GO-GJRSK Model with Application to Higher Order Risk-Based Portfolio

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Abstract: There are three distinguishing features in the financial time series, such as stock prices, are as follows: (1) Non-normality, (2) serial correlation, and (3) leverage effect. All three points need to be taken into account to model the financial time series. However, multivariate financial time series modeling involves a large number of stocks, with many parameters to be estimated. Therefore, there are few examples of multivariate financial time series modeling that explicitly deal with higher-order moments. Furthermore, there is no multivariate financial time series model that takes all three characteristics above into account. In this study, we propose the generalized orthogonal (GO)-Glosten, Jagannathan, and Runkle GARCH (GJR) model which extends the GO-generalized autoregressive conditional heteroscedasticity (GARCH) model and incorporates the three features of the financial time series. We confirm the effectiveness of the proposed model by comparing the performance of risk-based portfolios with higher-order moments. The results show that the performance with our proposed model is superior to that with baseline methods, and indicate that estimation methods are important in risk-based portfolios with higher moments.

Keywords: GO-GJRSK model; risk-based portfolio; higher order moment

1. Introduction

The financial time series is continually brought to our attention. Daily news reports inform us for instance of the latest stock market index values, currency exchange rates, oil prices, and interest rates. It is often desirable to monitor their price behaviors frequently and to try to understand the probable development of the prices in the future. Institutional and individual investors involved in international trades can all benefit from a deeper understanding of price behaviors. Therefore, modeling and understanding the financial time series is a very important task.

There are three distinguishing features in the financial time series, such as stock prices, as follows: (1) Non-normality, (2) serial correlation, and (3) leverage effect. (1) Non-normality means that the returns of financial time series does not follow the normal distribution because of negative skewness and large kurtosis [1,2]. Therefore, we need to model not only mean and variance but also higher-order moments such as skewness and kurtosis. (2) The serial correlation of return is known as momentum or reversal effect. The momentum [3] (reversal [4]) effect means that stock returns with a large (small) return in the past will continue to increase (decrease) during a certain period thereafter. In the case of volatility, the serial correlation is widely known as the volatility clustering [5]. It is a phenomenon that large changes in returns tend to cluster together, resulting in the persistence of the amplitudes of returns. The volatility clustering has intrigued many researchers and oriented in a major way the development of financial time series models. For example, the autoregressive conditional heteroscedasticity (ARCH) [5] and the generalized ARCH (GARCH) model [6] are intended primarily to model this phenomenon. The authors of [7] demonstrated that skewness and kurtosis also have

serial correlations, and they developed the GARCH with skewness and kurtosis (GARCHSK) model to capture these correlations of skewness and kurtosis. (3) The leverage effect corresponds to a negative correlation between returns and volatility. The Glosten, Jagannathan, and Runkle GARCH (GJR) model [8] is a representative model that captures the leverage effect. Recently, Many studies [9,10] demonstrated a negative correlation between returns, and skewness and kurtosis.

We need to take all three points into account to model the financial time series. However, when considering multivariate financial time series modeling involving a large number of stocks, there are many parameters to be estimated. Even when we only model the time-varying covariance matrix, it is very difficult to estimate it completely [11]. The generalized orthogonal GARCH (GO-GARCH) model [12] is proposed to estimate the time-varying covariance matrix using independent component analysis (ICA) to reduce the parameters to be estimated.

Therefore, there are few examples of multivariate financial time series modeling that explicitly deal with higher-order moments [13]. Furthermore, there is no multivariate financial time series that takes all three characteristics above into account.

In this study, we propose the GO-GJRSK model which extends the GO-GARCH and incorporates the three features of the financial time series. To define GO-GJRSK model, we first introduce a univariate GJR with skewness and kurtosis (GJRSK) model which is the GARCHSK model with a leverage effect. Furthermore, we extend the GJRSK model to a multivariate model with ICA.

Our proposed method describes the conditional volatility (skewness and kurtosis) in a straightforward form: The linear sum of the previous time volatility (skewness and kurtosis) and the shock with leverage effect. As with the GO-GARCH model, the estimation of the GO-GJRSK model is also easy when the number of stocks is large enough.

We confirm the effectiveness of the proposed methodology by comparing the performance of risk-based portfolios with higher-order moments. We use a representative benchmark dataset to test whether differences in estimation methods improve the performance of risk-based portfolios.

The remainder of the paper is organized as follows. Section 2 gives a brief explanation of risk-based portfolios and the volatility modeling methods. Section 3 describes our proposed GO-GJRSK model, which consists of an ICA and GJRSK model. Section 4 experiments with various risk-based portfolios with different estimation methods. Lastly, Section 5 concludes the paper.

