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The Dual Orlicz–Aleksandrov–Fenchel Inequality

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Abstract: In this paper, the classical dual mixed volume of star bodies $\widetilde{V}(K_1, \cdots, K_n)$ and dual Aleksandrov–Fenchel inequality are extended to the Orlicz space. Under the framework of dual Orlicz-Brunn-Minkowski theory, we put forward a new affine geometric quantity by calculating first order Orlicz variation of the dual mixed volume, and call it *Orlicz multiple dual mixed volume*. We generalize the fundamental notions and conclusions of the dual mixed volume and dual Aleksandrov-Fenchel inequality to an Orlicz setting. The classical dual Aleksandrov-Fenchel inequality and dual Orlicz-Minkowski inequality are all special cases of the new dual Orlicz-Aleksandrov-Fenchel inequality. The related concepts of L_p-dual multiple mixed volumes and L_p-dual Aleksandrov-Fenchel inequality are first derived here. As an application, the dual Orlicz–Brunn–Minkowski inequality for the Orlicz harmonic addition is also established.

Keywords: dual mixed volume; dual Aleksandrov–Fenchel inequality; Orlicz harmonic radial addition; Orlicz dual mixed volume; Orlicz dual Minkowski inequality; dual Orlicz–Brunn–Minkowski theory

1. Introduction

It is well known that vector addition is one of the important operators in convex geometry. As an operation between sets K and L, defined by

$$K + L = \{x + y : x \in K, y \in L\},$$

it is called Minkowski addition and plays an important role in the convex geometry. During the last few decades, the theory has been extended to L_p–Brunn–Minkowski theory. L_p addition of K and L was introduced by Firey in [1,2], denoted by $+_p$, and defined by

$$h(K +_p L, x)^p = h(K, x)^p + h(L, x)^p,$$

for $p \geq 1$, $x \in \mathbb{R}^n$ and compact convex sets K and L in $\mathbb{R}^n$ containing the origin. Here, function $h(K, \cdot)$ denotes the support function of K. If K is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then

$$h(K, x) = \max\{x \cdot y : y \in K\},$$

for $x \in \mathbb{R}^n$. A nonempty closed convex set is uniquely determined by its support function. L_p-addition is the fundamental and core content in the L_p–Brunn–Minkowski theory. For recent important results and more information from this theory, we refer to [3–23] and the references therein.

In recent years, the study turned to an Orlicz–Brunn–Minkowski theory, initiated by Lutwak, Yang, and Zhang [24,25]. Gardner, Hug, and Weil [26] introduced a corresponding Orlicz addition and established first the Orlicz–Minkowski, and Orlicz–Brunn–Minkowski inequalities. The same concepts and inequalities are derived by Xi, Jin and Leng [27] using a new geometric symmetry technique. Other articles on this theory can be found in the literature [28–35].

The radial addition $K\widetilde{+}L$ of star sets (compact sets that are star-shaped at o and contain o) K and L can be defined by

$$\rho(K\widetilde{+}L,\cdot) = \rho(K,\cdot) + \rho(L,\cdot),$$

where $\rho(K,\cdot)$ denotes the radial function of star set K. The radial function of star set K is defined by

$$\rho(K,u) = \max\{c \geq 0 : cu \in K\},$$

for $u \in S^{n-1}$. The origin and history of the radial addition can be referred to [36], p. 235. When $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies about the origin in $\mathbb{R}^n$. The radial addition and volume are the core and essence of the classical dual Brunn–Minkowski theory and played an important role in the theory (see, e.g., [20,37–42] for recent important contributions). Lutwak [43] introduced the concept of dual mixed volumes that laid the foundation of the dual Brunn–Minkowski theory. What is particularly important is that this theory plays a very important and key role in solving the Busemann–Petty problem in [38,44–46].

For any $p \neq 0$, the L_p-radial addition $K\widetilde{+}_pL$ defined by (see [47] and [48])

$$\rho(K\widetilde{+}_pL,x)^p = \rho(K,x)^p + \rho(L,x)^p,$$

for $x \in \mathbb{R}^n$ and $K, L \in \mathcal{S}^n$. Obviously, when $p = 1$, the L_p-radial addition $\widetilde{+}_p$ becomes the radial addition $\widetilde{+}$. The L_p-harmonic radial addition was defined by Lutwak [9]: If K, L are star bodies, the L_p-harmonic radial addition, defined by

$$\rho(K\widehat{+}_pL,x)^{-p} = \rho(K,x)^{-p} + \rho(L,x)^{-p}, \tag{1}$$

for $p \geq 1$ and $x \in \mathbb{R}^n$. The L_p-harmonic radial addition of convex bodies was first studied by Firey [1]. The operation of the L_p-harmonic radial addition and L_p-dual Minkowski, Brunn–Minkwski inequalities are the basic concept and inequalities in the L_p-dual Brunn–Minkowski theory. The latest information and important results of this theory can be referred to [32,37,39,40,47–51] and the references therein. For a systematic investigation on the concepts of the addition for convex body and star body, we refer the reader to [26,48,50]. L_p-dual Brunn–Minkowski theory has been extended to dual Orlicz–Brunn–Minkowski theory. The dual Orlicz–Brunn–Minkowski theory has also attracted attention, see [52–57]. The Orlicz harmonic radial addition $K\widehat{+}_\phi L$ of two star bodies K and L, defined by (see [57])

$$\rho(K\widehat{+}_\phi L,u)) = \sup\left\{\lambda > 0 : \phi\left(\frac{\rho(K,u)}{\lambda}\right) + \phi\left(\frac{\rho(L,u)}{\lambda}\right) \leq \phi(1)\right\}, \tag{2}$$

where $u \in S^{n-1}$, and $\phi : (0,\infty) \to (0,\infty)$ is a convex and decreasing function such that $\phi(0) = \infty$, $\lim_{t\to\infty}\phi(t) = 0$ and $\lim_{t\to 0}\phi(t) = \infty$. Let $\mathcal{C}$ denote the class of the convex and decreasing functions ϕ with $\phi(0) = \infty$, $\lim_{t\to\infty}\phi(t) = 0$, and $\lim_{t\to 0}\phi(t) = \infty$. Obviously, if $\phi(t) = t^{-p}$ and $p \geq 1$, then the Orlicz harmonic radial addition becomes the L_p-harmonic radial addition. The dual Orlicz mixed volume, denoted by $\widetilde{V}_\phi(K,L)$, defined by

$$\begin{aligned}\widetilde{V}_\phi(K,L) &:= \frac{\phi_r'(1)}{n}\lim_{\varepsilon\to 0^+}\frac{V(K\widehat{+}_\phi\varepsilon\cdot L) - V(K)}{\varepsilon}\\ &= \frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^n dS(u),\end{aligned} \tag{3}$$

where $K\widehat{+}_\phi\varepsilon\cdot L$ is the Orlicz harmonic linear combination of K and L (see Section 3), and the right derivative of a real-valued function ϕ is denoted by ϕ_r'. When $\phi(t)=t^{-p}$ and $p\geq 1$, the dual Orlicz mixed volume $\widetilde{V}_\phi(K,L)$ becomes the L_p-dual mixed volume $\widetilde{V}_{-p}(K,L)$, defined by (see [9])

$$\widetilde{V}_{-p}(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n+p}\rho(L,u)^{-p}dS(u). \tag{4}$$

If $K_1,\ldots,K_n\in\mathcal{S}^n$, the dual mixed volume of star bodies $K_1,\ldots,K_n$, denoted by $\widetilde{V}(K_1,\ldots,K_n)$, defined by Lutwak (see [43])

$$\widetilde{V}(K_1,\ldots,K_n)=\frac{1}{n}\int_{S^{n-1}}\rho(K_1,u)\cdots\rho(K_n,u)dS(u). \tag{5}$$

Lutwak's dual Aleksandrov–Fenchel inequality is the following: If $K_1,\ldots,K_n\in\mathcal{S}^n$ and $1\leq r\leq n$, then

$$\widetilde{V}(K_1,\cdots,K_n)\leq\prod_{i=1}^{r}\widetilde{V}(K_i\ldots,K_i,K_{r+1},\ldots,K_n)^{1/r},$$

with equality if and only if $K_1,\ldots,K_r$ are all dilations of each other.

As we all know, the dual mixed volume $\widetilde{V}_{-1}(K,L)$ of star bodies K and L has been extended to the L_p-space. Following this, the L_p-dual mixed volume $\widetilde{V}_{-p}(K,L)$ has been extended to the Orlicz space and becomes dual Orlicz mixed volume $\widetilde{V}_\phi(K,L)$. However, the classical dual mixed volume $\widetilde{V}(K_1,\ldots,K_n)$ has not been extended to the Orlicz space, and this question becomes a difficult research in convex geometry. Why? We all know that the history of geometric research has always followed the order from general convex geometric space to L_p-space, and then from L_p-space to Orlicz space. The dual mixed volume $\widetilde{V}(K_1,\ldots,K_n)$ has not been extended to L_p-space. In other words, there is nothing in the L_p-space about the dual mixed volume $\widetilde{V}(K_1,\ldots,K_n)$, which can be used as the basis for our further study. As a result, directly extend it to the Orlicz space. Its difficulty can be imagined. In this paper, our main aim is to generalize direct the classical dual mixed volumes $\widetilde{V}(K_1,\ldots,K_n)$ and dual Aleksandrov–Fenchel inequality to the Orlicz space without passing through L_p-space. Amazingly, all the corresponding concepts and inequalities of the L_p-space of the dual mixed volume $\widetilde{V}(K_1,\ldots,K_n)$ are all derived, which subverts the order of historical research on the issue, directly deriving the results of Orlicz space, saving a lot of time and resources. This is also unimaginable.

