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Laguerre-Type Exponentials, Laguerre Derivatives and Applications. A Survey

Paolo Emilio Ricci

Department of Mathematics, International Telematic University UniNettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy; paoloemilioricci@gmail.com or p.ricci@uninettunouniversity.net

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Abstract: Laguerrian derivatives and related autofunctions are presented that allow building new special functions determined by the action of a differential isomorphism within the space of analytical functions. Such isomorphism can be iterated every time, so that the resulting construction can be re-submitted endlessly in a cyclic way. Some applications of this theory are made in the field of population dynamics and in the solution of Cauchy's problems for particular linear dynamical systems.

Keywords: Laguerre-type derivative; Laguerre-type exponentials; Laguerre-type special functions; multivariable and multi-index Laguerre polynomials; population dynamics models; Laguerre-type linear dynamical systems

MSC: 33C45; 33C99; 30D05; 33B10; 33C10; 92D25; 34A30

1. Introduction

This survey article is dedicated to a topic that has received little attention in the past, and therefore seems not to be very well known by the mathematical community.

Recently the role of the Laguerre derivative was considered in a few papers.

In [1], the authors introduce an interesting application of Wright functions of the first kind to solve fractional ordinary differential equations, with variable coefficients, generalizing the Bessel-type equations.

In [2], the authors use the same tool in Combinatorics, a completely different area, and in [3] an operational approach to the subject has been examined, in the framework of Clifford algebras.

Actually, in past time, the Laguerre-type exponentials and the related Laguerre derivative were introduced and studied in several articles (see [4–15]) and applications to Special functions, have been obtained. In particular, Laguerre-type functions of Bessel, Appell, Bell and multivariate functions were defined.

The operator $DxD = D + x\,D^2$ determines a linear differential isomorphism, acting onto the space of analytic functions of the x variable. By using this isomorphism, a sort of parallel structure is created within this space, in such a way that the differentiation properties have their counterpart, which can be immediately derived.

Furthermore, iterations of the Laguerre derivative can be defined, so that this parallelism with the space of analytic functions can be iterated too, in an endless way.

Therefore, a cyclic construction is created within the space that repeats the same structure at a higher level of differentiation order. It is one of the great cycles that sometimes occur within mathematical theories: for example, in Number theory the Fibonacci numbers F_n with Fibonacci indexes constitute a higher sequence of Fibonacci numbers which still satisfies the same recursion, i.e., $F_{F_{n+2}} = F_{F_{n+1}} + F_{F_n}$, and this property can be iterated at infinity.

However the operators $D_L = DxD$ and its iterates as $D_{nL} = DxDxDx \cdots DxD$ are not completely new, since they can be considered to be particular cases of the hyper-Bessel differential operators when $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 1$ (the special case considered in operational calculus by Ditkin and Prudnikov [16]). In general, the *Bessel-type differential operators of arbitrary order n* were introduced by Dimovski, in 1966 [17] and later called by Kiryakova *hyper-Bessel operators*, because are closely related to their eigenfunctions, called hyper-Bessel by Delerue [18], in 1953. These operators were studied in 1994 by Kiryakova in her book [19] (Ch. 3).

Since the Laguerrian exponentials on the positive semi-axis of the abscissas are convex increasing functions, with a growth lower than the exponential one, in Section 7 a natural application was made in the context of population dynamics.

Laguerre-type linear dynamical systems were also considered in Section 8.

2. The Laguerre Derivative and the Relevant Exponentials

The *Laguerre derivative*, is defined by

$$D_L := DxD = D + xD^2, \tag{1}$$

where $D = D_x = d/dx$.