2. Related Work

2.1. Risk Based Portfolio

Portfolio construction is one of the most important tasks in finance. Mean Variance Optimization (MVO) proposed by [14] has been the most used in practice for portfolio construction and asset allocation for a long time. The MVO is still a very important framework for investment decision making today. The MVO provides economical justification because it is not only simple and practically tractable but also because it is consistent with the expected utility maximization principle. The goal of MVO is to find portfolios that optimally diversify risk without reducing expected return and to facilitate portfolio construction.

However, there are some practical difficulties with the MVO. First, it is very difficult to estimate the expected return as inputs of the MVO. In addition, the MVO is referred to as "Error Maximization" [15,16]. The reason is that even small changes in the estimated expected returns can result in huge changes in the whole portfolio structure.

Therefore, risk-based portfolio construction methods, which do not require the expected return, which is difficult to estimate, but are based only on risk, are attracting attention as an alternative to the MVO. Minimum variance portfolio [17], risk parity portfolio [18], and maximum diversification portfolio [19] are known as the typical risk-based portfolio construction methods.

In fact, many empirical studies of equity portfolios and asset allocations [17–20] showed that the risk-based portfolio construction methods perform better than the MVO and market

capitalization-weighted portfolios. In [21], it was demonstrated that risk-based portfolios are not very sensitive to the accuracy of estimating covariance matrices, unlike MVO.

More recently, the covariance matrix has been estimated using Gaussian process latent variable models [22], and Student's *t*-process latent variable models [23] for the minimum variance portfolio. And new risk parity portfolios are proposed by using the Hilbert transform to make the covariance matrix complex [24] or extended to quaternions [25].

However, these risk-based portfolio construction methods capture risk only in terms of covariance matrix (second moments) and do not capture the non-normality, i.e., skewness and kurtosis. Therefore, risk-based portfolios with higher-order moments have been proposed and their effectiveness has been confirmed [13,26,27]. We examine whether the estimation method improves the performance of risk-based portfolios with higher moments.

2.2. Volatility Modeling

Volatility modeling is important for risk management and the portfolio selection. In [5], it was proposed that the ARCH model could model the serial correlation of volatility. In [6], the GARCH model was proposed, which is a generalization of the ARCH model. The GARCH model assumes that the volatility at time t is formulated with only variables whose values are already known in time $t - 1$. The GARCH model has been applied to the empirical analysis of many asset prices because the parameters can easily be estimated by the maximum likelihood method. The GARCH model has been modified and extended in a number of ways. An important aspect of improving the GARCH model is the introduction of a leverage effect.

The GJR model [8] is a representative model that incorporates a leverage effect. In another direction of extension of GARCH model, León et al. [7] proposed the GARCHSK model, which also incorporates conditional skewness and conditional kurtosis variations. The feature of the GARCHSK model is that the conditional skewness and conditional kurtosis variations can be explicitly captured with an easy-to-understand structure equivalent to the GARCH model, and the easy to estimate.

Furthermore, extensions to multivariate models are also proposed. In order to efficiently parameterize multivariate GARCH, some methods have been developed based on univariate GARCH models in the following two directions. One is to construct the conditional covariance matrix by explicitly modeling both the volatility and conditional correlation matrix. For example, Engle [28] proposed the dynamic conditional correlation (DCC) model. The other direction is to exploit the idea of factor models to construct multivariate GARCH models. It is widely believed in finance that there exist some common factors, which may be determined empirically or constructed by some statistical methods, driving the evolution of the return series [22,23,29,30]. This approach has commonly been used in much research to model the conditional covariance of financial time series due to its feasibility in estimating large covariance matrices.

Orthogonal GARCH (OGARCH) was first proposed by [31]. The observed time series can be linearly transformed into a set of uncorrelated time series using a principal component analysis. In [12], the concept of ICA is applied to propose the GO-GARCH model for volatility modeling. It consists of a set of conditionally uncorrelated univariate GARCH and a linear map that allows the linkage between these components and observed data. Furthermore, a modified GO-GARCHSK model [32] that incorporates the GO-GARCH model with conditional skewness and conditional kurtosis variation has been proposed. However, there are no multivariate models that have a conditional skewness and kurtosis variation and leverage effect.

Hence, as a multivariate extension of the GJRSK, we propose GO-GJRSK model that has higher order moment variations and a leverage effect (Table 1).

Table 1. Comparison between our proposed and the models of related work.

