left\{\mathbb{E}\left[\left|S^\pi(t) + S^\pi(t_j)\right| \cdot \right|\right]^{2p}\right\}^{1/2} \\ &\qquad \left\{\mathbb{E}\left|\int_{t_j}^t f(v_2(s))ds + \sum_{t_j \le s \le t} \ln(1 + g(v_2(s))Y_{N(s)})\right|^{2p}\right\}^{1/2} \\ &\le K_p\left\{\mathbb{E}\left|\int_{t_j}^t f(v_2(s))ds\right|^{2p} + \mathbb{E}\left|\sum_{t_j \le s \le t} \ln(1 + g(v_2(s))Y_{N(s)})\right|^{2p}\right\}^{1/2} \\ &\le K_p\left\{\Delta^{2p} + \mathbb{E}\left|\sum_{t_j \le s \le t} \ln(1 + g(v_2(s))Y_{N(s)})\right|^{2p}\right\}^{1/2}. \end{aligned} \tag{18}$$

Now, we want to bound

$$I := \mathbb{E}\left|\sum_{t_j \le s \le t} \ln(1 + g(v_2(s))Y_{N(s)})\right|^{2p}.$$

(we use the same notation I to denote different quantities in different occasions and this does not cause ambiguity). We write the above sum as an integral:

$$\begin{aligned} I &= \mathbb{E}\Big|\int_{\mathbb{J}}\int_{t_j}^t \ln(1 + zg(v_2(s)))N(ds, dz)\Big|^{2p} \\ &= \mathbb{E}\Big|\int_{\mathbb{J}}\int_{t_j}^t \ln(1 + zg(v_2(s)))\tilde{N}(ds, dz) \\ &\qquad + \int_{\mathbb{J}}\int_{t_j}^t \ln(1 + zg(v_2(s)))\nu(dz)ds\Big|^{2p} \\ &\le C_p\left(\Delta^{2p} + \mathbb{E}\Big|\int_{\mathbb{J}}\int_{t_j}^t \ln(1 + zg(v_2(s)))\tilde{N}(ds, dz)\Big|^{2p}\right). \end{aligned}$$

By the Burkholder–Davis–Gundy inequality, we have

$$\begin{aligned} &\mathbb{E}\Big|\int_{\mathbb{J}}\int_{t_j}^t \ln(1 + zg(v_2(s)))\tilde{N}(ds, dz)\Big|^{2p} \\ &\qquad \le \mathbb{E}\left(\int_{\mathbb{J}}\int_{t_j}^t \Big|\ln(1 + zg(v_2(s)))\Big|^2 \nu(dz)ds\right)^p \\ &\qquad \le K_p\Delta^p. \end{aligned}$$

Thus, we have

$$I \le K_{p,T}\Delta^p.$$

Inserting this bound into (18) yields the lemma. □

Our next objective is to obtain the rate of convergence of our logarithmic Euler–Maruyama approximation $S^\pi(t)$ to the true solution $S(t)$.

Theorem 3. *Assume Assumptions (A1)–(A4) are true. Let $S^\pi(t)$ be the solution to (13) and let $S(t)$ be the solution to (10). Then, there is a constant $K_{p,T}$, independent of π such that*

$$\mathbb{E}_{\mathbb{Q}}\Big[\sup_{0\le t\le T}|S(t)-S^\pi(t)|^p\Big]\le K_{p,T}\Delta^{p/2}. \tag{19}$$

Proof. We write $S(t)=\phi(0)\exp\left(X(t)\right)$ and $S^\pi(t)=\phi(0)\exp\left(p(t)\right)$. Then,

$$\Big|S(t)-S^\pi(t)\Big|^p\le\Big|S(t)+S^\pi(t)\Big|^p\Big|X(t)-p(t)\Big|^p.$$

Hence, by Lemma 1, we have, for any $r\in[0,T]$,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le r}|S(t)-S^\pi(t)|^p\Big]\\
&\le\mathbb{E}\Big[\sup_{0\le t\le r}\Big|S(t)+S^\pi(t)\Big|^{2p}\Big]^{1/2}\mathbb{E}\Big[\sup_{0\le t\le r}\Big|X(t)-p(t)\Big|^{2p}\Big]^{1/2}\\
&\le 2^{2p-1}\Big(\mathbb{E}\Big[\sup_{0\le t\le r}\Big|S(t)\Big|^{2p}\Big]+\mathbb{E}\Big[\sup_{0\le t\le r}\Big|S^\pi(t)\Big|^{2p}\Big]\Big)^{1/2}\Big[\mathbb{E}\sup_{0\le t\le r}\Big|X(t)-p(t)\Big|^{2p}\Big]^{1/2}\\
&\le K_p\Big[\mathbb{E}\sup_{0\le t\le r}\Big|X(t)-p(t)\Big|^{2p}\Big]^{1/2}=K_pI^{1/2}.
\end{aligned}\tag{20}$$

Thus, we need only to bound the above expectation I, which is given by the following.