It is an interesting operator. In fact, as the exponential function e^{ax} (a constat) is an eigenfunction of the derivative operator $D = D_x$, i.e.,

$$De^{ax} = ae^{ax}\,, \tag{2}$$

equally the function

$$e_1(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} = C_0(-x), \tag{3}$$

where $C_0(x)$ is the Tricomi function of order zero, is an eigenfunction of the Laguerre derivative D_L, since:

$$D_L\, e_1(ax) = ae_1(ax)\,. \tag{4}$$

The proof easily follows, by noting that:

$$\begin{aligned} D_L\, e_1(ax) &= \left(D + xD^2\right) \sum_{k=0}^{\infty} a^k \frac{x^k}{(k!)^2} = \\ = \sum_{k=1}^{\infty} (k + k(k-1))\, a^k \frac{x^{k-1}}{(k!)^2} &= \sum_{k=1}^{\infty} k^2\, a^k \frac{x^{k-1}}{(k!)^2} = \\ = a \sum_{k=0}^{\infty} a^k \frac{x^k}{(k!)^2} &= ae_1(ax). \end{aligned} \tag{5}$$

For this reason, *the function $e_1(x)$ is called the Laguerre-type exponential* (of order 1).

In preceding articles, the role of the Laguerre derivative, in connection with the *monomiality principle*—an important technique introduced by G. Dattoli [20]—and its application to the multidimensional Hermite (Hermite-Kampé de Fériet or Gould-Hopper polynomials, see [21–23]) or Laguerre polynomials [14,24], has been shown.

The above technique can be iterated, producing Laguerre classes of exponential-type functions, of higher order, called *L-exponentials*, and the relevant *L-circular*, *L-hyperbolic*, *L-Gaussian functions* (see [4]).

Similar generalized hypergeometric functions, called trigonometric/Bessel type, exponential/confluent type and Gauss/Beta-distribution, can be found in a book by Kiryakova [19] and also in [25].

Before going on, we notice that the Laguerre derivative verifies [26]:

$$(DxD)^n = D^n x^n D^n \,, \tag{6}$$

an equation which can be easily proven by recursion.

2.1. L-Exponentials of Higher Order

We consider the operator:

$$D_{2L} := DxDxD = D\left(xD + x^2D^2\right) = D + 3xD^2 + x^2D^3, \tag{7}$$

and the function:

$$e_2(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^3}. \tag{8}$$

The following theorem holds:

Theorem 1. *The function* $e_2(ax)$ *is an eigenfunction of the operator* D_{2L}*, i.e.,*

$$D_{2L}\, e_2(ax) = a e_2(ax) \tag{9}$$

The proof (see [4]) depends on the identity: $k + 3k(k-1) + k(k-1)(k-2) = k^3$, so that, it can be recognized that the coefficients of the combination in Equation (7) are the *Stirling numbers of the second kind*, $S(3,1), S(3,2), S(3,3)$, (see [27], and [28] (p. 835 for an extended table)).

In general, we can state the following theorem:

Theorem 2. *The function*

$$e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}}. \tag{10}$$

is an eigenfunction of the operator

$$\begin{aligned} D_{nL} &:= Dx \cdots DxDxD = D\left(xD + x^2D^2 + \cdots + x^nD^n\right) = \\ &= S(n+1,1)D + S(n+1,2)xD^2 + \cdots + S(n+1,n+1)x^nD^{n+1}\,, \end{aligned} \tag{11}$$

i.e., for every constant a it results:

$$D_{nL}\, e_n(ax) = a e_n(ax)\,. \tag{12}$$

Remark 1. *The above results show that, for every positive integer* n*, we can define a Laguerre-exponential function, satisfying an eigenfunction property, which is an analog of the elementary property* (2) *of the exponential. The function* $e_n(x)$ *reduces to the exponential function when* $n = 0$*, so that we put by definition:*

$$e_0(x) := e^x, \qquad D_{0L} := D.$$

Obviously, $D_{1L} := D_L$.

Examples of the L-exponential functions are given in Figure 1.

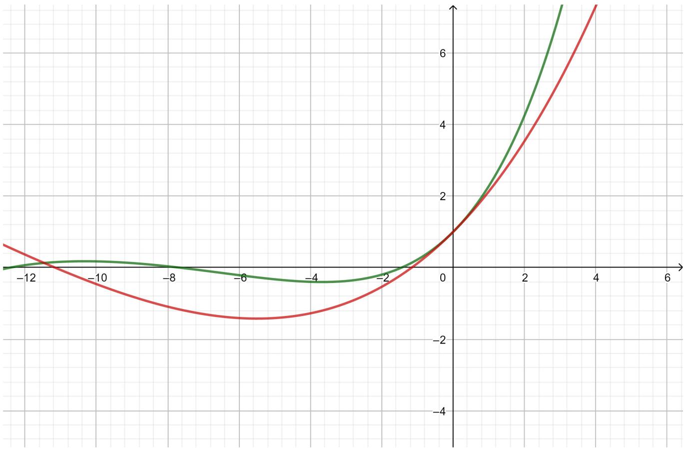

Figure 1. $e_1(x)$, (green) and $e_2(x)$, (red).