	Non-Normality	Higher Moment	Leverage Effect
GO-GARCH [12]	○		
GO-GARCHSK [32]	○	○	
GO-GJRSK (Our proposed)	○	○	○

3. GO-GJRSK Model

3.1. Univariate GJRSK Model

We first define a univariate GJRSK model. This model introduces leverage effects in the GARCHSK model. Here, r_t is the asset return, h_t is conditional volatility, s_t is conditional skewness, and k_t is conditional kurtosis at time t. The formulation of the GJRSK model is given as follows.

$$r_t = \alpha_0 + \alpha_1 r_{t-1} + \varepsilon_t \tag{1}$$

$$h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1} + \beta_3 \varepsilon_{t-1}^2 I_{\{\eta_{t-1}<0\}} \tag{2}$$

$$s_t = \gamma_0 + \gamma_1 \eta_{t-1}^3 + \gamma_2 s_{t-1} + \gamma_3 \eta_{t-1}^3 I_{\{\eta_{t-1}<0\}} \tag{3}$$

$$k_t = \delta_0 + \delta_1 \eta_{t-1}^4 + \delta_2 k_{t-1} + \delta_3 \eta_{t-1}^4 I_{\{\eta_{t-1}<0\}} \tag{4}$$

$$\eta_t = h_t^{-\frac{1}{2}} \varepsilon_t \tag{5}$$

$$\eta_t | I_{t-1} \sim g(0, 1, s_t, k_t) \tag{6}$$

Here α_0, α_1 are parameters of the AR model and $\beta_i, \gamma_i, \delta_i, i = 0, 1, 2, 3$ are parameters of the GJRSK model. I_A is an indicator function that returns 1 if A is true and 0 otherwise. g is a probability density function with mean 0, variance 1, skewness s_t, and kurtosis k_t.

As with GARCHSK model, the probability density function $g(0, 1, s_t, k_t)$ of the GJRSK model can be given by Gram-Charlier expansion using Chebyshev-Hermite polynomials as follows:

$$g(\eta_t | I_{t-1}) = \frac{\phi(\eta_t)\psi^2(\eta_t)}{\Gamma_t}, \tag{7}$$

$$\phi(\eta_t) = \frac{1}{\sqrt{2\pi h_t}} \exp(\eta_t^2 - h_t), \tag{8}$$

$$\psi(\eta_t) = 1 + \frac{s_t}{3!}(\eta_t^3 - 3\eta_t) + \frac{k_t - 3}{4!}(\eta_t^4 - 6\eta_t^2 + 3), \tag{9}$$

$$\Gamma_t = 1 + \frac{s_t^2}{3!} + \frac{(k_t - 3)^2}{4!}. \tag{10}$$

The probability density function of ε_t is $f(\eta_t | I_{t-1}) = h_t^{\frac{1}{2}} g(\eta_t | I_{t-1})$ because $\eta_t = h_t^{-\frac{1}{2}} \varepsilon_t$. Therefore, the log-likelihood function l_t without the constant term is obtained as:

$$l_t = -\frac{1}{2} \ln h_t - \frac{1}{2}\eta_t^2 + \ln(\psi^2(\eta_t)) - \Gamma_t. \tag{11}$$

We can estimate each parameter of the GJRSK model by maximizing log-likelihood function l_t.

3.2. GO-GJRSK Model

In this section, we introduce the GO-GJRSK model. We define the co-skewness matrix M_3 and co-kurtosis matrix M_4. Let co-skewness of assets i, j, k be s_{ijk} and S_{ijk} be the matrix that can be made by fixing i of s_{ijk}. The co-skewness matrix M_3 is defined as follows.

$$S_{ijk} = \begin{pmatrix} s_{i11} & \cdots & s_{i1k} \\ \vdots & \ddots & \vdots \\ s_{ij1} & \cdots & s_{ijk} \end{pmatrix} \tag{12}$$

$$M_3 = [S_{1jk}, S_{2jk}, \cdots, S_{Njk}]$$

Let co-kurtosis of assets i, j, k, l be k_{ijkl} and K_{ijkl} be the matrix that can be made by fixing i, j of k_{ijk}. The co-kurtosis matrix M_4 is defined as follows:

$$K_{ijkl} = \begin{pmatrix} k_{ij11} & \cdots & k_{ij1k} \\ \vdots & \ddots & \vdots \\ k_{ijk1} & \cdots & k_{ijkl} \end{pmatrix} \tag{13}$$

$$M_4 = [K_{11kl}, \cdots, K_{1Nkl} | K_{21kl}, \cdots, K_{2Nkl} |, \cdots, | K_{N1kl}, \cdots, K_{NNkl}].$$

