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Best Subordinant for Differential Superordinations of Harmonic Complex-Valued Functions

Georgia Irina Oros

Department of Mathematics and Computer Sciences, Faculty of Informatics and Sciences, University of Oradea, 410087 Oradea, Romania; georgia_oros_ro@yahoo.co.uk or goros@uoradea.ro

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Abstract: The theory of differential subordinations has been extended from the analytic functions to the harmonic complex-valued functions in 2015. In a recent paper published in 2019, the authors have considered the dual problem of the differential subordination for the harmonic complex-valued functions and have defined the differential superordination for harmonic complex-valued functions. Finding the best subordinant of a differential superordination is among the main purposes in this research subject. In this article, conditions for a harmonic complex-valued function p to be the best subordinant of a differential superordination for harmonic complex-valued functions are given. Examples are also provided to show how the theoretical findings can be used and also to prove the connection with the results obtained in 2015.

Keywords: differential subordination; differential superordination; harmonic function; analytic function; subordinant; best subordinant

MSC: 30C80; 30C45

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1. Introduction and Preliminaries

Since Miller and Mocanu [1] (see also [2]) introduced the theory of differential subordination, this theory has inspired many researchers to produce a number of analogous notions, which are extended even to non-analytic functions, such as strong differential subordination and superordination, differential subordination for non-analytic functions, fuzzy differential subordination and fuzzy differential superordination.

The notion of differential subordination was adapted to fit the harmonic complex-valued functions in the paper published by S. Kanas in 2015 [3]. In that paper, considering Ω and Δ any sets in the complex plane $\mathbb{C}$ and taking the functions $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ and p, a harmonic complex-valued function in the unit disc U of the form $p(z) = p_1(z) + \overline{p_2(z)}$, where p_1 and p_2 are analytic in U properties of the function p were determined such that p satisfies the differential subordination

$$\psi(p(z), Dp(z), D^2p(z); z) \subset \Omega \Rightarrow p(U) \subset \Delta.$$

Inspired by the idea provided by Miller and Mocanu [1], and following the research in [3,4] , the notion of differential superordination for harmonic complex—valued functions was introduced in [5]. In that paper, properties of the harmonic complex-valued function p of the form $p(z) = p_1(z) + \overline{p_2(z)}$, with p_1 and p_2 analytic in U, such that p satisfies the differential superordination

$$\Omega \subset \psi(p(z), Dp(z), D^2p(z); z) \Rightarrow \Delta \subset p(U).$$

Continuing the study on differential superordinations for harmonic complex-valued functions started in paper [5], the problem of finding the best subordinant of a differential superordination for harmonic complex-valued functions is studied in the present paper and a method for finding the best subordinant is provided in a theorem and few corollaries in the Main Results section. Examples are also given using those original and new theoretical findings.

The well-known definitions and notations familiar to the field of complex analysis are used. The unit disc of the complex plane is denoted by U. $\mathcal{H}(U)$ stands for the class of analytic functions in the unit disc and the classical definition for class A_n is applied, and it is known that it contains all functions from class $\mathcal{H}(U)$, which have the specific form

$$f(z) = z + a_{n+1}z^{n+1} + \dots,$$

with $z \in U$ and A_1 written simply A. All the functions in class A which are univalent in U form the class denoted by S. In particular, the functions in class A who have the property that

$$\operatorname{Re} \frac{zf''(z)}{zf'(z)} + 1 > 0$$

represent the class of convex functions K.

A harmonic complex-valued mapping of the simply connected region Ω is a complex-valued function of the form

$$f(z) = h(z) + \overline{g(z)}, \tag{1}$$

where h and g are analytic in Ω, with $g(z_0) = 0$, for some prescribed point $z_0 \in \Omega$.

We call h and g analytic and co-analytic parts of f, respectively. If f is (locally) injective, then f is called (locally) univalent. The Jacobian and the second complex dilatation of f are given by

$$Jf(z) = |h'(z)|^2 - |g'(z)|^2 \quad \text{and} \quad w(z) = \frac{g'(z)}{h'(z)},\ z \in U,$$

respectively. If $Jf(z) > 0$, $z \in U$, then f is a local sense-preserving diffeomorphism.

A function $f \in C^2(\Omega)$, $f(z) = u(z) + iv(z)$, which satisfies

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

is called harmonic function.

By $Har(U)$ we denote the class of complex-valued, sense-preserving harmonic mappings in U.