2.2. L-Circular and L-Hyperbolic Functions

Starting from the equation

$$e_1(ix) = \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h}}{((2h)!)^2} + i \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h+1}}{((2h+1)!)^2}, \tag{13}$$

we can define the $1L$-*circular functions* as follows

$$\cos_1(x) := \Re\left(e_1(ix)\right) = \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h}}{((2h)!)^2}, \tag{14}$$

$$\sin_1(x) := \Im\left(e_1(ix)\right) = \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h+1}}{((2h+1)!)^2}, \tag{15}$$

so that we find the Euler-type formulas

$$\cos_1(x) = \frac{e_1(ix) + e_1(-ix)}{2}, \qquad \sin_1(x) = \frac{e_1(ix) - e_1(-ix)}{2i}, \tag{16}$$

Recalling Equation (6), we find the result:

Theorem 3. *The* $1L$-*circular functions* (14) *and* (15) *are solutions of the differential equation*

$$D_L^2\, v + v = \left(D^2 x^2 D^2\right) v + v = 0. \tag{17}$$

The above results hold even for the generalized case.
Write the nL-exponential in the form:

$$e_n(ix) = \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h}}{((2h)!)^{n+1}} + i \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h+1}}{((2h+1)!)^{n+1}}. \tag{18}$$

Then we can define the nL-*circular functions* by putting

Definition 1.

$$\cos_n(x) := \Re\left(e_n(ix)\right) = \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h}}{((2h)!)^{n+1}}, \tag{19}$$

$$\sin_n(x) := \Im\left(e_n(ix)\right) = \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h+1}}{((2h+1)!)^{n+1}}, \tag{20}$$

and we find again the Euler-type formulas:

$$\cos_n(x) = \frac{e_n(ix) + e_n(-ix)}{2}, \qquad \sin_n(x) = \frac{e_n(ix) - e_n(-ix)}{2i}. \tag{21}$$

Theorem 3 becomes, in general:

Theorem 4. *The nL-circular functions* (18) *and* (19) *are solutions of the differential equation*

$$D_{nL}^2\, v + v = 0.$$

and satisfy the conditions:

$$\cos_n(0) = 1, \qquad \sin_n(0) = 0.$$

Furthermore, we find:

Theorem 5. *The nL-circular functions satisfy*

$$D_{nL}\, \cos_n(x) = -\sin_n(x), \qquad D_{nL}\, \sin_n(x) = \cos_n(x)\,. \tag{22}$$

Examples of the L-circular functions are given in Figures 2 and 3.

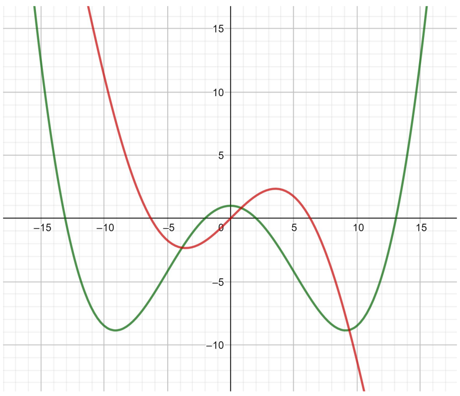

Figure 2. $\cos_1(x)$, (green) and $\sin_1(x)$, (red).