The GO-GJRSK model assumes that $\varepsilon_t \in R^n$ is defined by mixing matrix $Z \in R^{n \times n}$ and n independent factors $y_t = (y_{1,t}, \ldots, y_{n,t})^T \in R^n$. When the n assets returns are $r_t \in R^n$, the GO-GJRSK model can be written as follows:

$$r_t = A_0 + A_1 r_{t-1} + \varepsilon_t, \tag{14}$$

$$\varepsilon_t = Z y_t, \tag{15}$$

$$\eta_{i,t} = m_{i,t}^{H\,-\frac{1}{2}} y_{i,t}, \tag{16}$$

$$\eta_{i,t} | I_{t-1} \sim g(0, 1, m_{i,t}^S, m_{i,t}^K), \tag{17}$$

$$m_{i,t}^H = \beta_{i,0} + \beta_{i,1} y_{i,t-1}^2 + \beta_{i,2} m_{i,t-1}^H + \beta_{i,3} y_{i,t-1}^2 I_{\{\eta_{i,t-1} < 0\}}, \tag{18}$$

$$m_{i,t}^S = \gamma_{i,0} + \gamma_{i,1} \eta_{i,t-1}^3 + \gamma_{i,2} m_{i,t-1}^S + \gamma_{i,3} \eta_{i,t-1}^3 I_{\{\eta_{i,t-1} < 0\}}, \tag{19}$$

$$m_{i,t}^K = \delta_{i,0} + \delta_{i,1} \eta_{i,t-1}^4 + \delta_{i,2} m_{i,t-1}^K + \delta_{i,3} \eta_{i,t-1}^4 I_{\{\eta_{i,t-1} < 0\}}, \tag{20}$$

where $A_0 \in R^n$ and $A_1 \in R^{n \times n}$ are parameters of the VAR model and $\beta_{i,j}, \gamma_{i,j}, \delta_{i,j}, i = 1, \ldots, n$, $j = 0, \ldots, 3$ are parameters of the GJRSK model. The GO-GJRSK model follows a probability density function $p(0, H_t, S_t, K_t)$ with a conditional covariance matrix H_t, a co-skewness matrix S_t, and a co-kurtosis matrix K_t. Here, the following relationship holds:

$$H_t = Z M_t^H Z^T \tag{21}$$

$$S_t = Z M_t^S (Z^T \otimes Z^T) \tag{22}$$

$$K_t = Z M_t^K (Z^T \otimes Z^T \otimes Z^T) \tag{23}$$

$$M_t^H = \text{diag}^H \{ m_{1,t}^H, \ldots, m_{n,t}^H \} \tag{24}$$

$$M_t^S = \text{diag}^S \{ m_{1,t}^S, \ldots, m_{n,t}^S \} \tag{25}$$

$$M_t^K = \text{diag}^K \{ m_{1,t}^K, \ldots, m_{n,t}^K \}. \tag{26}$$

The operator $\otimes$ represents the Kronecker product. $\text{diag}^H(e_1, \ldots, e_n)$ makes a $(n \times n)$ sparse covariance matrix with elements e_i when $i = j$ and zero otherwise. $\text{diag}^S(e_1, \ldots, e_n)$ makes a $(n \times n^2)$ sparse co-skewness matrix with elements e_i when $i = j = k$ and zero otherwise. $\text{diag}^K(e_1, \ldots, e_n)$ makes a $(n \times n^3)$ sparse co-kurtosis matrix with elements e_i when $i = j = k = l$ and zero otherwise. Using this relationship, we can obtain the conditional covariance matrix H_t, the co-skewness matrix S_t, and the co-kurtosis matrix K_t.

3.3. Estimation Procedure

The parameters of GO-GJRSK model are estimated by the following procedure.

Step1 VAR Model:

First, we estimate the vector autoregression model (VAR) model in Equation (15). After estimating the parameter of VAR model, the residuals ε_t have been collected for further GO-GJRSK modeling.

Step2 Independent Component Analysis:

ICA is a signal processing method to express a set of random variables as linear combinations of statistically independent component variables [33]. If yi, t and $y_{j,t}, (i \neq j)$ are mutually statistically independent, and mixing matrix Z exists, such that:

$$\varepsilon_t = Zy_t \tag{27}$$

then y_t are n independent component of the residuals ε_t. The mixing matrix Z can be solved by fast fixed-point algorithm given by [34].

Step3 Estimate Univariate GJRSK Model:

Once the independent components y_t are obtained, the conditional variance, conditional skewness, and conditional kurtosis of independent components can be estimated by the univariate GJRSK model. The parameters of the univariate GARCHSK model can be estimated by maximizing the log-likelihood function in Equation (11).

4. Experiment

In this section, we report the results of our empirical studies with well-known benchmarks. We compared the out-of-sample performance among several portfolio strategies to check the effectiveness of our GO-GJRSK model.