$$\begin{aligned}
I\ =\ &\mathbb{E}\Big[\sup_{0\le t\le r}|X(t)-p(t)|^{2p}\Big]\\
&\le\mathbb{E}\sup_{0\le t\le r}\Big|\int_0^t(f(S(u-b))-f(v_2(u)))du\\
&+\sum_{0\le u\le t,\Delta Z(u)\ne0}\ln(1+g(S(u-b))Y_{N(u)})-\ln(1+g(v_2(u))Y_{N(u)})\Big|^{2p}.
\end{aligned}\tag{21}$$

By the Lipschitz conditions, we have

$$\begin{aligned}
I\ &\le K_p\mathbb{E}\int_0^r\Big|S(u-b)-v_2(u)\Big|^{2p}du\\
&+K_p\mathbb{E}\sup_{0\le t\le r}\Big|\textstyle\sum_{0\le u\le t,\Delta Z(u)\ne0}\ln(1+g(S(u-b))Y_{N(u)})-\ln(1+g(v_2(u))Y_{N(u)})\Big|^{2p}\\
&\le K_p\Big[\mathbb{E}\int_0^r\Big|S(u-b)-S^\pi(u-b)\Big|^{2p}du+\mathbb{E}\int_0^r\Big|S^\pi(u-b)-v_2(u)\Big|^{2p}du\Big]\\
&+K_p\mathbb{E}\sup_{0\le t\le r}\Big|\textstyle\sum_{0\le u\le t,\Delta Z(u)\ne0}\ln(1+g(S(u-b))Y_{N(u)})-\ln(1+g(v_2(u))Y_{N(u)})\Big|^{2p}\\
&=I_1+I_2+I_3.
\end{aligned}\tag{22}$$

By Lemma 2 and Assumption (A1) about the Hölder continuity of the initial data ϕ, we have

$$I_2\le K_{p,T}\Delta^p. \tag{23}$$

We write the above sum I_3 with jumps as a stochastic integral:

$$\begin{aligned} I_3 &= \mathbb{E} \sup_{0\le t\le r} \Big| \sum_{0\le u\le t, \Delta Z(u)\neq 0} \ln(1+g(S(u-b))Y_{N(u)}) - \ln(1+g(v_2(u))Y_{N(u)}) \Big|^{2p} \\ &= \mathbb{E} \sup_{0\le t\le r} \Big| \int_{\mathbb{J}} \int_0^t [\ln(1+zg(S(u-b))) - \ln(1+zg(v_2(u)))]\, \tilde{N}(du,dz) \\ &\qquad + \int_{\mathbb{J}} \int_0^t [\ln(1+zg(S(u-b))) - \ln(1+zg(v_2(u)))]\, \nu(dz)du \Big|^{2p} \\ &= 4^p \mathbb{E} \sup_{0\le t\le r} \Big| \int_{\mathbb{J}} \int_0^t [\ln(1+zg(S(u-b))) - \ln(1+zg(v_2(u)))]\, \tilde{N}(du,dz) \Big|^{2p} \\ &\qquad + 4^p \mathbb{E} \sup_{0\le t\le r} \Big| \int_{\mathbb{J}} \int_0^t [\ln(1+zg(S(u-b))) - \ln(1+zg(v_2(u)))]\, \nu(dz)du \Big|^{2p} \\ &=: I_{31} + I_{32}\,. \end{aligned}$$

Using the Lipschitz condition on g and Assumption (A3), we have

$$\begin{aligned} I_{32} &\le K_p \mathbb{E}\Big(\int_0^r \Big| g(S(u-b)) - g(v_2(u)) \Big| du \Big)^{2p} \\ &\le K_{p,T} \mathbb{E} \sup_{0\le t\le r} |S(t-b)) - S^{\pi}(t-b)|^{2p}\,. \end{aligned}$$

Using the Burkholder–Davis–Gundy inequality, we have

$$I_{31} \le K_p \mathbb{E}\Big(\int_{\mathbb{J}} \int_0^r \Big| \ln(1+zg(S(u-b))) - \ln(1+zg(v_2(u))) \Big|^2 \nu(dz)du \Big)^p\,.$$

Similar to the bound for I_{32}, we have

$$I_{31} \le K_{p,T} \mathbb{E} \sup_{0\le t\le r} |S(t-b)) - S^{\pi}(t-b)|^{2p}\,.$$

Combining the estimates for I_{31} and $_{32}$, we see

$$I_3 \le K_{p,T} \mathbb{E} \sup_{0\le t\le r} |S(t-b)) - S^{\pi}(t-b)|^{2p}\,. \tag{24}$$

It is easy to verify

$$I_1 \le K_{p,T} \mathbb{E} \sup_{0\le t\le r} |S(t-b)) - S^{\pi}(t-b)|^{2p}\,. \tag{25}$$

Inserting the bounds obtained in (23)–(26) into (22), we see that

$$I \le K_{p,T} \mathbb{E} \sup_{0\le t\le r} |S(t-b)) - S^{\pi}(t-b)|^{2p} + K_{P,T}\Delta^p\,. \tag{26}$$