Under the framework of dual Orlicz–Brunn–Minkowski theory, we introduce the affine geometric quantity by calculating the first order Orlicz variation of the dual mixed volumes, and call it Orlicz multiple dual mixed volumes, denoted by $\widetilde{V}_\phi(K_1,\ldots,K_n,L_n)$, which involves $(n+1)$ star bodies in $\mathbb{R}^n$. The fundamental notions and conclusions of the dual mixed volume $\widetilde{V}(K_1,\ldots,K_n)$ and the dual Minkowski, and Aleksandrov–Fenchel inequalities are extended to an Orlicz setting. The related concepts and conclusions of L_p-multiple dual mixed volume $\widetilde{V}_{-p}(K_1,\ldots,K_n,L_n)$ and L_p-dual Aleksandrov–Fenchel inequality are first derived here. The new dual Orlicz–Aleksandrov–Fenchel inequality in special cases yields the dual Aleksandrov–Fenchel inequality and the Orlicz dual Minkowski inequality for the dual quermassintegrals, respectively. As an application, a new dual Orlicz–Brunn–Minkowski inequality for the Orlicz harmonic radial addition is established, which implies the dual Orlicz–Brunn–Minkowski inequality for the dual quermassintegrals.

Complying with the spirit of introduction of Aleksandrov, Fenchel and Jessen's mixed quermassintegrals, and introduction of Lutwak's L_p-mixed quermassintegrals, we calculate the first order Orlicz variational of dual mixed volumes. If convex bodies $K_2,\ldots,K_n$ are given, we often use the abbreviations $\Im:=K_2,\ldots,K_n$; $\Im\in\mathcal{S}^n:=K_2,\ldots,K_n\in\mathcal{S}^n$ and $\rho_\Im:=\rho(K_2,u)\cdots\rho(K_n,u)$. In Section 4, we prove that the first order Orlicz variation of the dual mixed volumes can be expressed as:

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}\widetilde{V}(L_1\widehat{+}_\phi\varepsilon\cdot K_1,\Im)=\frac{1}{\phi_r'(1)}\cdot\widetilde{V}_\phi(L_1,K_1,\Im),$$

where $L_1, K_1, \Im \in \mathcal{S}^n$, $\phi \in \mathcal{C}$ and $\varepsilon > 0$. In the above first order variational equation, we find a new geometric quantity. Based on this, we extract the required geometric quantity, denoted by $\widetilde{V}_\phi(L_1, K_1, \Im)$ and call it Orlicz multiple dual mixed volume of $(n+1)$ star bodies $L_1, K_1, \Im$, defined by

$$\widetilde{V}_\phi(L_1, K_1, \Im) := \phi'_r(1) \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \widetilde{V}(L_1 \widehat{+}_\phi \varepsilon \cdot K_1, \Im).$$

We also prove the new affine geometric quantity $\widetilde{V}_\phi(L_1, K_1, \Im)$ has an integral representation.

$$\widetilde{V}_\phi(L_1, K_1, \Im) = \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{\rho(K_1, u)}{\rho(L_1, u)}\right) \rho(L_1, u) \rho_\Im dS(u), \tag{6}$$

Obviously, the dual mixed volume $\widetilde{V}(K_1, \ldots, K_n)$ and dual Orlicz mixed volume $\widetilde{V}_\phi(K, L)$ are all special cases of $\widetilde{V}_\phi(L_1, K_1, \Im)$. When $\phi(t) = t^{-p}$, $p \geq 1$, Orlicz multiple dual mixed volume $\widetilde{V}_\phi(L_1, K_1, \Im)$ becomes a new dual mixed volume in L_p-place, denoted by $\widetilde{V}_{-p}(K_1, \ldots, K_n, L_n)$, call it L_p multiple dual mixed volume. From (6), we have

$$\widetilde{V}_{-p}(L_1, K_1, \Im) = \frac{1}{n} \int_{S^{n-1}} \left(\frac{\rho(L_1, u)}{\rho(K_1, u)}\right)^p \rho(L_1, u) \rho_\Im dS(u). \tag{7}$$

The following harmonic mixed p-quermassintegral $\widetilde{W}_{-p,i}(K, L)$ is a special case of $\widetilde{V}_{-p}(L_1, K_1, \Im)$, defined by (see Section 2)

$$\widetilde{W}_{-p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i+p} \rho(L, u)^{-p} dS(u). \tag{8}$$

In Section 5, we establish the following dual Orlicz–Aleksandrov–Fenchel inequality for the Orlicz multiple dual mixed volumes.

The dual Orlicz–Aleksandrov–Fenchel inequality *If $L_1, K_1, \Im \in \mathcal{S}^n$, $\phi \in \mathcal{C}$ and $1 \leq r \leq n$, then*

$$\widetilde{V}_\phi(L_1, K_1, \Im) \geq \widetilde{V}(L_1, \Im) \cdot \phi\left(\frac{\prod_{i=1}^r \widetilde{V}(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{1/r}}{\widetilde{V}(L_1, \Im)}\right). \tag{9}$$

If ϕ is strictly convex, equality holds if and only if $L_1, K_1, \ldots, K_r$ are all dilations of each other.

Obviously, Lutwak's dual Aleksandrov–Fenchel inequality is a special case of (9). If $K_1, \Im \in \mathcal{S}^n$ and $1 \leq r \leq n$, then

$$\widetilde{V}(K_1, \cdots, K_n) \leq \prod_{i=1}^r \widetilde{V}(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{1/r}, \tag{10}$$

with equality if and only if $K_1, \ldots, K_r$ are all dilations of each other. When $\phi(t) = t^{-p}$, $p \geq 1$, the dual Orlicz–Aleksandrov–Fenchel inequality (9) becomes the following L_p-dual Aleksandrov–Fenchel inequality.

The L_p-dual Aleksandrov–Fenchel inequality *If $L_1, K_1, \Im \in \mathcal{S}^n$, $p \geq 1$ and $1 \leq r \leq n$, then*

$$\widetilde{V}_{-p}(L_1, K_1, \Im) \geq \widetilde{V}(L_1, \Im)^{p+1} \cdot \prod_{i=1}^r \widetilde{V}(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{-p/r}. \tag{11}$$

If ϕ is strictly convex, equality holds if and only if $L_1, K_1, \ldots, K_r$ are all dilations of each other.

The following dual Orlicz–Minkowski inequality (see [57]) is a special case of the dual Orlicz–Aleksandrov–Fenchel inequality (9). If $K, L \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then

$$\widetilde{V}_\phi(K, L) \geq V(K) \cdot \phi\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right). \tag{12}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates. In Section 5, we show also the Orlicz–Aleksandrov–Fenchel inequality (9) in special case yields also the following result. If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $\phi \in \mathcal{C}$, then

$$\widetilde{W}_{\phi,i}(K,L) \geq \widetilde{W}_i(K)\phi\left(\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\right)^{1/(n-i)}\right). \tag{13}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates. Here, $\widetilde{W}_i(K)$ is the usually dual quermassintegral of K, and $\widetilde{W}_{\phi,i}(K,L)$ is the Orlicz dual mixed quermassintegral of K and L, defined by (see Section 4)

$$\widetilde{W}_{\phi,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^{n-i}dS(u). \tag{14}$$

In Section 6, we establish the following dual Orlicz Brunn–Minkowski type inequality. If $L_1, K_1, \Im \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then

$$\phi(1) \geq \phi\left(\frac{\widetilde{V}(K_1,\Im)}{\widetilde{V}(K_1\widehat{+}_\phi L_1,\Im)}\right) + \phi\left(\frac{\widetilde{V}(L_1,\Im)}{\widetilde{V}(K_1\widehat{+}_\phi L_1,\Im)}\right). \tag{15}$$

If ϕ is strictly convex, equality holds if and only if $L_1, K_1, \Im$ are all dilations of each other. A special case of (15) is the following inequality.

$$\phi(1) \geq \phi\left(\left(\frac{V(K_1)\cdots V(K_n)}{\widetilde{V}(K_1\widehat{+}_\phi L_1,\Im)^n}\right)^{1/n}\right) + \phi\left(\left(\frac{V(L_1)V(K_2)\cdots V(K_n)}{\widetilde{V}(K_1\widehat{+}_\phi L_1,\Im)^n}\right)^{1/n}\right). \tag{16}$$

If ϕ is strictly convex, equality holds if and only if $L_1, K_1, \ldots, K_n$ are all dilations of each other. Putting $K_1 = K, L_1 = L$ and $K_2 = \cdots = K_n = K_1\widehat{+}_\phi L_1$ in (16), it follows the Orlicz dual Brunn–Minkowski inequality established in [57]. In Section 6, we show also the dual Orlicz–Brunn–Minkowski inequality (16) in a special case yields the following result. If $K, L \in \mathcal{S}^n$, $\phi \in \mathcal{C}$ and $0 \leq i < n-1$, then

$$\phi(1) \geq \phi\left(\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K\widehat{+}_\phi L)}\right)^{1/(n-i)}\right) + \phi\left(\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K\widehat{+}_\phi L)}\right)^{1/(n-i)}\right). \tag{17}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

2. Preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. A body in $\mathbb{R}^n$ is a compact set equal to the closure of its interior. For a compact set $K \subset \mathbb{R}^n$, we write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the volume of K. The unit ball in $\mathbb{R}^n$ and its surface are denoted by B and S^{n-1}, respectively. Let $\mathcal{K}^n$ denote the class of nonempty compact convex subsets containing the origin in their interiors in $\mathbb{R}^n$. Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin and contains the origin, its radial function is $\rho(K,\cdot): S^{n-1} \to [0,\infty)$, defined by

$$\rho(K,u) = \max\{\lambda \geq 0 : \lambda u \in K\}.$$

Two star bodies K and L are dilates if $\rho(K,u)/\rho(L,u)$ is independent of $u \in S^{n-1}$. If $\lambda > 0$, then

$$\rho(\lambda K,u) = \lambda\rho(K,u).$$

From the definition of the radial function, it follows immediately that for $A \in GL(n)$ the radial function of the image $AK = \{Ay : y \in K\}$ of K is given by (see e.g., [36])

$$\rho(AK, u) = \rho(K, A^{-1}u),$$

for all $u \in S^{n-1}$. Namely, the radial function is homogeneous of degree -1. Let $\tilde{\delta}$ denote the radial Hausdorff metric, as follows, if $K, L \in \mathcal{S}^n$, then (see e.g., [58])

$$\tilde{\delta}(K, L) = |\rho(K, u) - \rho(L, u)|_\infty.$$

2.1. Dual Mixed Volumes

The polar coordinate formula for volume of a compact set K is

$$V(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^n dS(u). \tag{18}$$

The first dual mixed volume, $\widetilde{V}_1(K, L)$, defined by

$$\widetilde{V}_1(K, L) = \frac{1}{n}\lim_{\varepsilon\to 0^+}\frac{V(K\tilde{+}\varepsilon\cdot L) - V(K)}{\varepsilon},$$

where $K, L \in \mathcal{S}^n$. The integral representation for first dual mixed volume is proved: For $K, L \in \mathcal{S}^n$,

$$\widetilde{V}_1(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n-1}\rho(L, u) dS(u). \tag{19}$$

The Minkowski inequality for first dual mixed volume is the following: If $K, L \in \mathcal{S}^n$, then

$$\widetilde{V}_1(K, L)^n \le V(K)^{n-1}V(L),$$

with equality if and only if K and L are dilates. (see [45]) If $K_1, \ldots, K_n \in \mathcal{S}^n$, the dual mixed volume $\widetilde{V}(K_1, \ldots, K_n)$ is defined by (see [43])

$$\widetilde{V}(K_1, \ldots, K_n) = \frac{1}{n}\int_{S^{n-1}} \rho(K_1, u)\cdots\rho(K_n, u) dS(u). \tag{20}$$

If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the dual mixed volume $\widetilde{V}(K_1, \ldots, K_n)$ is written as $\widetilde{V}_i(K, L)$. If $L = B$, the dual mixed volume $\widetilde{V}_i(K, L) = \widetilde{V}_i(K, B)$ is written as $\widetilde{W}_i(K)$ and called dual quermassintegral of K. For $K \in \mathcal{S}^n$ and $0 \le i < n$,

$$\widetilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n-i} dS(u). \tag{21}$$

If $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = B$ and $K_n = L$, the dual mixed volume $\widetilde{V}(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i}, L)$ is written as $\widetilde{W}_i(K, L)$ and called dual mixed quermassintegral of K and L. For $K, L \in \mathcal{S}^n$ and $0 \le i < n$, it is easy that ([33])

$$\widetilde{W}_i(K, L) = \lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\tilde{+}\varepsilon\cdot L) - \widetilde{W}_i(K)}{\varepsilon} = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n-i-1}\rho(L, u) dS(u). \tag{22}$$

The fundamental inequality for dual mixed quermassintegral stated that: If $K, L \in \mathcal{S}^n$ and $0 \le i < n$, then

$$\widetilde{W}_i(K, L)^{n-i} \le \widetilde{W}_i(K)^{n-1-i}\widetilde{W}_i(L), \tag{23}$$

with equality if and only if K and L are dilates. The Brunn–Minkowski inequality for dual quermassintegral is the following: If $K, L \in \mathcal{S}^n$ and $0 \leq i < n$, then

$$\widetilde{W}_i(K\widetilde{+}L)^{1/(n-i)} \leq \widetilde{W}_i(K)^{1/(n-i)} + \widetilde{W}_i(L)^{1/(n-i)}, \tag{24}$$

with equality if and only if K and L are dilates.

2.2. L_p-dual Mixed Volume

The dual mixed volume $\widetilde{V}_{-1}(K, L)$ of star bodies K and L is defined by ([9])

$$\widetilde{V}_{-1}(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K) - V(K\widehat{+}\varepsilon \cdot L)}{\varepsilon}, \tag{25}$$

where $\widehat{+}$ is the harmonic addition. The following is a integral representation for the dual mixed volume $\widetilde{V}_{-1}(K, L)$:

$$\widetilde{V}_{-1}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n+1} \rho(L, u)^{-1} dS(u). \tag{26}$$

The dual Minkowski inequality for the dual mixed volume states that

$$\widetilde{V}_{-1}(K, L)^n \geq V(K)^{n+1} V(L)^{-1}, \tag{27}$$

with equality if and only if K and L are dilates. (see ([42]))

The dual Brunn–Minkowski inequality for the harmonic addition states that

$$V(K\widehat{+}L)^{-1/n} \geq V(K)^{-1/n} + V(L)^{-1/n}, \tag{28}$$

with equality if and only if K and L are dilates (This inequality is due to Firey [1]).

The L_p-dual mixed volume $\widetilde{V}_{-p}(K, L)$ of K and L is defined by ([9])

$$\widetilde{V}_{-p}(K, L) = -\frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K\widehat{+}_p\varepsilon \cdot L) - V(K)}{\varepsilon}, \tag{29}$$

where $K, L \in \mathcal{S}^n$ and $p \geq 1$.

The following is an integral representation for the L_p-dual mixed volume: For $K, L \in \mathcal{S}^n$ and $p \geq 1$,

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n+p} \rho(L, u)^{-p} dS(u). \tag{30}$$

L_p-dual Minkowski and Brunn-Minkowski inequalities were established by Lutwak [9]: If $K, L \in \mathcal{S}^n$ and $p \geq 1$, then

$$\widetilde{V}_{-p}(K, L)^n \geq V(K)^{n+p} V(L)^{-p}, \tag{31}$$

with equality if and only if K and L are dilates, and

$$V(K\widehat{+}_pL)^{-p/n} \geq V(K)^{-p/n} + V(L)^{-p/n}, \tag{32}$$

with equality if and only if K and L are dilates.

2.3. Mixed p-harmonic Quermassintegral

From (1), it is easy to see that if $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, then

$$-\frac{p}{n-i} \lim_{\varepsilon \to 0^+} \frac{\widetilde{W}_i(K\widehat{+}_p\varepsilon \cdot L) - \widetilde{W}_i(L)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} \rho(K.u)^{n-i+p} \rho(L.u)^{-p} dS(u). \tag{33}$$

Let $K, L \in \mathcal{S}^n$, $0 \le i < n$ and $p \ge 1$, the mixed p-harmonic quermassintegral of star K and L, denoted by $\widetilde{W}_{-p,i}(K, L)$, defined by (see [59])

$$\widetilde{W}_{-p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i+p}\rho(L,u)^{-p}dS(u). \tag{34}$$

Obviously, when $K = L$, the p-harmonic quermassintegral $\widetilde{W}_{-p,i}(K, L)$ becomes the dual quermassintegral $\widetilde{W}_i(K)$. The Minkowski and Brunn–Minkowski inequalities for the mixed p-harmonic quermassintegral are following (see [59]): If $K, L \in \mathcal{S}^n$, $0 \le i < n$, and $p \ge 1$, then

$$\widetilde{W}_{-p,i}(K,L)^{n-i} \ge \widetilde{W}_i(K)^{n-i+p}\widetilde{W}_i(L)^{-p}, \tag{35}$$

with equality if and only if K and L are dilates. If $K, L \in \mathcal{S}^n$, $0 \le i < n$, and $p \ge 1$, then

$$\widetilde{W}_i(K \widehat{+}_p L)^{-p/(n-i)} \ge \widetilde{W}_i(K)^{-p/(n-i)} + \widetilde{W}_i(L)^{-p/(n-i)}, \tag{36}$$

with equality if and only if K and L are dilates.

Inequality (36) is a Brunn–Minkowski type inequality for the p-harmonic addition. For different variants of dual Brunn–Minkowski inequalities, we refer to [16,46,60–65] and the references therein.

3. Orlicz Harmonic Linear Combination

Throughout the paper, the standard orthonormal basis for $\mathbb{R}^n$ will be $\{e_1, \dots, e_n\}$. Let $\mathcal{C}_m$, $m \in \mathbb{N}$, denote the set of convex function $\phi : [0,\infty)^m \to (0,\infty)$ that are strictly decreasing in each variable and satisfy $\phi(0) = \infty$. When $m = 1$, we shall write $\mathcal{C}$ instead of $\mathcal{C}_1$. Orlicz harmonic radial addition is defined below.

Definition 1. *Let $m \ge 2, \phi \in \mathcal{C}_m$, $K_j \in \mathcal{S}^n$, and $j = 1, \dots, m$, define the Orlicz harmonic addition of $K_1, \dots, K_m$, denoted by $\widehat{+}_\phi(K_1, \dots, K_m)$, defined by*

$$\rho(\widehat{+}_\phi(K_1,\dots,K_m),x) = \sup\left\{\lambda > 0 : \phi\left(\frac{\rho(K_1,x)}{\lambda},\dots,\frac{\rho(K_m,x)}{\lambda}\right) \le \phi(1)\right\}, \tag{37}$$

for $x \in \mathbb{R}^n$.