4.1. Dataset

In the experiments, we used well-known academic benchmarks called Fama and French (FF) datasets (https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html) to ensure the reproducibility of the experiment. This FF dataset is public and is readily available to anyone. The FF datasets have been recognized as standard datasets and heavily adopted in finance research because of their extensive coverage of asset classes and very long historical data series. We used the FF17 dataset and FF25 dataset. For example, the FF17 dataset contains monthly returns of 17 portfolios representing different industrial sectors while the FF25 dataset includes 25 portfolios formed on the basis of size and book-to-market ratio. We used both datasets as monthly data from January 1989 to December 2018.

4.2. Higher Order Risk-Based Portfolio

Here, we formulate the risk-based portfolio construction methodology in the experiment. In this study, two risk-based portfolios with higher-order moments are used to confirm the effectiveness of our proposed model. One is a minimum variance portfolio and the other is a risk parity portfolio.

Let $w = (w_1, \cdots, w_n)^{\mathrm{T}}$ be the weights of each of the n assets, M_2 is the covariance matrix, M_3 is the co-skewness matrix, and M_4 is the co-kurtosis matrix. Then, the portfolio variance σ_P^2, skewness s_P^3, and kurtosis k_P^4 can be expressed as follows [35]:

$$\sigma_P^2 = w^{\mathrm{T}} M_2 w \tag{28}$$

$$s_P^3 = w^{\mathrm{T}} M_3 (w \otimes w) \tag{29}$$

$$k_P^4 = w^{\mathrm{T}} M_4 (w \otimes w \otimes w) \tag{30}$$

We then use these portfolio characteristics to extend the minimum variance portfolio and the risk-based portfolio to include higher-order moments.

4.3. Minimum Variance Portfolio with Higher Moments

The minimum variance portfolio is extended with the skewness in Equation (29) and kurtosis in Equation (30) as follows [26]:

$$\min_{w} \lambda_1 \sigma_P^2 - \lambda_2 s_P^3 + \lambda_3 k_P^4, \tag{31}$$

$$s.t. \sum_{i=1}^{N} w_i = 1, w_i > 0,$$

where $\lambda_i > 0, i = 1, 2, 3$ are weights to account for the trade-offs between the moments of each order. Here, if $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0$, Equation (31) is the minimum variance portfolio.

4.4. Risk Parity Portfolio with Higher Moments

The extension of risk parity portfolios to higher-order moments is discussed in [27]. First, as in the risk parity portfolio, we define from the second to fourth order marginal risk contribution (MRC). $MRC_i, i = 2, 3, 4$ are derived by differentiating the portfolio variance of Equation (28), skewness of Equation (29), and the kurtosis of Equation (30) by the weight w, respectively.

$$MRC_2 = \frac{1}{2} \frac{\partial \sigma_P^2}{\partial w} = M_2 w \tag{32}$$

$$MRC_3 = \frac{1}{3} \frac{\partial s_P^3}{\partial w} = M_3 (w \otimes w) \tag{33}$$

$$MRC_4 = \frac{1}{4} \frac{\partial k_P^3}{\partial w} = M_4 (w \otimes w \otimes w). \tag{34}$$

Then, we can decompose the portfolio variance, skewness and kurtosis using MMC_i as follows:

$$\sigma_P^2 = \sum_{i=1}^{N} w_i \times MRC_{2,i} = w^{\mathrm{T}} MRC_2 \tag{35}$$

$$s_P^3 = \sum_{i=1}^{N} w_i \times MRC_{3,i} = w^{\mathrm{T}} MRC_3 \tag{36}$$

$$k_P^4 = \sum_{i=1}^{N} w_i \times MRC_{4,i} = w^{\mathrm{T}} MRC_4. \tag{37}$$

We define the contribution of the moments (RM) by dividing Equations (35)–(37) by each portfolio moment, respectively.

$$RM_{2,i} = \frac{w_i \times MRC_{2,i}}{\sigma_P^2} \tag{38}$$

$$RM_{3,i} = \frac{w_i \times MRC_{3,i}}{s_P^3} \tag{39}$$

$$RM_{4,i} = \frac{w_i \times MRC_{4,i}}{k_P^4} \tag{40}$$

Based on the above, the risk parity portfolio is extended as follows [27]:

$$\min_{w} \lambda_1 f_1(w) + \lambda_2 f_2(w) + \lambda_3 f_3(w) \tag{41}$$

$$s.t. \sum_{i=1}^{N} w_i = 1, w_i > 0,$$

$$f_1(w) = \sum_{i=1}^{N}\sum_{j=1}^{N} \left(MC_{2,i} - MC_{2,j}\right)^2,$$

$$f_2(w) = \sum_{i=1}^{N}\sum_{j=1}^{N} \left(MC_{3,i} - MC_{3,j}\right)^2,$$

$$f_3(w) = \sum_{i=1}^{N}\sum_{j=1}^{N} \left(MC_{4,i} - MC_{4,j}\right)^2,$$

where $\lambda_i > 0, i = 1, 2, 3$ are weights to account for the trade-offs among the MC_i. Here, if $\lambda_1 = 1$, $\lambda_2 = 0, \lambda_3 = 0$, Equation (41) is the risk parity portfolio.