Combining this estimate with (20), we see

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\le t\le r} |S(t) - S^{\pi}(t)|^p \Big] \\ &\quad \le K_{p,T} \Big[\mathbb{E} \sup_{0\le t\le r} |S(t-b) - S^{\pi}(t-b)|^{2p} \Big]^{1/2} + K_{p,T}\Delta^{p/2} \end{aligned} \tag{27}$$

for any $p \geq 2$ and for any $r \in [0, T]$. Now, we use (27) to prove the theorem on the interval $[0, kb]$ recursively for $k = 1, 2, \cdots, [\frac{T}{b}] + 1$. Since $S^\pi(t) = S(t) = \phi(t)$ for $t \in [-b, 0]$. Taking $r = b$, we have

$$\mathbb{E}\Big[\sup_{0\leq t\leq b} |S(t) - S^\pi(t)|^p\Big] \leq K_{p,T}\Delta^{p/2} \tag{28}$$

for any $p \geq 2$. Now, taking $r = 2b$ in (27), we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup\nolimits_{0\leq t\leq 2b} |S(t) - S^\pi(t)|^p\Big] \\
&\quad \leq K_{p,T}\left[\mathbb{E}\sup\nolimits_{-b\leq t\leq b} |S(t)) - S^\pi(t)|^{2p}\right]^{1/2} + K_{P,T}\Delta^{p/2} \\
&\quad \leq K_{p,T}\left[K_{2p,T}\Delta^p\right]^{1/2} + K_{P,T}\Delta^{p/2} \leq K_{p,T}\Delta^{p/2}.
\end{aligned} \tag{29}$$

Continuing this way, we obtain for any positive integer $k \in \mathbb{N}$,

$$\mathbb{E}\Big[\sup_{0\leq t\leq kb} |S(t) - S^\pi(t)|^p\Big] \leq K_{k,p,T}\Delta^{p/2}. \tag{30}$$

Now, since T is finite, we can choose a k such that $(k-1)b < T \leq kb$. This completes the proof of the theorem. □

4. Option Pricing in Delayed Black–Scholes Market with Jumps

In this section, we consider the problem of option pricing in a delayed Black–Scholes market which consists of two assets. One is risk free, whose price is described by

$$dB(t) = rB(t)dt, \quad \text{or} \quad B(t) = e^{rt}, t \geq 0. \tag{31}$$

Another asset is a risky one, whose price is described by the delayed Equation (4) or (10), namely,

$$dS(t) \;=\; f(S(t-b))S(t)dt + g(S(t-b))S(t-)dZ(t), \tag{32}$$

where $Z(t) = \sum_{i=1}^{N_t} Y_i$ is a Lévy process, N_t is a Poisson process with intensity λ, and $Y_1, Y_2, \cdots,$ are iid random variables. As in Section 2, we introduce the Poisson random measure $N(dt, dz)$ and its compensator $\tilde{N}(dt, dz)$. The above delayed equation can be written as

$$\begin{aligned}
dS(t) \;=\; &\left[f(S(t-b)) + g(S(t-b))\int_{\mathbb{J}} z\nu(dz)\right] S(t)dt \\
&+ g(S(t-b))S(t-)\int_{\mathbb{J}} z\tilde{N}(dt, dz).
\end{aligned}$$

Denote

$$L = \int_{\mathbb{J}} z f_Y(z)dz, \tag{33}$$

where f_Y is the probability density of Y_i (whose support is $\mathbb{J}$). Then,

$$\int_{\mathbb{J}} z\nu(dz) = \lambda L.$$

Set

$$\tilde{S}(t) = \frac{S(t)}{B(t)}.$$

Then, by Itô's formula, we have

$$d\tilde{S}(t) = \tilde{S}(t-)g(S(t-b))\Big(\int_{\mathbb{J}} z\big[\theta(t)\nu(dz)dt + \tilde{N}(dt,dz)\big]\Big)\,, \tag{34}$$

where $\theta(t) = \frac{f(S(t-b))+g(S(t-b))-r}{\lambda L g(S(t-b))}$. We keep Assumptions (A1)–(A4) made in the previous section and we need to make an additional assumption:

(A5) There is a constant $\alpha_1 \in (1,\infty)$ such that $\int_{\mathbb{J}} \nu(dz) \geq \alpha_1 \Big|\frac{f(s)+g(s)-r}{g(t)}\Big| \quad \forall\, s,t \in [0,\infty)$

To find the risk neutral probability measure, we apply Girsanov theorem for Lévy process (see [19] (Theorem 12.21)). The $\theta(t)$ is predictable for $t \in [0,T]$. From the assumptions above, we also have that $0 < \theta(s) \leq \frac{1}{\alpha_1}$. Thus,

$$\int_{[0,T]\times\mathbb{J}} \Big(|\log(1+\theta(s))| + \theta^2(s)\Big)\nu(dz)ds \leq K < \infty\,.$$