Equivalently, the Orlicz multiple harmonic addition $\widehat{+}_\phi(K_1, \dots, K_m)$ can be defined implicitly by

$$\phi\left(\frac{\rho(K_1,x)}{\rho(\widehat{+}_\phi(K_1,\dots,K_m),x)},\dots,\frac{\rho(K_m,x)}{\rho(\widehat{+}_\phi(K_1,\dots,K_m),x)}\right) = \phi(1), \tag{38}$$

for all $x \in \mathbb{R}^n$. An important special case is obtained when

$$\phi(x_1,\dots,x_m) = \sum_{j=1}^m \phi(x_j),$$

for $\phi(t) \in \mathcal{C}_m$. We then write $\widehat{+}_\phi(K_1, \dots, K_m) = K_1 \widehat{+}_\phi \cdots \widehat{+}_\phi K_m$. This means that $K_1 \widehat{+}_\phi \cdots \widehat{+}_\phi K_m$ is defined either by

$$\rho(K_1\widehat{+}_\phi\cdots\widehat{+}_\phi K_m,x) = \sup\left\{\lambda > 0 : \sum_{j=1}^m \phi\left(\frac{\rho(K_j,x)}{\lambda}\right) \le \phi(1)\right\}, \tag{39}$$

for all $x \in \mathbb{R}^n$, or by the corresponding special case of (37). From (39), it follows easy that

$$\sum_{j=1}^{m} \phi\left(\frac{\rho(K_j, x)}{\lambda}\right) = \phi(1),$$

if and only if

$$\lambda = \rho(K_1 \widehat{+}_\phi \cdots \widehat{+}_\phi K_m, x). \tag{40}$$

Next, define a new Orlicz dual harmonic linear combination on the case $m = 2$.

Definition 2. *The Orlicz dual harmonic linear combination is denoted $\widehat{+}_\phi(K, L, \alpha, \beta)$, defined by*

$$\alpha\phi\left(\frac{\rho(K, x)}{\rho(\widehat{+}_\phi(K, L, \alpha, \beta), x)}\right) + \beta\phi\left(\frac{\rho(L, x)}{\rho(\widehat{+}_\phi(K, L, \alpha, \beta), x)}\right) = \phi(1), \tag{41}$$

for $K, L \in \mathcal{S}^n$, $x \in \mathbb{R}^n$, and $\alpha, \beta \geq 0$ (not both zero).

When $\phi(t) = t^{-p}$ and $p \geq 1$, then Orlicz harmonic linear combination $\widehat{+}_\phi(K, L, \alpha, \beta)$ changes to the L_p-harmonic linear combination $\alpha \cdot K \widehat{+}_p \beta \cdot L$. We shall write $K \widehat{+}_\phi \varepsilon \cdot L$ instead of $\widehat{+}_\phi(K, L, 1, \varepsilon)$, for $\varepsilon \geq 0$ and assume throughout that this is defined by (41), where $\alpha = 1, \beta = \varepsilon$, and $\phi \in \mathcal{C}$. It is easy that $\widehat{+}_\phi(K, L, 1, 1) = K \widehat{+}_\phi L$.

4. Orlicz Multiple Dual Mixed Volumes

Let us introduce the Orlicz multiple dual mixed volumes.

Definition 3. *For $L_1, K_1, \Im \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, the Orlicz multiple dual mixed volume of $L_1, K_1, \Im$, denoted by $\widetilde{V}_\phi(L_1, K_1, \Im)$, defined by*

$$\widetilde{V}_\phi(L_1, K_1, \Im) =: \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{\rho(K_1, u)}{\rho(L_1, u)}\right) \rho(L_1, u) \cdot \rho_\Im dS(u).$$

Lemma 1 ([57])**.** *If $K_1, L_1 \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then*

$$L_1 \widehat{+}_\phi \varepsilon \cdot K_1 \to L_1 \tag{42}$$

as $\varepsilon \to 0^+$.

Lemma 2. *If $L_1, K_1, \Im \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} \widetilde{V}(L_1 \widehat{+}_\phi \varepsilon \cdot K_1, \Im) = \frac{1}{n\phi_r'(1)} \int_{S^{n-1}} \phi\left(\frac{\rho(K_1, u)}{\rho(L_1, u)}\right) \rho(L_1, u)\rho_\Im dS(u). \tag{43}$$

Proof. Suppose $\varepsilon > 0$, $K_1, L_1 \in \mathcal{S}^n$, and $u \in S^{n-1}$, let

$$\rho_\varepsilon = \rho(L_1 \widehat{+}_\phi \varepsilon \cdot K_1, u).$$

From Lemma 1, and noting that ϕ is a continuous function, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{(\rho_\varepsilon-\rho(L_1,u))\rho_{\Im}}{\varepsilon} &= \rho_{\Im}\lim_{\varepsilon\to 0^+}\frac{\rho_\varepsilon}{\varepsilon}\cdot\frac{\rho_\varepsilon-\rho(L_1,u)}{\rho_\varepsilon}\\ &= \rho_{\Im}\lim_{\varepsilon\to 0^+}\rho_\varepsilon\cdot\phi\left(\frac{\rho(K_1,u)}{\rho_\varepsilon}\right)\\ &\times\ \frac{1-\phi^{-1}\left(\phi(1)-\varepsilon\phi\left(\frac{\rho(K_1,u)}{\rho_\varepsilon}\right)\right)}{\phi(1)-\left(\phi(1)-\varepsilon\phi\left(\frac{\rho(K_1,u)}{\rho_\varepsilon}\right)\right)}.\end{aligned}$$

Noting that $y\to 1^+$ as $\varepsilon\to 0^+$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{(\rho_\varepsilon-\rho(L_1,u))\rho_{\Im}}{\varepsilon} &= \rho(L_1,u)\rho_{\Im}\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\lim_{y\to 1^+}\frac{1-y}{\phi(1)-\phi(y)}\\ &= \frac{1}{\phi_r'(1)}\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\rho(L_1,u)\rho_{\Im}, \qquad (44)\end{aligned}$$

where

$$y=\phi^{-1}\left(\phi(1)-\varepsilon\phi\left(\frac{\rho(K_1,u)}{\rho_\varepsilon}\right)\right).$$

The Equation (43) follows immediately from (20) with (44). □

Second proof Since

$$\frac{d\rho_\varepsilon}{d\varepsilon}=\frac{\rho(L_1,u)\dfrac{d\phi^{-1}(y)}{dy}\phi\left(\dfrac{\rho(K_1,u)}{\rho_\varepsilon}\right)}{\left(\phi^{-1}\left(\phi(1)-\varepsilon\phi\left(\dfrac{\rho(K_1,u)}{\rho_\varepsilon}\right)\right)\right)^2+\varepsilon\cdot\dfrac{\rho(K_1,u)\rho(L_1,u)}{\rho_\varepsilon^2}\dfrac{d\phi^{-1}(y)}{dy}\dfrac{d\phi(z)}{dz}}, \qquad (45)$$

where

$$y=\phi(1)-\varepsilon\phi\left(\frac{\rho(K_1,u)}{\rho_\varepsilon}\right),$$

and

$$z=\frac{\rho(K_1,u)}{\rho_\varepsilon}.$$

Hence,

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{(\rho_\varepsilon-\rho(L_1,u))\rho_{\Im}}{\varepsilon} &= \rho_{\Im}\lim_{\varepsilon\to 0^+}\frac{\rho_\varepsilon-\rho(L_1,u)}{\varepsilon}\\ &= \rho_{\Im}\lim_{\varepsilon\to 0^+}\frac{d\rho_\varepsilon}{d\varepsilon}. \qquad (46)\end{aligned}$$

On the other hand,

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{d\phi^{-1}(y)}{dy} &= \lim_{\triangle y\to 0^+}\frac{\phi^{-1}(\phi(1)+\triangle y)-1}{\triangle y}\\ &= \lim_{\omega\to 1^+}\frac{\omega-1}{\phi(\omega)-\phi(1)}\\ &= \frac{1}{\phi_r'(1)}, \qquad (47)\end{aligned}$$

where $\omega=\phi^{-1}(\phi(1)+\triangle y)$.

From (45), (46), (47), and Lemma 1, we obtain

$$\lim_{\varepsilon\to 0^+}\frac{(\rho_\varepsilon-\rho(L_1,u))\rho_{\Im}}{\varepsilon}=\frac{1}{\phi_r'(1)}\cdot\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\rho(L_1,u)\rho_{\Im}. \tag{48}$$

From (20) and (48), the Equation (43) follows easy. □

For any $L_1, K_1, \Im \in \mathcal{S}^n$, and $\phi \in \mathcal{C}$, the integral on the right-hand side of (43) denoting by $\widetilde{V}_\phi(L_1, K_1, \Im)$, and hence this new Orlicz multiple dual mixed volume $\widetilde{V}_\phi(L_1, K_1, \Im)$ has been born.