4.5. Experimental Settings

In our empirical studies, the tested portfolios have the following meanings:

- "EW" stands for equally-weighted portfolio [36];
- "MV" stands for minimum-variance portfolio [14];
- "RP" stands for risk parity portfolio [18];
- "HMV" stands for higher order minimum-variance portfolio [26]. Here, $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$ in Equation (31);
- "HRP" stands for higher order risk parity portfolio [27]. Here, $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$ in Equation (41).

The following method is used to estimate the covariance, co-skewness, and co-kurtosis matrix:

- "MLE" stands for maximum likelihood estimation of covariance, co-skewness and co-kurtosis matrix;
- "GO" stands for covariance matrix in GO-GARCH model [12];
- "GO-SK" stands for covariance, co-skewness and co-kurtosis matrix in GO-GARCHSK model [32];
- "GO-GJRSK" stands for covariance, co-skewness, and co-kurtosis matrix in our proposed GO-GJRSK model.

We used the latest 10 years (120 months) to compute for the parameter estimation, and used each dataset from July 1936 to March 2020, as the out-of-sample period. Each portfolio was updated by sliding one-month-ahead.

We performed our experiment with R language.

4.6. Performance Measures

We compared the out-of-sample performance of the portfolios. In evaluating the portfolio strategy, we used the following measures.

Here, the portfolio return R_t at time t is defined as:

$$R_t = \sum_{i=1}^{n} r_{i,t} w_{i,t-1}, \tag{42}$$

where $r_{i,t}$ is the return of i asset at time t, $w_{i,t-1}$ is the weight of i asset in the portfolio at time $t-1$, and n is the number of assets.

We evaluated the portfolio strategy using its annualized return (AR), the downside risk as the standard deviation of negative return (DR), and the return/downside risk (R/R) (return divided by downside risk) as the portfolio strategy. As a measure of risk, standard deviation is usually used [14]. However, the standard deviation as a risk measure has several drawbacks. It punishes both upside

and downside deviations of return. Moreover, the variance is justified only when the joint distribution of asset returns is symmetric. In this study, we assume that the asset returns are asymmetric and non-normal as several empirical study shows. On the other hand, DR only penalizes the downside deviations of returns. Hence, it is intuitively much more appealing.

$$AR = \frac{12}{T} \sum_{t=1}^{T} R_t \tag{43}$$

$$DR = \sqrt{\frac{12}{T-1} \sum_{t=1}^{T} (R_t \times 1_{\{R_t \leq 0\}})^2} \tag{44}$$

$$R/R = AR/DR \tag{45}$$

Here, $\mu = (1/T) \sum_{t=1}^{T} R_t$ is the average return of the portfolio.

4.7. Results

The results are shown in Table 2. For both FF17 and FF25, the AR and R/R of HMV portfolio using the GO-GJRSK model are the highest in MV and HMV portfolios. The DR of MV portfolio using the GO-GARCH model has the lowest in MV and HMV portfolios.

Furthermore, interestingly, extending MV to HMV using a simple MLE does not improve R/R. Furthermore, when the GO-GARCHSK model is used in HMV, the R/R is subordinated to the HMV using MLE.

On the other hand, for both FF17 and FF25, HRP using the GO-GJRSK model gives the best results in terms of AR, DR, and R/R in RP and HRP portfolios. As in the result of MV portfolio, extending RP to HRP with a simple MLE does not improve R/R. This is a different result from the previous study [27]. Furthermore, when the GO-GARCHSK model is used in HRP, the R/R is inferior to the HMV using MLE. As in [21], risk-based portfolios without higher moments have little impact on performance due to differences in estimation methods.

We can confirm the effectiveness of our proposed GO-GJRSK model for simple MLE and GO-GARCHSK.

Table 2. AR, DR, and R/R for an EW, MV, RP, HMV, and HRP portfolios using three methods. In the rows labeled AR, DR, and R/R, the best results appear in bold for MV, HMV portfolios, and RP, HRP portfolios.