Now, define

$$\begin{aligned} S^{\theta}(t) \quad := \quad & \exp\Big(\int_{[0,t]} \{\log\left(1-\theta(s)\right) + \theta(s)\}\nu(dx)ds \\ & + \int_{[0,t]} \log\left(1-\theta(s)\right)\tilde{N}(dx,ds)\Big)\,. \end{aligned}$$

In order for us to obtain an equivalent martingale measure, we need to verify the following Novikov condition:

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_{[0,T]\times\mathbb{J}} \{(1-\theta(s))\log(1-\theta(s)) + \theta(s)\}\nu(dz)ds\Big)\Big] < \infty \tag{35}$$

This is a consequence of our Assumption (A5). In fact, we have first

$$|\theta(s)| \quad = \quad \frac{|f(S(t-b)) - r|}{\lambda L g(S(t-b))} \leq \frac{1}{\alpha_1} < 1\,.$$

Hence, we have

$$\int_{[0,T]} \{(1-\theta(s))\log(1-\theta(s)) + \theta(s)\}ds \quad < \quad \infty\,.$$

However, $\nu(dz) = \lambda f_Y(z)dz$, we have

$$\int_{\mathbb{J}} \nu(dz) \quad = \quad \int_{\mathbb{J}} \lambda f_Y(z)dz < \infty\,.$$

Thus, we have (35).

Now, since we have verified the Novikov condition (35) we have then $\mathbb{E}[S^{\theta}(T)] = 1$. Define an equivalent probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ by

$$d\mathbb{Q} := S^{\theta}(T)d\mathbb{P}\,. \tag{36}$$

On the new probability space $(\Omega, \mathcal{F}_T, \mathbb{Q})$ (new probability $\mathbb{Q}$), the random measure

$$\tilde{N}_{\mathbb{Q}}(dz,ds) = \theta(t)\nu(dz)ds + \tilde{N}(dz,ds)\,, \tag{37}$$

is a compensated Poisson random measure. The corresponding Lévy measure is denoted by $\nu_{\mathbb{Q}}$. With this new Poisson random measure, we can write (34) as

$$d\tilde{S}(t) = \tilde{S}(t-) \int_{\mathbb{J}} zg(S(t-b))\tilde{N}_{\mathbb{Q}}(dt, dz)\,. \tag{38}$$

The following result gives the fair price formula for the European call option as well as the corresponding hedging portfolio.

Theorem 4. *Let the market be given by (31) and (32), where the coefficients f and g satisfy Assumptions (A1)–(A5). Then, the market is complete. Let T be the maturity time of the European call option on the stock with payoff function given by $X = (S_T - K)^+$. Then, at any time $t \in [0, T]$, the fair price V(t) of the option is given by the formula*

$$V(t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\Big((S_T - K)^+|\mathcal{F}_t\Big) \tag{39}$$

where $\mathbb{Q}$ is the martingale measure on $(\Omega, \mathcal{F}_T)$ given by (36).

Moreover, if $\int_{\mathbb{J}} z^j \nu_{\mathbb{Q}}(dz) < \infty$, $\int_{\mathbb{R}_+} g(t)^j dt < \infty$ for $j = 1, 2, 3, 4$, there is an adapted and square integrable process $\psi(z, t) \in \mathcal{L}^2(\mathbb{J} \times [0, T])$ such that

$$\mathbb{E}_{\mathbb{Q}}\Big(e^{-rT}(S_T - K)^+|\mathcal{F}_t\Big) = \mathbb{E}_{\mathbb{Q}}\Big(e^{-rT}(S_T - K)^+\Big) + \int_{[0,t]\times\mathbb{J}} \psi(z, s)\tilde{N}_{\mathbb{Q}}((dz, ds)$$

and the hedging strategy is given by

$$\pi_S(t) := \frac{\int_{\mathbb{J}} \psi(z, t)\tilde{N}_{\mathbb{Q}}(dz, t)}{\tilde{S}(t)g(S(t-b))}, \qquad \pi_B(t) := U(t) - \pi_S(t)\tilde{S}(t), \quad t \in [0, T]\,, \tag{40}$$

where $U(t) = \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T - K)^+|\mathcal{F}_t)$.

Proof. Applying the Itô formula to (38), we get

$$\begin{aligned} \tilde{S}(T) \quad &= \exp\Big(\int_{[0,T]\times\mathbb{J}} \{\ln(1 + zg(S(t-b))) - zg(S(t-b)\}\nu_{\mathbb{Q}}(dz)dt \\ &\quad + \int_{[0,T]\times\mathbb{J}} \ln(1 + zg(S(t-b)))\tilde{N}_{\mathbb{Q}}(dt, dz)\Big) \end{aligned} \tag{41}$$

Denote $X = (S_T - K)^+$ and consider

$$U(t) := \mathbb{E}_{\mathbb{Q}}(e^{-rT}X|\mathcal{F}_t)\,.$$

To apply martingale representation theorem for Lévy process (see, e.g., [18] (Theorem 5.3.5)), we first show that $U_t \in \mathcal{L}^2$, which is implied by $\mathbb{E}_{\mathbb{Q}}[S_T^2] < \infty$.