For $f \in Har(U)$, let the differential operator D be defined as follows

$$Df = z \cdot \frac{\partial f}{\partial z} - \overline{z}\, \frac{\partial f}{\partial \overline{z}} = zh'(z) - \overline{zg'(z)}, \tag{2}$$

where $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ are the formal derivatives of function f

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right). \tag{3}$$

The conditions (3) are satisfied for any function $f \in C'(\Omega)$ not necessarily harmonic, nor analytic.

Moreover, we define the n-th order differential operator by recurrence relation

$$D^2 f = D(Df) = Df + z^2h'' - \overline{z^2 q''}, \quad D^n f = D(D^{n-1} f). \tag{4}$$

Remark 1. *If $f \in \mathcal{H}(U)$ (i.e., $g(z) = 0$) then $Df(z) = zf'(z)$.*

In order to prove the main results of this paper, we use the following definitions and lemmas:

Definition 1 ([3] Definition 2.2)**.** *By Q, we denote the set of functions*

$$q(z) = q_1(z) + \overline{q_2(z)},$$

harmonic complex-valued and univalent on $\overline{U} - E(q)$, where

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \right\}.$$

Moreover, we assume that $D(q(\zeta)) \neq 0$, for $\zeta \in \partial U \setminus E(q)$.

The set $E(q)$ is called an exception set. We note that the functions

$$q(z) = \overline{z}, \quad q(z) = \frac{1+\overline{z}}{1-\overline{z}}$$

are in Q, therefore Q is a nonempty set.

Definition 2 ([5] Definition 2.2)**.** *Let $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be harmonic univalent in U. If p and $\varphi(p(z), Dp(z), D^2(p(z)))$ are harmonic univalent in U, and satisfy the second-order differential superordination for harmonic complex-valued functions*

$$h(z) \prec \varphi(p(z), Dp(z), D^2p(z); z) \tag{5}$$

then p is called a solution of the differential superordination.

A harmonic univalent function q is called a subordinant of the solutions of the differential superordination for harmonic complex-valued functions, or more simply a subordinant if $q \prec p$, for all p satisfying (5).

An univalent harmonic subordinant $\overline{q}$ that satisfies $q \prec \overline{q}$ for all subordinants q of (5) is said to be the best subordinant. The best subordinant is unique up to a rotation of U.

Lemma 1 ([5] Theorem 3.2)**.** *Let h, q be harmonic and univalent functions in U, $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$, and suppose that*

$$\varphi(q(z), tDq(z); \zeta) \in h(U),$$

for $z \in U$, $\zeta \in \partial U$ and $0 < t \leq \dfrac{1}{m} \leq 1$, $m \geq 1$.

If $p \in Q$ and $\varphi(p(z), D(p(z)); z \in U)$ is univalent in U, then

$$h(z) \prec \varphi(p(z), Dp(z); z \in U)$$

implies

$$q(z) \prec p(z),\ z \in U.$$

Furthermore, if $\varphi(q(z), Dq(z); z \in U) = h(z)$, has an univalent solution $q \in Q$, then q is the best subordinant.

Let $f : U \to \mathbb{C}$. We consider the special set

$$E(U) = \{f :\ f \in C'(U),\ Df \in C'(U)\} \supset C^2(U).$$

Lemma 2 ([6] Theorem 7.2.2, p. 131)**.** *If the function $f \in E(U)$ satisfies*

(i) $f(0) = 0,\ f(z) \cdot Df(z) \neq 0,\ z \in \dot{U};$

(ii) $Jf(z) = \left|\dfrac{\partial f}{\partial z}\right|^2 - \left|\dfrac{\partial f}{\partial \overline{z}}\right|^2 > 0,\ z \in U;$

(iii) $\operatorname{Re} \dfrac{D^2 f(z)}{Df(z)} > 0,\ z \in \dot{U}$

then the function f is convex in U. Furthermore $f(U_r)$ is a convex domain for any $r \in (0,1)$.

2. Main Results

In Definitions 1 and 2, just like in the hypothesis of Lemma 1, the function q must have a "nice" behavior on the border of the unit disc. If this condition is not satisfied or if the behavior of function q on the border of the domain is unknown, then the superordination $q(z) \prec p(z)$ can be proven by using a limiting procedure.

The next theorem and the corollaries give the sufficient conditions for obtaining the best subordinant for the differential superordination.