FF17	EW	MV		HMV			RP		HRP		
		MLE	GO	MLE	GO-SK	GO-GJRSK	MLE	GO	MLE	GO-SK	GO-GJRSK
AR [%]	11.31	10.69	10.26	11.24	11.02	**12.03**	11.36	11.36	11.35	11.01	**12.37**
DR [%]	12.06	9.17	**9.07**	11.55	11.81	9.47	11.52	11.52	11.53	11.64	**10.61**
R/R	0.94	1.17	1.13	0.97	0.93	**1.27**	0.99	0.99	0.98	0.95	**1.17**

FF25	EW	MV		HMV			RP		HRP		
		MLE	GO	MLE	GO-SK	GO-GJRSK	MLE	GO	MLE	GO-SK	GO-GJRSK
AR [%]	13.20	11.58	11.69	12.83	12.45	**13.81**	13.06	13.07	12.97	12.71	**14.05**
DR [%]	14.34	10.94	**10.82**	13.91	14.32	11.54	13.89	13.89	13.88	14.20	**11.38**
R/R	0.92	1.06	1.08	0.92	0.87	**1.20**	0.94	0.94	0.93	0.90	**1.23**

5. Conclusions

In this study, we proposed the GO-GJRSK model which extends the GO-GARCH and incorporates the three features of a financial time series: (1) non-normality, (2) serial correlation, and (3) leverage effect.

We confirmed the effectiveness of the proposed method by comparing the performance of risk-based portfolios with higher-order moments. The results showed that the performance of the proposed method was superior to that of the MLE and GO-GARCHSK. We showed that extending MV to HMV and RP to HRP with a simple MLE did not improve R/R. This result indicated that estimation methods are important in risk-based portfolios with higher moments as with MVO.

A future challenge is to extend our model to stochastic volatility, skewness, and kurtosis models because stochastic volatility model might provide a better alternative to the GARCH models [37].

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Abbreviations

The following abbreviations are used in this manuscript:

ARCH	autoregressive conditional heteroscedasticity
GARCH	generalized ARCH
GARCHSK	GARCH with skewness and kurtosis
GJR	Glosten, Jagannathan and Runkle GARCH
GJRSK	GJR with skewness and kurtosis
DCC	dynamic conditional correlation
O-GARCH	orthogonal GARCH
GO-GARCH	generalized O-GARCH
GO-GARCHSK	GO-GARCH with skewness and kurtosis
GO-GJRSK	GO-GJR with skewness and kurtosis
ICA	independent component analysis
VAR	vector autoregression
FF17	Fama French dataset for 17 sectors
FF25	Fama French dataset for 25 factors
EW	equally weight portfolio
MV	minimum variance portfolio
HMV	minimum variance portfolio with higher moments
RP	risk parity portfolio
HRP	risk parity portfolio with higher moments
MLE	maximum likelihood estimator
AR	annualized return
DR	annualized downside risk
R/R	AR/DR

References

1. Mandelbrot, B.B. The variation of certain speculative prices. In *Fractals and Scaling in Finance*; Springer: Berlin/Heidelberg, Germany, 1997; pp. 371–418.
2. Cont, R. Empirical properties of asset returns: Stylized facts and statistical issues. *Quant. Financ.* **2001**, *1*, 223–236. [CrossRef]
3. Jegadeesh, N.; Titman, S. Returns to buying winners and selling losers: Implications for stock market efficiency. *J. Financ.* **1993**, *48*, 65–91. [CrossRef]
4. Jegadeesh, N. Evidence of predictable behavior of security returns. *J. Financ.* **1990**, *45*, 881–898. [CrossRef]
5. Engle, R.F. Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econom. J. Econom. Soc.* **1982**, *50*, 987–1007. [CrossRef]

6.   Bollerslev, T. Generalized autoregressive conditional heteroskedasticity. *J. Econom.* **1986**, *31*, 307–327. [CrossRef]

7.   León, Á.; Rubio, G.; Serna, G. Autoregresive conditional volatility, skewness and kurtosis. *Q. Rev. Econ. Financ.* **2005**, *45*, 599–618. [CrossRef]

8.   Glosten, L.R.; Jagannathan, R.; Runkle, D.E. On the relation between the expected value and the volatility of the nominal excess return on stocks. *J. Financ.* **1993**, *48*, 1779–1801. [CrossRef]

9.   Boyer, B.; Mitton, T.; Vorkink, K. Expected idiosyncratic skewness. *Rev. Financ. Stud.* **2010**, *23*, 169–202. [CrossRef]

10.   Blau, B.M.; Whitby, R.J. Idiosyncratic kurtosis and expected returns. *J. Investig.* **2017**, *26*, 81–88. [CrossRef]

11.   Engle, R.F.; Ledoit, O.; Wolf, M. Large dynamic covariance matrices. *J. Bus. Econ. Stat.* **2019**, *37*, 363–375. [CrossRef]

12.   Van der Weide, R. GO-GARCH: A multivariate generalized orthogonal GARCH model. *J. Appl. Econom.* **2002**, *17*, 549–564. [CrossRef]

13.   Boudt, K.; Carl, P.; Peterson, B.G. Asset allocation with conditional value-at-risk budgets. *J. Risk* **2012**, *15*, 39–68. [CrossRef]

14.   Markowitz, H. Portfolio selection. *J. Financ.* **1952**, *7*, 77–91.

15.   Michaud, R.O. The Markowitz optimization enigma: Is 'optimized'optimal? *Financ. Anal. J.* **1989**, *45*, 31–42. [CrossRef]

16.   Chopra, V.K.; Ziemba, W.T. The effect of errors in means, variances, and covariances on optimal portfolio choice. In *Handbook of the Fundamentals of Financial Decision Making: Part I*; World Scientific: Singapore, 2013; pp. 365–373.