Write $h = g(S(t-b))$. Then, we can write

$$\begin{aligned} \tilde{S}_T^2 \quad &= \exp\Big(\int_{[0,T]\times\mathbb{J}} \{\ln(1 + zh)^2 - 2zh\}\nu_{\mathbb{Q}}(dz)dt \\ &\quad + \int_{[0,T]\times\mathbb{J}} \ln(1 + zh)^2\tilde{N}_{\mathbb{Q}}(dt, dz)\Big)\,. \end{aligned} \tag{42}$$

Denoting $\mathbb{T} = [0, T] \times \mathbb{J}$ and taking $\tilde{h} = \frac{(1+zh)^4-1}{z}$, we have

$$\begin{aligned}\tilde{S}_T^2 &= \exp\Big(\frac{1}{2}\int_{\mathbb{T}}\{\ln(1+z\tilde{h}) - z\tilde{h}\}\nu_{\mathbb{Q}}(dz)dt + \frac{1}{2}\int_{\mathbb{T}}\ln(1+z\tilde{h})\tilde{N}_{\mathbb{Q}}(dt,dz)\Big)\cdot \\ &\qquad \exp\Big(\int_{\mathbb{T}}\Big(\frac{z\tilde{h}}{2} - zh\Big)\nu_{\mathbb{Q}}(dz)dt\Big)\,.\end{aligned}$$

Applying the Hölder inequality, we have

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}[\tilde{S}_T^2] \\ &\le \Big[\mathbb{E}_{\mathbb{Q}}\exp\Big(\int_{\mathbb{T}}\{\ln(1+z\tilde{h}) - z\tilde{h}\}\nu_{\mathbb{Q}}(dz)dt + \int_{\mathbb{T}}\ln(1+z\tilde{h})\tilde{N}_{\mathbb{Q}}(dt,dz)\Big)\Big]^{1/2} \\ &\qquad \cdot\Big[\mathbb{E}_{\mathbb{Q}}\exp\Big(2\int_{\mathbb{T}}\Big(\frac{z\tilde{h}}{2} - zh\Big)\nu_{\mathbb{Q}}(dz)dt\Big)\Big]^{1/2} \\ &= \Big[\mathbb{E}_{\mathbb{Q}}\exp\Big(2\int_{\mathbb{T}}\Big(\frac{z\tilde{h}}{2} - zh\Big)\nu_{\mathbb{Q}}(dz)dt\Big)\Big]^{1/2}.\end{aligned}$$

From the definition of $\tilde{h}$, we have $z\tilde{h} = (1+zh)^4 - 1$. Then,

$$z\tilde{h} - 2zh = (1+zh)^4 - 1 - 2zh = z^4h^4 + 4z^3h^3 + 6z^2h^2 + 2zh\,.$$

Thus,

$$\mathbb{E}_{\mathbb{Q}}[\tilde{S}_T^2] \le \exp\Big(\int_{\mathbb{T}}\Big(z^4h^4 + 4z^3h^3 + 6z^2h^2 + 2zh\Big)\nu_{\mathbb{Q}}(dz)dt\Big)$$

which is finite by the assumptions of the theorem.

From the martingale representation theorem (see, e.g., [18] (Theorem 5.3.5)), there exists a square integrable predictable mapping $\psi : \mathbb{T} \times \Omega \to \mathbb{R}$ such that

$$U(t) = \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T - K)^+) + \int_0^t\int_{\mathbb{J}}\psi(s,z)\tilde{N}(ds,dz).$$

Define

$$\begin{aligned}\pi_S(t) &:= \frac{\int_{\mathbb{J}}\psi(z,t)\tilde{N}_{\mathbb{Q}}(dz,t)}{\tilde{S}(t)g(S(t-b))} \\ &= \frac{\int_{\mathbb{J}}\psi(z,t)\tilde{S}(t)g(S(t-b))d\tilde{S}(t)}{\tilde{S}(t)g(S(t-b))}, \\ \pi_B(t) &:= U(t) - \pi_S(t)\tilde{S}(t), \quad t \in [0,T]\,.\end{aligned}$$

Consider the strategy $\{(\pi_B(t), \pi_S(t)) : t \in [0,T]\}$ to invest $\pi_B(t)$ units in the riskyless asset $B(t)$ and $\pi_S(t)$ units in the risky asset $S(t)$ at time t. Then, the value of the portfolio at time t is given by

$$V(t) := \pi_B(t)e^{rt} + \pi_S(t)S(t) = e^{rt}U(t)$$

By the definition of the strategy, we see that

$$dV(t) = \pi_B(t)de^{rt} + \pi_S(t)dS(t) = e^{rt}dU(t) + U(t)de^{rt}\,.$$

Hence, the strategy is self-financing. Moreover, we have

$$V(T) = e^{rT}U(T) = (S_T - K)^+.$$

Hence, the claim (referring to the European call option) is attainable stand, therefore the market $\{S(t), B(t) : t \in [0, T]\}$ is complete. □

The pricing Formula (39) is hard to evaluate analytically and we use a general Monte-Carlo method to find the approximate values. However, when the time fall in the last delay period, namely, when $t \in [T-b, T]$, we have the following analytic expression for the price.