Lemma 3. *If $L_1, K_1, \Im \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then*

$$\widetilde{V}_\phi(L_1,K_1,\Im)=\phi_r'(1)\cdot\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}\widetilde{V}(L_1\widehat{+}_\phi\varepsilon\cdot K_1,\Im). \tag{49}$$

Proof. This yields immediately from the Definition 3 and the variational formula of volume (43). □

Lemma 4. *Let $K, L \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then*

$$\lim_{\varepsilon\to 0^+}\frac{\widetilde{V}_1(K,K\widehat{+}_\phi\varepsilon\cdot L)-V(K)}{\varepsilon}=\frac{1}{n}\lim_{\varepsilon\to 0^+}\frac{V(K\widehat{+}_\phi\varepsilon\cdot L)-V(K)}{\varepsilon}. \tag{50}$$

Proof. Suppose $\varepsilon > 0$, $K, L \in \mathcal{S}^n$, and $u \in S^{n-1}$, let

$$\bar{\rho}_\varepsilon=\rho(K\widehat{+}_\phi\varepsilon\cdot L,u).$$

From (3), (18), (19), and (45), we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\frac{\widetilde{V}_1(K,K\widehat{+}_\phi\varepsilon\cdot L)-V(K)}{\varepsilon} &= \frac{1}{n}\int_{S^{n-1}}\lim_{\varepsilon\to 0^+}\frac{\rho(K\widehat{+}_\phi\varepsilon\cdot L),u)\rho(K,u)^{n-1}-\rho(L,u)^n}{\varepsilon}dS(u)\\
&= \frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-1}\lim_{\varepsilon\to 0^+}\frac{d\bar{\rho}_\varepsilon}{d\varepsilon}dS(u)\\
&= \frac{1}{n\phi_r'(1)}\int_{S^{n-1}}\phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^n dS(u)\\
&= \frac{1}{n}\lim_{\varepsilon\to 0^+}\frac{V(K\widehat{+}_\phi\varepsilon\cdot L)-V(K)}{\varepsilon}.
\end{aligned}$$

□

Lemma 5. *Let $L_1, K_1, \Im \in \mathcal{S}^n$, and $\phi \in \mathcal{C}$, then*

$$\widetilde{V}_\phi(L_1,K_1,\Im)=\widetilde{V}_\phi(K,L), \tag{51}$$

if $K_2 = \cdots = K_n = K$, $L_1 = K$ and $K_1 = L$.

Proof. On the one hand, putting $K_2 = \cdots = K_n = K$, $L_1 = K$ and $K_1 = L$ in (49), and noting Lemma 4 and (3), it follows that

$$\begin{aligned}
\widetilde{V}_\phi(L_1,K_1,\Im) &= \phi_r'(1)\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}\widetilde{V}(L_1\widehat{+}_\phi\varepsilon\cdot K_1,\Im)\\
&= \phi_r'(1)\lim_{\varepsilon\to 0^+}\frac{\widetilde{V}_1(K,K\widehat{+}_\phi\varepsilon\cdot L)-V(K)}{\varepsilon}\\
&= \frac{\phi_r'(1)}{n}\lim_{\varepsilon\to 0^+}\frac{V(K\widehat{+}_\phi\varepsilon\cdot L)-V(K)}{\varepsilon}
\end{aligned}$$

$$= \widetilde{V}_{\phi}(K,L). \tag{52}$$

On the other hand, let $K_2 = \cdots = K_n = K$, $L_1 = K$, and $K_1 = L$, from Definition 3 and (3), then

$$\begin{aligned}\widetilde{V}_{\phi}(L_1,K_1,\lambda\Im) &= \frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\rho(L_1,u)\rho_{\Im}dS(u)\\ &= \frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^n dS(u)\\ &= \widetilde{V}_{\phi}(K,L).\end{aligned} \tag{53}$$

Combining (52) and (53), this shows that

$$\widetilde{V}_{\phi}(L_1,K_1,\Im) = \widetilde{V}_{\phi}(K,L),$$

if $K_2 = \cdots = K_n = K$, $L_1 = K$ and $K_1 = L$. □

Lemma 6 ([57]). *If $K_i, L_i \in \mathcal{S}^n$, and $K_i \to K$, $L_i \to L$ as $i \to \infty$, then*

$$a \cdot K_i \widehat{+}_{\phi} b \cdot L_i \to a \cdot K \widehat{+}_{\phi} b \cdot L, \ \ as \ \ i \to \infty, \tag{54}$$

for all a and b.

Lemma 7. *If $L_1, K_1, \Im, K, L \in \mathcal{S}^n$, $\lambda_1, \cdots, \lambda_n \geq 0$, and $\phi \in \mathcal{C}$, then*

(1) $\widetilde{V}_{\phi}(L_1,K_1,\Im) \geq 0.$

(2) $\widetilde{V}_{\phi}(K_1,K_1,\Im) = \phi(1)\widetilde{V}(K_1,\Im).$

(3) $\widetilde{V}_{\phi}(K,K,\cdots,K) = \phi(1)V(K).$

(4)

$$\widetilde{V}_{\phi}(L_1,K_1,\lambda_i\Im) = \lambda_2\cdots\lambda_n\widetilde{V}_{\phi}(L_1,K_1,\Im),$$

where $\lambda_i\Im$ denotes $\lambda_2K_2, \cdots, \lambda_nK_n$.

(5)

$$\begin{aligned}&\widetilde{V}_{\phi}(L_1,K_1,\lambda_1K\widehat{+}\lambda_2L,K_3\cdots,K_n)\\ &\quad= \lambda_1V_{\phi}(L_1,K_1,K,K_3,\cdots,K_n) + \lambda_2V_{\phi}(L_1,K_1,L,K_3,\cdots,K_n).\end{aligned}$$

This shows the Orlicz multiple mixed volume $V_{\phi}(L_1,K_1,\Im)$ is linear in its back $(n-1)$ variables.

(6) $\widetilde{V}_{\phi}(L_1,K_1,\Im)$ *is continuous.*

Proof. From Definition 3, it immediately gives (1), (2), (3), and (4).

From Definition 3, combining the following fact

$$\rho(\lambda_1K\widetilde{+}\lambda_2L,\cdot) = \lambda_1\rho(K,\cdot) + \lambda_2\rho(L,\cdot),$$

it yields (5) directly.

Suppose $L_{i1} \to L_1$, $K_{ij} \to K_j$ as $i \to \infty$ where $j = 1,\ldots,n$, combining Definition 3 and Lemma 6 with the following facts

$$\widetilde{V}(L_{i1}\widehat{+}_{\phi}\varepsilon \cdot K_{i1},\Im_i) \to \widetilde{V}(L_1\widehat{+}_{\phi}\varepsilon \cdot K_1,\Im)$$

and

$$\widetilde{V}(L_{i1},\Im_i) \to \widetilde{V}(L_1,\Im)$$

as $i \to \infty$, where $\Im_i$ denotes $K_{i2},\cdots,K_{in}$. It yields (6) directly. □

Lemma 8. *([57]) Suppose $K, L \in \mathcal{S}^n$ and $\varepsilon > 0$. If $\phi \in \mathcal{C}$, then for $A \in GL(n)$*

$$A(K\widehat{+}_\phi\varepsilon \cdot L) = AK\widehat{+}_\phi\varepsilon \cdot AL. \tag{55}$$

We easily find that Orlicz multiple dual mixed volume $\widetilde{V}_\phi(L_1, K_1, \Im)$ is invariant under simultaneous unimodular centro-affine transformation.

Lemma 9. *If $L_1, K_1, \Im \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then for $A \in SL(n)$,*

$$\widetilde{V}_\phi(AL_1, AK_1, A\Im) = \widetilde{V}_\phi(L_1, K_1, \Im), \tag{56}$$

where $A\Im$ denotes $AK_2, \ldots, AK_n$.

Proof. From (49) and Lemma 8, we have, for $A \in SL(n)$,

$$\begin{aligned}
\widetilde{V}_\phi(AL_1, AK_1, A\Im) &= \phi_r'(1) \lim_{\varepsilon\to0^+} \frac{\widetilde{V}(AL_1\widehat{+}_\phi\varepsilon \cdot AK_1, A\Im) - \widetilde{V}(AL_1, A\Im)}{\varepsilon} \\
&= \phi_r'(1) \lim_{\varepsilon\to0^+} \frac{\widetilde{V}(A(L_1\widehat{+}_\phi\varepsilon \cdot K_1), A\Im) - \widetilde{V}(L_1, \Im)}{\varepsilon} \\
&= \phi_r'(1) \lim_{\varepsilon\to0^+} \frac{\widetilde{V}(L_1\widehat{+}_\phi\varepsilon \cdot K_1, \Im) - \widetilde{V}(L_1, \Im)}{\varepsilon} \\
&= \widetilde{V}_\phi(L_1, K_1, \Im).
\end{aligned}$$

This completes the proof. □

For the convenience of writing, when $K_1 = \cdots = K_i = K$, $K_{i+1} = \cdots = K_n = L$, $L_n = M$, the Orlicz multiple dual mixed volume $\widetilde{V}_\phi(K, \cdots, K, L, \cdots, L, M)$, with i copies of K, $n-i$ copies of L, and 1 copy of M, will be denoted by $\widetilde{V}_\phi(K\,[i], L\,[n-i], M)$.

Lemma 10. *If $K, L \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, and $0 \le i < n$ then*

$$\widetilde{V}_\phi(K, L, K\,[n-i-1], B\,[i]) = \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^{n-i} dS(u). \tag{57}$$

Proof. On the one hand, putting $L_1 = K$, $K_1 = L$,$K_2 = \cdots = K_{n-i} = K$, and $K_{n-i+1} = \cdots = K_n = B$ in (49), from (21), (22), (45), and (47), we obtain for $\varphi_1, \varphi_2 \in \Phi$

$$\begin{aligned}
\widetilde{V}_\phi(K, L, K\,[n-i-1], B\,[i]) &= \phi_r'(1) \lim_{\varepsilon\to0^+} \frac{\widetilde{W}_i(K, K\widehat{+}_\phi\varepsilon \cdot L) - \widetilde{W}_i(K)}{\varepsilon} \\
&= \frac{1}{n}\phi_r'(1) \int_{S^{n-1}} \lim_{\varepsilon\to0^+} \frac{\rho(K\widehat{+}_\phi\varepsilon \cdot L)\rho(K,u)^{n-i-1} - \rho(K,u)^{n-i}}{\varepsilon} dS(u) \\
&= \frac{1}{n}\phi_l'(1) \int_{S^{n-1}} \rho(K,u)^{n-i-1} \lim_{\varepsilon\to0^+} \frac{d\overline{\rho}_\varepsilon}{\varepsilon} dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^{n-i} dS(u).
\end{aligned} \tag{58}$$

On the other hand, putting $L_1 = K$, $K_1 = L$,$K_2 = \cdots = K_{n-i} = K$, and $K_{n-i+1} = \cdots = K_n = B$ in Definition 3, we have

$$\widetilde{V}_\phi(K, L, K\,[n-i-1], B\,[i]) = \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^{n-i} dS(u). \tag{59}$$

Combining (58) and (59), (57) yields easy. □

Here, we denote the Orlicz multiple dual mixed volume $\widetilde{V}_\phi(K, L, K\,[n-i-1], B\,[i])$ by $\widetilde{W}_{\phi,i}(K, L)$, and call $\widetilde{W}_{\phi,i}(K, L)$ as Orlicz dual quermassintegral of star bodies K and L. When $i = 0$, Orlicz dual quermassintegral $\widetilde{W}_{\phi,i}(K, L)$ becomes Orlicz dual mixed volume $\widetilde{V}_\phi(K, L)$.