17.   Clarke, R.G.; De Silva, H.; Thorley, S. Minimum-variance portfolios in the US equity market. *J. Portf. Manag.* **2006**, *33*, 10–24. [CrossRef]

18.   Maillard, S.; Roncalli, T.; Teïletche, J. The properties of equally weighted risk contribution portfolios. *J. Portf. Manag.* **2010**, *36*, 60–70. [CrossRef]

19.   Choueifaty, Y.; Coignard, Y. Toward maximum diversification. *J. Portf. Manag.* **2008**, *35*, 40–51. [CrossRef]

20.   Lee, W. Risk-Based Asset Allocation: A New Answer to an Old Question? *J. Portf. Manag.* **2011**, *37*, 11. [CrossRef]

21.   Nakagawa, K.; Imamura, M.; Yoshida, K. Risk-based portfolios with large dynamic covariance matrices. *Int. J. Financ. Stud.* **2018**, *6*, 52. [CrossRef]

22.   Nirwan, R.S.; Bertschinger, N. Applications of gaussian process latent variable models in finance. In Proceedings of the SAI Intelligent Systems Conference, London, UK, 5–6 September 2019; Springer: Berlin/Heidelberg, Germany, 2019; pp. 1209–1221.

23.   Uchiyama, Y.; Nakagawa, K. TPLVM: Portfolio Construction by Student's t-Process Latent Variable Model. *Mathematics* **2020**, *8*, 449. [CrossRef]

24.   Uchiyama, Y.; Kadoya, T.; Nakagawa, K. Complex Valued Risk Diversification. *Entropy* **2019**, *21*, 119. [CrossRef]

25.   Sugitomo, S.; Maeta, K. Quaternion Valued Risk Diversification. *Entropy* **2020**, *22*, 390. [CrossRef]

26.   Martellini, L.; Ziemann, V. Improved estimates of higher-order comoments and implications for portfolio selection. *Rev. Financ. Stud.* **2010**, *23*, 1467–1502. [CrossRef]

27.   Baitinger, E.; Dragosch, A.; Topalova, A. Extending the risk parity approach to higher moments: Is there any value added? *J. Portf. Manag.* **2017**, *43*, 24–36. [CrossRef]

28.   Engle, R. Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *J. Bus. Econ. Stat.* **2002**, *20*, 339–350. [CrossRef]

29.   Fama, E.F.; French, K.R. Common risk factors in the returns on stocks and bonds. *J. Financ. Econ.* **1993**, *33*, 3–56. [CrossRef]

30.   Fama, E.F.; French, K.R. A five-factor asset pricing model. *J. Financ. Econ.* **2015**, *116*, 1–22. [CrossRef]

31.   Ding, Z. *Time Series Analysis of Speculative Returns*; Department of Economics, University of California: San Diego, CA, USA, 1994.

32.   Xu, Q.F.; Jiang, C.X.; Kang, P. Dynamic portfolio selection under higher moments. In Proceedings of the 2007 International Conference on Machine Learning and Cybernetics, Hong Kong, China, 19–22 August 2007; IEEE: Piscataway, NJ, USA, 2007; Volume 5, pp. 2488–2493.

33.   Comon, P. Independent component analysis, a new concept? *Signal Process.* **1994**, *36*, 287–314. [CrossRef]

34. Hyvärinen, A.; Oja, E. A fast fixed-point algorithm for independent component analysis. *Neural Comput.* **1997**, *9*, 1483–1492. [CrossRef]

35. Jondeau, E.; Rockinger, M. Optimal portfolio allocation under higher moments. *Eur. Financ. Manag.* **2006**, *12*, 29–55. [CrossRef]

36. DeMiguel, V.; Garlappi, L.; Uppal, R. Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy? *Rev. Financ. Stud.* **2009**, *22*, 1915–1953. [CrossRef]

37. Chan, J.C.; Grant, A.L. Modeling energy price dynamics: GARCH versus stochastic volatility. *Energy Econ.* **2016**, *54*, 182–189. [CrossRef]