Theorem 5. *Assume the conditions of Theorem 4. When $t \in [T-b, T]$, then price for the European Call option is given by*

$$
\begin{aligned}
V(t) \quad = \quad & e^{rt} \lim_{v\to\infty} \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\xi}(e^{iv\xi} - e^{iw\xi})A(t) \cdot \tilde{S}(t) \exp\left\{ \int_t^T \int_{\mathbb{J}} \Big((1 + zg(S(u-b)))^{(1-i\xi)} \right. \\
& \left. - (1-i\xi)\ln(1 + zg(S(u-b))) - 1 \Big) \nu_{\mathbb{Q}}(dz)du \right\} \\
& - Ke^{rt} \lim_{v\to\infty} \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\xi}(e^{iv\xi} - e^{iw\xi})A(t) \cdot \tilde{S}(t) \exp\left\{ \int_t^T \int_{\mathbb{J}} \Big((1 + zg(S(u-b)))^{-i\xi} \right. \\
& \left. + i\xi \ln(1 + zg(S(u-b))) - 1 \Big) \nu_{\mathbb{Q}}(dz)du \right\},
\end{aligned} \tag{43}
$$

where $w = \ln(K/A) - rT$ and

$$
A(t) = \exp\Big(\int_t^T \int_{\mathbb{J}} \{\ln(1 + zg(S(u-b))) - zg(S(u-b))\nu_{\mathbb{Q}}(dz)du \Big). \tag{44}
$$

Proof. By (39) for any time $t \in [0, T]$, we have

$$
\begin{aligned}
V(t) \quad &= \quad e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\Big((S(T) - K)^+ \mid \mathcal{F}_t \Big) \\
&= \quad e^{rt} \mathbb{E}_{\mathbb{Q}}\Big((\tilde{S}(T) - Ke^{-rT})^+ \mid \mathcal{F}_t \Big) \\
&= \quad e^{rt} \mathbb{E}_{\mathbb{Q}}\Big(\tilde{S}(T) \mathbb{I}_{\{\tilde{S}(T) \geq Ke^{-rT}\}} \mid \mathcal{F}_t \Big) - Ke^{rt} \mathbb{Q}(\tilde{S}(T) \geq Ke^{-rT}) \\
&=: \quad V_1(t) - V_2(t).
\end{aligned} \tag{45}
$$

First, let us compute $V_1(t)$ and $V_2(t)$ can be computed similarly. The solution $\tilde{S}(t)$ is given by (41), which we rewrite here:

$$
\begin{aligned}
\tilde{S}(T) \quad = \quad & \tilde{S}(t) \exp\Big\{ \int_t^T \int_{\mathbb{J}} \{\ln(1 + zg(S(u-b))) - zg(S(u-b))\} \nu_{\mathbb{Q}}(dz)du \\
& + \int_t^T \int_{\mathbb{J}} \ln(1 + zg(S(u-b))) \tilde{N}_{\mathbb{Q}}(dz, du) \Big\}.
\end{aligned} \tag{46}
$$

when $u \in [t, T]$ and $t \in [T-b, T]$, we see that $S(u-b)$ is $\mathcal{F}_t$-measurable. Hence, while computing the conditional expectation of $h(\tilde{S}(T))$ with respect to $\mathcal{F}_t$, we can consider the integrands $\ln(1 + zg(S(u-b)))$ and $\ln(1 + zg(S(u-b))) - zg(S(u-b))$ as "deterministic" functions. Thus, the analytic expression for the conditional expectation is possible. However, it is still complicated. To find the exact expression and to simplify the presentation, let us use the notation (44) and introduce

$$
Y = \int_t^T \int_{\mathbb{J}} \ln(1 + zg(S(u-b))) \tilde{N}_{\mathbb{Q}}(dz, du).
$$

With these notation we have

$$
\tilde{S}(T) = \tilde{S}(t) A \exp Y.
$$

To calculate $\mathbb{E}_{\mathbb{Q}}\Big(e^Y \mathbb{I}_{\{v \geq Y \geq w\}}\Big)$, we first express $\mathbb{I}_{[w,v]}$ as the (inverse) Fourier transform of exponential function because $\mathbb{E}(e^{i\xi Y})$ is computable. Since the Fourier transform of $\mathbb{I}_{\{w,v\}}$ is

$$\int_{-\infty}^{\infty} e^{ix\xi}\mathbb{I}_{[w,v]}dx = \frac{1}{i\xi}(e^{iv\xi} - e^{iw\xi}),$$

we can write

$$\mathbb{I}_{[w,v]}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{i\xi}(e^{i[v-x]\xi} - e^{i[w-x]\xi})d\xi\,.$$