Remark 1. *When $\phi(t) = t^{-p}$, $p = 1$, and $L_1 = K_1$, from (49) and noting that $\phi_r'(1) = -1$, hence*

$$\widetilde{V}(K_1, \Im) = \lim_{\varepsilon\to 0^+} \frac{\widetilde{V}(K_1, \Im) - \widetilde{V}(K_1 \widehat{+} \varepsilon \cdot K_1, \Im)}{\varepsilon}. \tag{60}$$

This is very interesting for the usually dual mixed volume of this form.

Remark 2. *When $\phi(t) = t^{-p}$, $p \geq 1$, write the Orlicz multiple dual mixed volume $\widetilde{V}_\phi(L_1, K_1, \Im)$ as $\widetilde{V}_{-p}(L_1, K_1, \Im)$ and call it the L_p-multiple dual mixed volume, from Definition 3, it easily yields*

$$\widetilde{V}_{-p}(L_1, K_1, \Im) = \frac{1}{n}\int_{S^{n-1}} \rho(K_1, u)^{-p} \rho(L_1, u)^{1+p} \rho_{\Im} dS(u). \tag{61}$$

When $\varphi(t) = t^{-p}$ and $p \geq 1$, from (49), we get the following expression of L_p-multiple dual mixed volume.

$$\frac{1}{-p}\widetilde{V}_{-p}(L_1, K_1, \Im) = \lim_{\varepsilon\to 0^+} \frac{\widetilde{V}(L_1 \widehat{+}_p \varepsilon \cdot K_1, \Im) - \widetilde{V}(L_1, \Im)}{\varepsilon}.$$

When $K_1 = L$ and $L_1 = K_2 = \cdots = K_n = K$, the Orlicz multiple dual mixed volume $\widetilde{V}_\phi(L_1, K_1, \Im)$ becomes the usual dual Orlicz mixed volume $\widetilde{V}_\phi(K, L)$. Putting $L_1 = K_1$ in (61), the L_p multiple dual mixed volume $\widetilde{V}_{-p}(L_1, K_1, \Im)$ becomes the usual dual mixed volume $\widetilde{V}(K_1, \Im)$. Putting $K_1 = L$ and $L_1 = K_2 = \cdots = K_n = K$ in (61), $\widetilde{V}_{-p}(L_1, K_1, \Im)$ becomes the L_p dual mixed volume $\widetilde{V}_{-p}(K, L)$. Putting $K_1 = L$, $L_1 = K_2 = \cdots = K_{n-i} = K$, and $K_{n-i+1} = \cdots = K_n = B$ in (61), $\widetilde{V}_{-p}(L_1, K_1, \Im)$ becomes the harmonic mixed p-quermassintegral $\widetilde{W}_{-p,i}(K, L)$,

Lemma 11. *(Jensen's inequality) Let μ be a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. If $\psi : I \to \mathbb{R}$ is a convex function, then*

$$\int_X \psi(g(x))d\mu(x) \geq \psi\left(\int_X g(x)d\mu(x)\right). \tag{62}$$

If ψ is strictly convex, equality holds if and only if $g(x)$ is constant for μ-almost all $x \in X$ (see [63]).

5. The Dual Orlicz–Aleksandrov–Fenchel Inequality

Theorem 1. *If $L_1, K_1, \ldots, K_n \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then*

$$\widetilde{V}_\phi(L_1, K_1, \ldots, K_n) \geq \widetilde{V}(L_1, K_2 \ldots, K_n)\phi\left(\frac{\widetilde{V}(K_1, \ldots, K_n)}{\widetilde{V}(L_1, K_2 \ldots, K_n)}\right). \tag{63}$$

If ϕ is strictly convex, equality holds if and only if K_1 and L_1 are dilates.

Proof. For $K_1, \Im \in \mathcal{S}^n$ and any $u \in S^{n-1}$, it is not difficult to see that $\frac{\rho(K_1, u)\rho_{\Im}}{n\widetilde{V}(K_1, \Im)} S(u)$ is a probability measure on S^{n-1}.

From Definition 3 and Jensen's inequality (43) and (20), it follows that

$$\begin{aligned}\frac{\widetilde{V}_\phi(L_1,K_1,\Im)}{\widetilde{V}(L_1,\Im)} &= \frac{1}{n\widetilde{V}(L_1,\Im)}\int_{S^{n-1}}\phi\left(\frac{\rho(K_1,u)}{\rho(L_1,u)}\right)\rho(L_1,u)\rho_\Im dS(u)\\ &\geq \phi\left(\frac{1}{n\widetilde{V}(L_1,\Im)}\int_{S^{n-1}}\rho(K_1,u)\rho_\Im dS(u)\right)\\ &= \phi\left(\frac{\widetilde{V}(K_1,\Im)}{\widetilde{V}(L_1,\Im)}\right). \end{aligned} \tag{64}$$

If ϕ is strictly convex, from the equality condition of Jensen's inequality, it follows that the equality in (64) holds if and only if K_1 and L_1 are dilates. □

Theorem 2. *(The dual Orlicz–Aleksandrov–Fenchel inequality) If $L_1, K_1, \ldots, K_n \in \mathcal{S}^n$, $1 \leq r \leq n$, and $\phi \in \mathcal{C}$, then*

$$\widetilde{V}_\phi(L_1,K_1,\ldots,K_n) \geq \widetilde{V}(L_1,K_2\ldots,K_n)\cdot\phi\left(\frac{\prod_{i=1}^{r}\widetilde{V}(K_i\ldots,K_i,K_{r+1},\ldots,K_n)^{1/r}}{\widetilde{V}(L_1,K_2\ldots,K_n)}\right). \tag{65}$$

If ϕ is strictly convex, equality holds if and only if $L_1, K_1, \ldots, K_r$ are all dilations of each other.

Proof. This follows immediately from Theorem 1 with the dual Aleksandrov–Fenchel inequality. □

Obviously, putting $L_1 = K_1$ in (65), (65) becomes the Lutwak's dual Aleksandrov–Fenchel inequality (11) stated in the introduction.

Corollary 1. *If $L_1, K_1, \Im \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then*

$$\widetilde{V}_\phi(L_1,K_1,\Im) \geq \widetilde{V}(L_1,\Im)\phi\left(\left(\frac{V(K_1)\cdots V(K_n)}{\widetilde{V}(L_1,\Im)^n}\right)^{1/n}\right). \tag{66}$$

If ϕ is strictly convex, equality holds if and only if $L_1, K_1, \ldots, K_n$ are all dilations of each other.

Proof. This follows immediately from Theorem 2 with $r = n$. □

Corollary 2. *If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $\phi \in \mathcal{C}$, then*

$$\widetilde{W}_{\phi,i}(K,L) \geq \widetilde{W}_i(K)\phi\left(\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\right)^{1/(n-i)}\right). \tag{67}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof. This follows immediately from Theorem 2 with $r = n - i$, $L_1 = K$, $K_1 = L$, $K_2 = \cdots = K_{n-i} = K$, and $K_{n-i+1} = \cdots = K_n = B$. □

The following inequality follows immediately from (67) with $\phi(t) = t^{-p}$ and $p \geq 1$. If $K, L \in \mathcal{S}^n$, $0 \leq i < n$, and $p \geq 1$, then

$$\widetilde{W}_{-p,i}(K,L)^{n-i} \geq \widetilde{W}_i(K)^{n-i+p}\widetilde{W}_i(L)^{-p}, \tag{68}$$

with equality if and only if K and L are dilates. Taking $i = 0$ in (68), this yields Lutwak's L_p-dual Minkowski inequality: If $K, L \in \mathcal{S}^n$ and $p \geq 1$, then

$$\widetilde{V}_{-p}(K,L)^n \geq V(K)^{n+p}V(L)^{-p}, \tag{69}$$

with equality if and only if K and L are dilates.

Theorem 3. *(Orlicz dual isoperimetric inequality) If $K \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, and $0 \leq i < n$ then*

$$\frac{\widetilde{V}_\phi(K,B,K\,[n-i-1],B\,[i])}{\widetilde{W}_i(K)} \geq \phi\left(\left(\frac{V(B)}{\widetilde{W}_i(K)}\right)^{1/(n-i)}\right). \tag{70}$$

If ϕ is strictly convex, equality holds if and only if K is a ball.