Theorem 1. *Let h be a convex harmonic complex-valued function in U, with $h(0) = a$, and let $\theta : D \subset \mathbb{C} \to \mathbb{C}$, $\phi : D \subset \mathbb{C} \to \mathbb{C}$ be a harmonic complex-valued function in a domain D. Suppose that the differential equation*

$$\theta[q(z)] + Dq(z) \cdot \phi[q(z)] = h(z),\ z \in U, \tag{6}$$

has an univalent harmonic solution q that satisfies $q(0) = a$, $q(U) \subset D$ and

$$\theta[q(z)] \prec h(z),\ z \in U. \tag{7}$$

Let p be a harmonic complex-valued univalent function with $p(0) = h(0) = \theta[p(0)]$, $p \in Q$ and $p(U) \subset D$. Then

$$h(z) \prec \theta[p(z)] + Dp(z) \cdot \phi[p(z)], \tag{8}$$

implies

$$q(z) \prec p(z),\ z \in U.$$

The function q is the best subordinant.

Proof. We can assume that h, p and q satisfy the conditions of the theorem on the closed disc $\overline{U}$, and $Dq(\zeta) \neq 0$, for $|\zeta| = 1$. If not, we can replace h, p and q by $h(\rho z)$, $p(\rho z)$ and $q(\rho z)$, where $0 < \rho < 1$.

These new functions have the desired properties on $\overline{U}$, and we can use them in the proof of the theorem. Theorem 1 would then follow by letting $\rho \to 1$. We will use Lemma A to prove this result.

Let $\varphi : \mathbb{C}^2 \times U \to \mathbb{C}$, where

$$\varphi(r,s) = \theta(r) + s \cdot \phi(r). \tag{9}$$

For $r = p(z)$, $s = Dp(z)$, relation (9) becomes

$$\varphi(p(z), Dp(z)) = \theta[p(z)] + Dp(z) \cdot \phi[p(z)], \tag{10}$$

and the superordination (8) becomes

$$h(z) \prec \varphi[p(z)] + Dp(z) \cdot \phi[p(z)]. \tag{11}$$

For $r = q(z)$ and $s = Dq(z)$, relation (9) becomes

$$\varphi(q(z), Dq(z)) = \theta[q(z)] + Dq(z) \cdot \phi[q(z)],\ z \in U \tag{12}$$

and (6) is equivalent to

$$\varphi(q(z), Dq(z)) = h(z),\ z \in U.$$

For $r = q(z)$ and $s = tDq(z)$, $0 \leq t \leq 1$, relation (9) becomes

$$\varphi(q(z), tDq(z)) = \theta[q(z)] + tDq(z) \cdot \phi[q(z)],\ 0 \leq t \leq 1. \tag{13}$$

From (6), we have

$$Dq(z) \cdot \phi[q(z)] = h(z) - \theta[q(z)]. \tag{14}$$

Using (14) in (13), we have

$$\varphi(q(z), tDq(z)) = (1-t)\theta[q(z)] + th(z), \ 0 \le t \le 1. \tag{15}$$

Since h is a convex function, $f(U)$ is a convex domain and using (7), we have

$$\varphi(q(z), tDq(z)) \in h(U), \ \text{for } 0 \le t \le 1.$$

Since the conditions from Lemma A are satisfied, we have

$$q(z) \prec p(z), \ z \in U.$$

Since q is the solution of Equation (6) we get that q is the best subordinant. □

In the special case when $\theta(w) = w$, and

$$\phi(w) = \frac{1}{\beta w + \gamma}, \ w = q(z), \ z \in U,$$

we obtain the following result for the Briot–Bouquet differential superordination.

Corollary 1. *Let $\beta, \gamma \in \mathbb{C}$, $\beta \neq 0$, and let h be a convex harmonic complex-valued function in U, with $h(0) = a$. Suppose that the differential equation*

$$q(z) + \frac{Dq(z)}{\beta q(z) + \gamma} = h(z), \ z \in U$$

has an univalent harmonic complex-valued solution q that satisfies $q(0) = a$ and $q(z) \prec h(z)$. If $p \in Q$ and $p(z) + \dfrac{Dp(z)}{\beta p(z) + \gamma}$ is harmonic complex-valued univalent in U, then

$$h(z) \prec p(z) + \frac{Dp(z)}{\beta p(z) + \gamma}$$

implies $q(z) \prec p(z)$, $z \in U$. The function q is the best subordinant.

If $\theta(w) = w$ and $\phi(w) = \beta w + \gamma$, $\beta \neq 0$, $w = q(z)$, $\gamma \in \mathbb{C}$, we obtain the following result.