Therefore, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(e^Y \mathbb{I}_{\{v \geq Y \geq w\}} \mid \mathcal{F}_t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{E}_{\mathbb{Q}}\Big(\frac{1}{i\xi}(e^{i[v-Y]\xi+Y} - e^{i[w-Y]\xi+Y}) \mid \mathcal{F}_t\Big)d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{i\xi}(e^{iv\xi} - e^{iw\xi})\mathbb{E}_{\mathbb{Q}}(e^{Y(1-i\xi)} \mid \mathcal{F}_t)d\xi\,. \end{aligned}$$

Denote $\mathbb{T}_t = [t, T] \times \mathbb{J}$. Then, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(e^{Y-iY\xi}) &= \mathbb{E}_{\mathbb{Q}}\Big(\exp\int_{\mathbb{T}_t}(1-i\xi)\ln(1+zg(S(u-b)))\tilde{N}(dz,du) \mid \mathcal{F}_t\Big) \\ &= \mathbb{E}_{\mathbb{Q}}\Big(\exp\int_{\mathbb{T}_t}(1-i\xi)\ln(1+zg(S(u-b)))\tilde{N}(dz,du)\Big) \\ &= \exp\Big(\int_{\mathbb{T}_t}\{e^{(1-i\xi)\ln(1+zg(S(u-b)))} \\ &\qquad -(1-i\xi)\ln(1+zg(S(u-b))) - 1\}\nu_{\mathbb{Q}}(dz)du\Big) \\ &= \exp\Big(\int_{\mathbb{T}_t}\{(1+zg(S(u-b)))^{(1-i\xi)} \\ &\qquad -\ln(1+zg(S(u-b)))^{(1-i\xi)} - 1\}\nu_{\mathbb{Q}}(dz)du\Big)\,. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(e^Y \mathbb{I}_{\{v \geq Y \geq w\}} \mid \mathcal{F}_t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{i\xi}(e^{iv\xi} - e^{iw\xi})\exp\Big(\int_{\mathbb{T}_t}\{(1+zg(S(u-b)))^{(1-i\xi)} \\ -\ln(1+zg(S(u-b)))^{(1-i\xi)} - 1\}\nu_{\mathbb{Q}}(dz)du\Big)d\xi\,. \end{aligned}$$

Taking $w = \ln(K/A) - rT$, $v \to \infty$ in the above formula, we can evaluate (45) as follows.

$$\begin{aligned} V_1(t) &= e^{rt}\mathbb{E}_{\mathbb{Q}}\Big(\tilde{S}(T)\mathbb{I}_{\{\tilde{S}(T) \geq Ke^{-rT}\}} \mid \mathcal{F}_t\Big) \\ &= e^{rt}\lim_{v\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{i\xi}(e^{iv\xi} - e^{iw\xi})A\cdot\tilde{S}(t)\cdot\exp\Big(\int_{\mathbb{T}_t}\{(1+zg(S(u-b)))^{(1-i\xi)} \\ &\quad -\ln(1+zg(S(u-b)))^{(1-i\xi)} - 1\}\nu_{\mathbb{Q}}(dz)du\Big)d\xi \\ &= e^{rt}\lim_{v\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{i\xi}(e^{iv\xi} - e^{iw\xi})A\cdot\tilde{S}(t).\exp\Big(\int_{\mathbb{T}_t}\{(1+zg(S(u-b)))^{(1-i\xi)} \\ &\quad -\ln(1+zg(S(u-b)))^{(1-i\xi)} - 1\}\nu_{\mathbb{Q}}(dz)du\Big)d\xi\,. \end{aligned}$$

Exactly in the same way (and now without the factor e^Y), we have

$$V_2(t) = Ke^{rt} \lim_{v\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\xi}(e^{iv\xi} - e^{iw\xi})A \cdot \tilde{S}(t). \exp\Big(\int_{\mathbb{T}_t} \{(1 + zg(S(u-b)))^{-i\xi)} - \ln(1 + zg(S(u-b)))^{-i\xi} - 1\}\nu_{\mathbb{Q}}(dz)du\Big)d\xi\,.$$

This gives (43). □

5. Numerical Attempt

In this section, we carry out some numerical computations of our Formula (39) against the American call options Microsoft stock traded on Questrade platform. To apply our model in the financial market, we need to estimate all the parameters including the delay factor b from the real data. To the best of our knowledge, the theory on the parameter estimation is still unavailable even in the case of the classical model of [1]. Motivated by the work of [7], we try our best guess of the parameters in the model (31) and (32).

The real market option prices we consider is for the American call option on Microsoft stock. The data we use are from Questrade trading/investment platform on 5 October 2020 at 12:25 (EDT). We take T to be one, three and six months active trading period, respectively. The real prices of the options of different strike prices are listed in the last column of the three tables below.