Proof. This follows immediately from (65) with $r = n-i$, $L_1 = K$, $K_1 = B$, $K_2 \cdots = K_{n-i} = K$, and $K_{n-i+1} = \cdots = K_n = B$. □

When $\phi(t) = t^{-p}$, $p \geq 1$, the Orlicz isoperimetric inequality (70) becomes the following L_p-dual isoperimetric inequality. If K is a star body, $p \geq 1$ and $0 \leq i < n$, then

$$\left(\frac{n\widetilde{V}_{-p}(K,B)}{\omega_n}\right)^{n-i} \geq \left(\frac{\widetilde{W}_i(K)}{\kappa_n}\right)^{n-i+p}, \tag{71}$$

with equality if and only if K is a ball, and where κ_n denotes volume of the unit ball B, and its surface area by ω_n.

Putting $p = 1$ and $i = 0$ in (71), (71) becomes the following dual isoperimetric inequality. If K is a star body, then

$$\left(\frac{n\widetilde{V}_{-1}(K,B)}{\omega_n}\right)^{n} \geq \left(\frac{V(K)}{\kappa_n}\right)^{n+1},$$

with equality if and only if K is a ball.

Theorem 4. *If $L_1, K_1, \Im \in \mathcal{M} \subset \mathcal{S}^n$, and $\phi \in \mathcal{C}$ be strictly convex, and if either*

$$\widetilde{V}_\phi(Q,K_1,\Im) = \widetilde{V}_\phi(Q,L_1,\Im), \ \ for\ all\ \ Q \in \mathcal{M}, \tag{72}$$

or

$$\frac{\widetilde{V}_\phi(K_1,Q,\Im)}{\widetilde{V}(K_1,\Im)} = \frac{\widetilde{V}_\phi(L_1,Q,\Im)}{\widetilde{V}(L_1,\Im)}, \ \ for\ all\ \ Q \in \mathcal{M}, \tag{73}$$

then $K_1 = L_1$.

Proof. Suppose (72) holds. Taking K_1 for Q, then from Definition 3 and Theorem 1, we obtain

$$\phi(1)\widetilde{V}(K_1,\Im) = \widetilde{V}_\phi(K_1,L_1,\Im) \geq \widetilde{V}(K_1,\Im)\phi\left(\frac{\widetilde{V}(L_1,\Im)}{\widetilde{V}(K_1,\Im)}\right),$$

with equality if and only if K_1 and L_1 are dilates. Hence,

$$\phi(1) \geq \phi\left(\frac{\widetilde{V}(L_1,\Im)}{\widetilde{V}(K_1,\Im)}\right),$$

with equality if and only if K_1 and L_1 are dilates. Since φ is a decreasing function on $(0, \infty)$, it follows that

$$\widetilde{V}(K_1, \Im) \leq \widetilde{V}(L_1, \Im),$$

with equality if and only if K_1 and L_1 are dilates. On the other hand, if taking L_1 for Q, we similarly get $\widetilde{V}(K_1, \Im) \geq \widetilde{V}(L_1, \Im)$, with equality if and only if K_1 and L_1 are dilates. Hence, $\widetilde{V}(K_1, \Im) = \widetilde{V}(L_1, \Im)$, and K_1 and L_1 are dilates, it follows that K_1 and L_1 must be equal.

Suppose (73) holds. Taking K_1 for Q, then from Definition 3 and Theorem 1, we obtain

$$\phi(1) = \frac{\widetilde{V}_\phi(L_1, K_1, \Im)}{\widetilde{V}(L_1, \Im)} \geq \phi\left(\frac{\widetilde{V}(K_1, \Im)}{\widetilde{V}(L_1, \Im)}\right),$$

with equality if and only if K_1 and L_1 are dilates. Since φ is an increasing function on $(0, \infty)$, this follows that

$$\widetilde{V}(L_1, \Im) \leq \widetilde{V}(K_1, \Im),$$

with equality if and only if K_1 and L_1 are dilates. On the other hand, if taking L_1 for Q, we similar get $\widetilde{V}(L_1, \Im) \geq \widetilde{V}(K_1, \Im)$, with equality if and only if K_1 and L_1 are dilates. Hence, $\widetilde{V}(L_1, \Im) = \widetilde{V}(K_1, \Im)$, and K_1 and L_1 are dilates, it follows that K_1 and L_1 must be equal. □

Corollary 3. *Let $K, L \in \mathcal{M} \subset \mathcal{S}^n$, $0 \leq i < n$, and $\phi \in \mathcal{C}$ be strictly convex, and if either*

$$\widetilde{W}_{\phi,i}(Q, K) = \widetilde{W}_{\phi,i}(Q, L), \text{ for all } Q \in \mathcal{M},$$

or

$$\frac{\widetilde{W}_{\phi,i}(K, Q)}{\widetilde{W}_i(K)} = \frac{\widetilde{W}_{\phi,i}(L, Q)}{\widetilde{W}_i(L)}, \text{ for all } Q \in \mathcal{M},$$

then $K = L$.

Proof. This yields immediately from Theorem 4 and Lemma 10. □

Remark 3. *When $\phi(t) = t^{-p}$ and $p = 1$, the dual Orlicz Aleksandrov–Fenchel inequality (65) becomes the following inequality. If $L_1, K_1, \Im \in \mathcal{S}^n$ and $1 \leq r \leq n$, then*

$$\widetilde{V}_{-1}(L_1, K_1, \Im) \geq \frac{\widetilde{V}(L_1, \Im)^2}{\prod_{i=1}^{r} \widetilde{V}(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{1/r}}, \tag{74}$$

with equality if and only if $L_1, K_1, \ldots, K_r$ are all dilations of each other.

Putting $L_1 = K_1$ in (74) and noting that $\widetilde{V}_{-1}(K_1, K_1, \Im) = \widetilde{V}(K_1, \Im)$, (74) becomes the dual Aleksandrov–Fenchel inequality (11). Putting $r = n$ in (74), (74) becomes the following inequality.

$$\widetilde{V}_{-1}(L_1, K_1, \Im)^n \geq \widetilde{V}(L_1, \Im)^{2n}(V(K_1) \cdots V(K_n))^{-1}, \tag{75}$$

with equality if and only if $L_1, K_1, \Im$ are all dilations of each other. Putting $L_1 = K$, $K_1 = L$ and $K_2 = \cdots = K_n = K$ in (75), (75) becomes the well-known Minkowski inequality. If $K, L \in \mathcal{S}^n$, then

$$\widetilde{V}_{-1}(K, L)^n \geq V(K)^{n+1}V(L)^{-1}, \tag{76}$$

with equality if and only if K and L are dilates. Obviously, inequality (74) in a special case yields also the following result. If $K_1, \Im \in \mathcal{S}^n$ and $0 \le i < n$, then

$$\widetilde{W}_{-1,i}(K,L)^{n-i} \ge \widetilde{W}_i(K)^{n-i+1} W_i(L)^{-1}, \tag{77}$$

with equality if and only if K and L are dilates. When $i = 0$, (77) becomes (76). On the other hand, putting $L_1 = K_1$ in (75), (75) becomes the well-known inequality. If $K_1, \Im \in \mathcal{S}^n$, then

$$\widetilde{V}(K_1, \dots, K_n)^n \le V(K_1) \cdots V(K_n),$$

with equality if and only if $K_1, \Im$ are all dilations of each other.

6. The Dual Orlicz–Brunn–Minkowski Inequality

Lemma 12. *If $L_1, K_1, \Im \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then*

$$\phi(1)\widetilde{V}(K_1 \widehat{+}_\phi L_1, \Im) = \widetilde{V}_\phi(K_1 \widehat{+}_\phi L_1, K_1, \Im) + \widetilde{V}_\phi(K_1 \widehat{+}_\phi L_1, L_1, \Im). \tag{78}$$

Proof. Suppose $\varepsilon > 0$, $K_1, L_1 \in \mathcal{S}^n$, let

$$Q = K_1 \widehat{+}_\phi \varepsilon \cdot L_1.$$

From Definition 3, (20), and (40), we have

$$\begin{aligned} \phi(1)\widetilde{V}(Q, \Im) &= \frac{1}{n}\int_{S^{n-1}} \left(\phi\left(\frac{\rho(K_1,u)}{\rho(Q,u)} \right) + \phi\left(\frac{\rho(L_1,u)}{\rho(Q,u)} \right) \right) \rho(Q,u)\rho_\Im dS(u) \\ &= \widetilde{V}_\phi(Q, K_1, \Im) + \widetilde{V}_\phi(Q, L_1, \Im). \end{aligned} \tag{79}$$

This completes the proof. □

Lemma 13. *([53]) Let $K, L \in \mathcal{S}^n$, $\varepsilon > 0$ and $\phi \in \mathcal{C}$.*
(1) If K and L are dilates, then K and $K \widehat{+}_\phi \varepsilon \cdot L$ are dilates.
(2) If K and $K \widehat{+}_\phi \varepsilon \cdot L$ are dilates, then K and L are dilates.

Theorem 5. *(The dual Orlicz–Brunn–Minkowski inequality) If $L_1, K_1, \dots, K_n \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then for $\varepsilon > 0$*

$$\phi(1) \ge \phi\left(\frac{\widetilde{V}(K_1, \dots, K_n)}{\widetilde{V}(K_1 \widehat{+}_\phi \varepsilon \cdot L_1, K_2 \dots, K_n)} \right) + \varepsilon \cdot \phi\left(\frac{\widetilde{V}(L_1, K_2 \dots, K_n)}{\widetilde{V}(K_1 \widehat{+}_\phi \varepsilon \cdot L_1, K_2 \dots, K_n)} \right), \tag{80}$$

If ϕ is strictly convex, equality holds if and only if K_1 and L_1 are dilates.