Corollary 2. *Let $\beta, \gamma \in \mathbb{C}$, $\beta \neq 0$, and let h be a convex harmonic-valued function in U, with $h(0) = a$. Suppose that the differential equation*

$$q(z) + Dq(z)[\beta q(z) + \gamma] = h(z), \ z \in U,$$

has an univalent harmonic complex-valued solution q that satisfies $q(0) = a$ and $q(z) \prec h(z)$.

If $p \in Q$ and $p(z) + Dp(z)[\beta p(z) + \gamma]$ is univalent harmonic complex valued in U, then

$$h(z) \prec p(z) + Dp(z)[\beta p(z) + \gamma]$$

implies $q(z) \prec p(z)$. The function q is the best subordinant.

If $\theta(w) = w$, $\phi(w) = \dfrac{1}{\gamma}$, $\gamma \neq 0$, $w = q(z)$, we obtain the following result.

Corollary 3. *Let h be a convex harmonic complex-valued function in U, with $h(0) = a$. Let $\gamma \neq 0$, with $\operatorname{Re}\gamma \geq 0$. Suppose that the differential equation*

$$q(z) + \frac{1}{\gamma} Dq(z) = h(z),\ z \in U$$

has an univalent harmonic complex-valued solution q that satisfies $q(0) = a$ and $q(z) \prec p(z)$.

If $p \in Q$ and $p(z) + \dfrac{1}{\gamma} Dp(z)$ is univalent harmonic complex-valued in U, then

$$h(z) \prec p(z) + \frac{1}{\gamma} \cdot Dp(z),\ z \in U$$

implies

$$q(z) \prec p(z),\ z \in U.$$

The function q is the best subordinant.

Example 1. *For $\gamma = 2$, the univalent harmonic complex-valued function $q(z) = 6z - 4\overline{z}$, is the solution of the equation*

$$q(z) + \frac{1}{2} Dq(z) = h(z) = 9z - 2\overline{z}.$$

We next prove that h is a harmonic non-analytic function.

$$h(z) = 9(x + iy) - 2(x - iy) = 7x + 11iy.$$

We have

$$\frac{\partial h}{\partial x} = 7,\ \frac{\partial^2 h}{\partial x^2} = 0,\ \frac{\partial h}{\partial y} = 11,\ \frac{\partial^2 h}{\partial y^2} = 0,\ \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

We obtain that h is univalent harmonic complex-valued function and since $\dfrac{\partial h}{\partial x} \neq \dfrac{\partial h}{\partial y}$, we conclude that it is not analytic.

We next prove that the harmonic function h is also convex.

In order to do that, we show that it satisfies the conditions in the hypothesis of Lemma 2.

We calculate:

$$Dh(z) = z\,\frac{\partial h}{\partial z} - \overline{z}\,\frac{\partial h}{\partial \overline{z}} = 9z + 2\overline{z},$$

$$D^2 h(z) = D(Dh(z)) = 9z - 2\overline{z},$$

(i) $h(0) = 0,\ h(z) \cdot Dh(z) = (9z - 2\overline{z})(9z + 2\overline{z}) \neq 0,\ z \in \dot{U};$

(ii) $Jh(z) = \left|\dfrac{\partial h}{\partial z}\right|^2 - \left|\dfrac{\partial h}{\partial \overline{z}}\right|^2 = 77 > 0;$

(iii) $\operatorname{Re}\dfrac{D^2 h(z)}{Dh(z)} = \operatorname{Re}\dfrac{9z - 2\overline{z}}{9z + 2\overline{z}} = \dfrac{77x^2 + 77y^2}{121x^2 + 49y^2} > 0, z \in \dot{U}.$

As can be seen, all the conditions in Lemma B are satisfied, hence h is a harmonic convex function.

Using Corollary 3, we have:

If $p \in Q$, $p(0) = q(0) = 0$ and $p(z) + \dfrac{Dp(z)}{z}$ is univalent harmonic complex-valued in U, then

$$9z - 2\overline{z} \prec p(z) + \frac{Dp(z)}{z}$$

implies

$$6z - 4\overline{z} \prec p(z),\ z \in U.$$

The function $q(z) = 6z - 4\overline{z}$ is the best subordinant.

Example 2. *For $\gamma = 1$, the univalent harmonic complex-valued function $q(z) = 1 + 2z - 4\overline{z}$ is the solution of the equation:*

$$q(z) + Dq(z) = 1 + 4z = h(z).$$

We next prove that h is a harmonic complex-valued function.

$$h(z) = 1 + 4(x + iy) = 1 + 4x + i \cdot 4y.$$

We have

$$\frac{\partial h}{\partial x} = 4, \ \frac{\partial^2 h}{\partial x^2} = 0, \ \frac{\partial h}{\partial y} = 4, \ \frac{\partial^2 h}{\partial y^2} = 0.$$

From $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$, we have $h(z) = 1 + 4z$, is a harmonic complex-valued function.