The readers may wonder that, since the option pricing formulas for both our model and the classical Black–Scholes model are for the European call option, why we use the market price for the American option. The reason is that we can only find the market price for the American option. On the other hand, as stated by [24] (p. 251), "There is no advantage to exercise an American call prematurely when the asset received upon early exercise does not pay dividends. The early exercise right is rendered worthless when the underlying asset does not pay dividends, so in this case the American call has the same value as that of its European counterpart". See also the work of [25] (p. 61, Theorem 6.1). This justifies our use of the market price for the American option.

Using Monte-Carlo simulation, we calculate the prices of European option given by (39) and the analogous Black–Scholes formula obtained from the model: $dS(t) = S(t)[\alpha dt + \sigma dW(t)]$. We simulate 2000 paths of the solutions to both equations using the logarithmic Euler–Maruyama scheme (for Black–Scholes model, the logarithmic Euler–Maruyama scheme is the same by replacing the jump process by Brownian motion). In the simulations, we take the time step Δ to be the trading unit minute. Thus, when $T = 1$ month, there are

$$n = \text{trading hours} \times 60 \times \text{trading days} = 6.5 \times 60 \times 22 = 8580$$

minutes. Thus, $\Delta = \frac{1}{8580}$. We do the same for $T = 3$ and $T = 6$.

In our calculation for the delayed jump model, we use the double exponential jump process as our Y_is with parameters $p = 0.60, q = 1 - p = 0.40, \eta = 12.8, \theta = 8.40$ with the intensity $\lambda = 0.03$. The interest rate $r = 0.01$ is the risk free rate. The delay factor was taken to be one day, which is $b = \frac{6.5\times 60}{8580}$ because there are trading 6.5 h in a trading day. The function $f(x)$ was taken to be a fixed constant $f(x) = 0.1$, $g(x) = 0.15 * \sin(x/209.11)$ and $\phi(x) = \exp(\alpha x/n)$ with $\alpha = 0.11$. We choose $\alpha = 0.11$ since the initial price we have taken is 209.11 and the predicted average price target of Microsoft stock for next one year (around 12 months from 5 October 2020) is 230 which is 11%.

For the simulation of the Black–Scholes model, based on stock prices for the year 2019, we take volatility of the Microsoft stock as $\sigma = 15\%$ to calculate Black–Scholes price. We take $r = 1\%$ since in the last one year the range of 10 year treasury rate has been between 0.52% and 1.92%.

The computations are summarized in the following tables. Table 1, Table 2, and Table 3 are for one, two, and three month call options respectively. From Tables 1–3 notice an interesting phenomenon that the price we obtain by using our formula is comparable to the Black–Scholes price for shorter maturities and is closer to the real market price for longer maturity. This may be because of our choice of the parameters by guessing.

Table 1. Call option price comparison for $T = 1$ month for Microsoft stock.

Strike Price	Black–Scholes Option Price (European) with One-Month Expiration (No Delay)	Option Price of Jump Model (European) with One-Month Expiration	Market Price of American Option with Expiration One Month
195	16.27	16.08	18.3
200	11.41	11.05	15.15
205	7.65	6.91	12
210	4.54	3.62	9.43
215	2.05	1.48	7
220	0.83	0.61	5.15

Table 2. Call option price comparison for $T = 3$ months for Microsoft stock.

Strike Price	Black–Scholes Option Price (European) with Three-Month Expiration (No Delay)	Option Price of Jump Model (European) with Three-Month Expiration	Market Price of American Option with Expiration Three Months
195	21.37	21.27	24.40
200	16.72	16.99	21.35
205	13.08	14.50	18.55
210	9.65	11.43	15.95
215	6.35	8.58	13.65
220	4.31	7.51	11.55

Table 3. Call option price comparison for $T = 6$ months for Microsoft stock.

Strike Price	Black–Scholes Option Price (European) with Six-Month Expiration (No Delay)	Option Price of Jump Model (European) with Six-Month Expiration	Market Price of American Option with Expiration Six Months
195	28.41	29.53	29.00
200	23.85	26.11	26.15
205	19.49	24.44	23.50
210	16.24	21.15	21.05
215	12.83	18.39	18.80
220	10.58	17.97	16.70

6. Conclusions

In this paper, we introduce and study a stochastic delay equation with jump and derive a formula for the fair price of the European call option. We assume that the jump is dictated by a compensated Lévy process, which includes a process similar to asymmetric double exponential or hyper-exponential jump process. In the numerical execution, we consider the asymmetric double exponential process. Furthermore, we propose a logarithmic Euler–Maruyama scheme (a variant of Euler–Maruyama scheme) which preserves the positivity of the approximate solutions and shows that the convergence rate of this scheme is 0.5 in any L^p norm, the optimal rate for the classical Euler–Maruyama scheme for the stochastic differential equations driven by standard Brownian motion (see, e.g., [17]). From the above tables, we see that the parameters guessed here may not be the best possible values but our formula still gives a good fit to the real market prices compared to the Black–Scholes formula. We note further that the potential research problem of parameter estimation is still open before we can come up with the best possible simulated results.

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