Proof. From Theorem 1 and Lemma 12, we have

$$\begin{aligned} \phi(1)\widetilde{V}(K_1 \widehat{+}_\phi \varepsilon \cdot L_1, \Im) &= \widetilde{V}_\phi(K_1 \widehat{+}_\phi \varepsilon \cdot L_1, K_1, \Im) \\ &+ \varepsilon \cdot \widetilde{V}_\phi(K_1 \widehat{+}_\phi \varepsilon \cdot L_1, L_1, \Im) \\ &\ge \widetilde{V}(K_1 \widehat{+}_\phi \varepsilon \cdot L_1, \Im) \left\{ \phi\left(\frac{\widetilde{V}(K_1, \Im)}{\widetilde{V}(K_1 \widehat{+}_\phi \varepsilon \cdot L_1, \Im)} \right) \right. \\ &+ \left. \phi\left(\frac{\widetilde{V}(L_1, \Im)}{\widetilde{V}(K_1 \widehat{+}_\phi \varepsilon \cdot L_1, \Im)} \right) \right\}. \end{aligned}$$

If ϕ is strictly convex, from the equality condition of Theorem 1, the equality in (80) holds if and only if K_1 and $K_1\widehat{+}_\phi L_1$, and L_1 and $K_1\widehat{+}_\phi L_1$ are dilates. Further, from Lemma 13, it follows that if ϕ is strictly convex, the equality in (80) holds if and only if K_1 and L_1 are dilates. □

Theorem 6. *(The dual Orlicz–Brunn–Minkowski type inequality) If $L_1, K_1, \Im \in \mathcal{S}^n$, $0 \le i, j < n$, $1 < r \le n$, and $\phi \in \mathcal{C}$, then*

$$\phi(1) \ge \phi\left(\frac{\prod_{i=1}^r \widetilde{V}(K_i,\ldots,K_i,K_{r+1},\ldots,K_n)^{1/r}}{\widetilde{V}(K_1\widehat{+}_\phi L_1,\Im)}\right) + \phi\left(\frac{M(r)\prod_{j=2}^r \cdot\widetilde{V}(K_j,\ldots,K_j,K_{r+1},\ldots,K_n)^{1/r}}{\widetilde{V}(K_1\widehat{+}_\phi L_1,\Im)}\right), \quad (81)$$

where $M(r) = \widetilde{V}(L_1,\ldots,L_1,K_{r+1},\ldots,K_n)^{1/r}$. If ϕ is strictly convex, equality holds if and only if $L_1, K_1, \ldots, K_r$ are all dilations of each other.

Proof. This follows immediately from Theorem 5 and the dual Aleksandrov–Fenchel inequality. □

Corollary 4. *(L_p-dual Brunn-Minkowski inequality) If $L_1, K_1, \Im \in \mathcal{S}^n$, $0 \le i, j < n$, $1 < r \le n$, and $p \ge 1$, then*

$$\widetilde{V}(K_1\widehat{+}_p L_1,\Im)^{-p} \ge \prod_{i=1}^r \widetilde{V}(K_i,\ldots,K_i,K_{r+1},\ldots,K_n)^{\frac{-p}{r}} + \prod_{j=1}^r M(r)^{-p}\cdot\widetilde{V}(K_j,\ldots,K_j,K_{r+1},\ldots,K_n)^{\frac{-p}{r}}, \quad (82)$$

with equality if and only if $L_1, K_1, \ldots, K_r$ are all dilations of each other, and $M(r)$ is as in Theorem 6.

Proof. This follows immediately from (81) with $\phi(t) = t^{-p}$ and $p \ge 1$. □

Corollary 5. *If $K, L \in \mathcal{S}^n$, $\phi \in \mathcal{C}$ and $0 \le i < n-1$, then*

$$\phi(1) \ge \phi\left(\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K\widehat{+}_\phi L)}\right)^{1/(n-i)}\right) + \phi\left(\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K\widehat{+}_\phi L)}\right)^{1/(n-i)}\right). \quad (83)$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof. This follows immediately from Theorem 6 with $r = n-i$, $K_2 = \cdots = K_{n-i} = K\widehat{+}_\phi L$, $K_{n-i+1} = \cdots = K_n = B$. □

The following inequality follows immediately from (83) with $\phi(t) = t^{-p}$ and $p \ge 1$. If $K, L \in \mathcal{S}^n$, $0 \le i < n$, and $p \ge 1$, then

$$\widetilde{W}_i(K\widehat{+}_p L)^{-p/(n-i)} \ge \widetilde{W}_i(K)^{-p/(n-i)} + \widetilde{W}_i(L)^{-p/(n-i)},$$

with equality if and only if K and L are dilates.

Corollary 6. *If $L_1, K_1, \Im \in \mathcal{S}^n$ and $\phi \in \mathcal{C}$, then*

$$\phi(1) \ge \phi\left(\left(\frac{V(K_1)\cdots V(K_n)}{\widetilde{V}(K_1\widehat{+}_\phi L_1,\Im)^n}\right)^{1/n}\right) + \phi\left(\left(\frac{V(L_1)V(K_2)\cdots V(K_n)}{\widetilde{V}(K_1\widehat{+}_\phi L_1,\Im)^n}\right)^{1/n}\right). \quad (84)$$

If ϕ is strictly convex, equality holds if and only if $L_1, K_1, \Im$ are all dilations of each other.

Proof. This follows immediately from Theorem 6 with $r = n$. □

Corollary 7. *If $L_1, K_1, \Im \in \mathcal{S}^n$ and $p \geq 1$, then*

$$\widetilde{V}(K_1 \widehat{+}_p L_1, \Im)^{-p} \geq (V(K_1) \cdots V(K_n))^{-p/n} + (V(L_1)V(K_2) \cdots V(K_n))^{-p/n}, \tag{85}$$

with equality if and only if $L_1, K_1, \Im$ are all dilations of each other.

Proof. This follows immediately from (84) with $\phi(t) = t^{-p}$ and $p \geq 1$. □

Putting $K_2 = \cdots = K_n = K_1 \widehat{+}_p L_1$ in (85), (85) becomes Lutwak's L_p-dual Brunn–Minkowski inequality

$$V(K \widehat{+}_p L)^{-p/n} \geq V(K)^{-p/n} + V(L)^{-p/n},$$

with equality if and only if K and L are dilates.

Corollary 8. *If $L_1, K_1, \Im \in \mathcal{S}^n$, $1 \leq r \leq n$, and $\phi \in \mathcal{C}$, then*

$$\widetilde{V}_\phi(L_1, K_1, \Im) \geq \widetilde{V}(L_1, \Im)\phi\left(\frac{\prod_{i=1}^r \widetilde{V}(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{1/r}}{\widetilde{V}(L_1, \Im)}\right). \tag{86}$$

If ϕ is strictly convex, equality holds if and only if $L_1, K_1, \ldots, K_r$ are all dilations of each other.

Proof. Let

$$K_\varepsilon = L_1 \widehat{+}_\phi \varepsilon \cdot K_1.$$

From (49), dual Orlicz–Brunn–Minkowski inequality (80), and dual Aleksandrov–Fenchel inequality, we obtain

$$\begin{aligned}
\frac{1}{\phi'_+(1)} \cdot \widetilde{V}_\phi(L_1, K_1, \Im) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} \widetilde{V}(K_\varepsilon, \Im) \\
&= \lim_{\varepsilon \to 0^+} \frac{\widetilde{V}(K_\varepsilon, \Im) - \widetilde{V}(L_1, \Im)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0^+} \frac{1 - \dfrac{\widetilde{V}(L_1, \Im)}{\widetilde{V}(K_\varepsilon, \Im)}}{\phi(1) - \phi\left(\dfrac{\widetilde{V}(L_1, \Im)}{\widetilde{V}(K_\varepsilon, \Im)}\right)} \cdot \frac{\phi(1) - \phi\left(\dfrac{\widetilde{V}(L_1, \Im)}{\widetilde{V}(K_\varepsilon, \Im)}\right)}{\varepsilon} \cdot \widetilde{V}(K_\varepsilon, \Im) \\
&= \lim_{t \to 0^+} \frac{1-t}{\phi(1) - (t)} \cdot \lim_{\varepsilon \to 0^+} \frac{\phi(1) - \phi\left(\dfrac{\widetilde{V}(L_1, \Im)}{\widetilde{V}(K_\varepsilon, \Im)}\right)}{\varepsilon} \cdot \lim_{\varepsilon \to 0^+} \widetilde{V}(K_\varepsilon, \Im) \\
&\geq \frac{1}{\phi'_+(1)} \cdot \lim_{\varepsilon \to 0^+} \phi\left(\frac{\widetilde{V}(K_1, \Im)}{\widetilde{V}(K_\varepsilon, \Im)}\right) \cdot \widetilde{V}(L_1, \Im) \\
&= \frac{1}{\phi'_+(1)} \cdot \phi\left(\frac{\widetilde{V}(K_1, \Im)}{\widetilde{V}(L_1, \Im)}\right) \cdot \widetilde{V}(L_1, \Im) \\
&\geq \frac{1}{\phi'_+(1)} \cdot \phi\left(\frac{\prod_{i=1}^r \widetilde{V}(K_i \ldots, K_i, K_{r+1}, \ldots, K_n)^{1/r}}{\widetilde{V}(L_1, \Im)}\right) \cdot \widetilde{V}(L_1, \Im).
\end{aligned} \tag{87}$$

From (87), inequality (86) easily follows. From the equality conditions of the dual Orlicz–Brunn–Minkowski inequality (80) and dual Aleksandrov–Fenchel inequality, it follows that if ϕ is strictly convex, the equality in (87) holds if and only if $L_1, K_1, \Im$ are all dilations of each other.

This proof is complete. □

From the proof of (87) and (80), it is not difficult to see that inequalities (63) and (80) are equivalent.

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