We next prove that the harmonic function is also convex.

In order to that, we show that it satisfies the conditions in the hypothesis of Lemma B.

We calculate:

$$Dh(z) = z \cdot \frac{\partial h}{\partial z} - \overline{z} \cdot \frac{\partial h}{\partial \overline{z}} = 4z,$$

$$D^2 h(z) = D(Dh(z)) = z \cdot \frac{Dh(z)}{z} - \overline{z} \cdot \frac{Dh(z)}{\partial \overline{z}} = 4z.$$

(i) $h(0) = 1, \ h(z) \cdot Dh(z) = 4z + 16z^2 \neq 0, \ z \in \dot{U}$;

(ii) $Jh(z) = \left|\frac{\partial h}{\partial z}\right|^2 - \left|\frac{\partial h}{\partial \overline{z}}\right|^2 = 16 > 0$;

(iii) $\operatorname{Re} \frac{D^2 h(z)}{Dh(z)} = \operatorname{Re} \frac{4z}{4z} = \operatorname{Re} 1 = 1 > 0, \ z \in \dot{U}$.

As can be seen, all the conditions in Lemma B are satisfied, hence h is a harmonic convex function.

Using Corollary 3 we have:

If $p \in Q$, $p(0) = q(0) = 1$ and $p(z) + Dp(z)$ is univalent harmonic complex-valued in U, then

$$1 + 4z \prec p(z) + Dp(z)$$

implies

$$1 + 2z - 4\overline{z} \prec p(z), \ z \in U.$$

The function $q(z) = 1 + 2z - 4\overline{z}$ is the best subordinant.

Remark 2. *Using Example 2 and Example 2.4 in [3], we can write the following sandwich type result:*

If $p \in Q$, $p(0) = q(0) = 1$ and $p(z) + Dp(q)$ is univalent harmonic complex-valued in U, then

$$1 + 4z \prec p(z) + Dp(q) \prec \frac{1+z}{1-z} + \frac{\overline{z}}{1-\overline{z}}$$

implies

$$1 + 2z - 4\overline{z} \prec p(z) \prec \frac{1+z}{1-z} + \frac{\overline{z}}{1-\overline{z}}, \ z \in U.$$

3. Conclusions

The notion of differential superordination for harmonic complex-valued functions is a new topic emerged in the theory of differential superordinations. It contributes to further developing the theory of differential superordinations. The study done related to the research of this topic is just starting, so the present paper provides essential means for continuing this idea. The original and new results contained in the Main Results section of the present paper are important, since the problem of finding the best subordinant of the differential superordination for harmonic complex-valued functions is essential for the study related to the topic as it is well-known from the classical theory of differential superordinations. No further findings can be done without having a method for finding the best subordinant. A method is given in Theorem 1 and in the corollaries that follow. Using those results, researchers interested in the topic should be able to obtain further original results. Two examples are also enclosed, giving a better view on the idea.

The second example contains a sandwich-type result which makes the connection of the original results in this paper with the results previously obtained by S. Kanas [3]. The examples are useful by inspiring researchers in using the theoretical results contained in the theorem and corollaries for further studies on the subject.

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References

1. Miller, S.S.; Mocanu, P.T. Subordinants of differential superordinations. *Complex Var.* **2003**, *48*, 815–826. [CrossRef]
2. Miller, S.S.; Mocanu, P.T. Briot-Bouquet differential superordinations and sandwich theorems. *J. Math. Anal. Appl.* **2007**, *329*, 327–335. [CrossRef]
3. Kanas, S. Differential subordinations for harmonic complex-valued functions. *arXiv* **2015**, arXiv:1509.037511V1.
4. Bulboacă, T. *Differential Subordinations and Superordinations. Recent Results*; Casa Cărţii de Ştiinţă: Cluj-Napoca, Romania, 2005.
5. Oros, G.I.; Oros, G. Differential superordination for harmonic complex-valued functions. *Stud. Univ. Babeş-Bolyai Math.* **2019**, *64*, 487–496. [CrossRef]
6. Mocanu, P.T.; Bulboacă, T.; Sălăgean, S.G. *Geometric Function Theory*; Casa Cărţii de Ştiinţă: Cluj-Napoca, Romania, 1999.