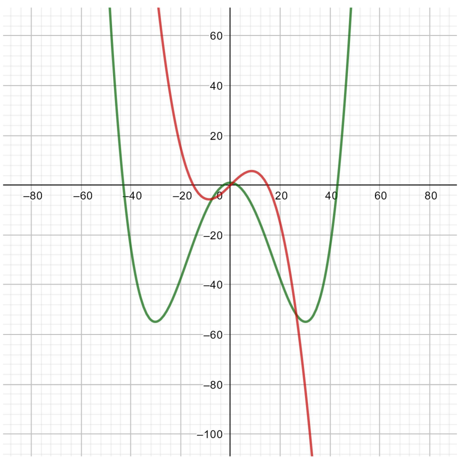

Figure 3. $\cos_2(x)$ (green) and $\sin_2(x)$ (red).

In a similar way we can define the nL-hyperbolic functions, putting

$$\cosh_n(x) := \sum_{h=0}^{\infty} \frac{x^{2h}}{((2h)!)^{n+1}},$$

$$\sinh_n(x) := \sum_{h=0}^{\infty} \frac{x^{2h+1}}{((2h+1)!)^{n+1}},$$

and the formulas analogues of that of the circular functions are easily derived (see [4]).

All the eigenfunctions $e_1(x), e_2(x), \ldots, e_n(x)$ can be expressed as generalized hypergeometric functions ${}_pF_q$, [29], namely: $e_1(x) = {}_0F_1(-x)$, $e_2(x) = {}_0F_2(x), \ldots, e_n(x) = {}_0F_n(x)$. In practice, starting from the Bessel function $e_1(x)$, all these eigenfunctions are special cases of the hyper-Bessel functions of Delerue [18], which are shown to be eigenfunctions of Dimovski's operators mentioned above.

Naturally, the $\cos_n$, $\sin_n$ functions, in Equations (19) and (20), and their hyperbolic variants, are special cases of the *trigonometric type generalized hypergeometric functions* considered in the Kiryakova book [19].

3. The Isomorphism $\mathcal{T}_x$ and Its Iterations

It was previously noted (see e.g., [14]) that, in the space $\mathcal{A}_x$ of analytic functions, it is possible to define an isomorphism $\mathcal{T}_x$ that preserves the differentiation properties, by means of correspondence:

$$D \to D_L, \qquad x\cdot \to D_x^{-1}, \tag{23}$$

where

$$D_x^{-1} f(x) = \int_0^x f(\xi)\, d\xi\,, \qquad D_x^{-n} f(x) = \frac{1}{(n-1)!} \int_0^x (x-\xi)^{n-1} f(\xi)\, d\xi\,, \tag{24}$$

so that

$$\mathcal{T}_x(x^n) = D_x^{-n}(1) = \frac{1}{(n-1)!} \int_0^x (x-\xi)^{n-1}\, d\xi = \frac{x^n}{n!}\,. \tag{25}$$

It is worth noting that this kind of isomorphism is widely used in operational calculus and differential equations also under the name of *Transmutation or Similarity operator*, since it transforms one operator into another, and eigenfunctions into each other.

In fact, in such an isomorphism we have the correspondences:

- The exponential function is transformed into the function $e_1(x)$, since

$$\mathcal{T}_x(e^x) = \sum_{k=0}^{\infty} \frac{\mathcal{T}_x(x^k)}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} = e_1(x)\,.$$

- The Hermite polynomial $H_n^{(1)}(x,y) := (x-y)^n$ becomes the Laguerre polynomial

$$\mathcal{L}_n(x,y) := n! \sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^r}{(n-r)!(r!)^2}$$

  and by using the *monomiality principle* we can prove thate all the relations valid in the polynomial space still hold after the substitutions stated in Equation (23).

Furthermore, an iterative application of Equation (23) gives in sequence the functions $e_1(x), e_2(x), e_3(x), \ldots$.

We have, for example:

$$\mathcal{T}_x^2(e^x) = \sum_{k=0}^{\infty} \frac{\mathcal{T}_x(x^k)}{(k!)^2} = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^3} = e_2(x)\,,$$

and so on.

We already noticed that the isomorphism connected with the Laguerre derivative can be iterated as many times as we wish.

Correspondently, the derivative operator is transformed into

$$\begin{aligned} &D_L = DxD\,, \qquad D_{2L} = D_L D_x^{-1} D_L = DxDxD\,, \\ &D_{3L} = D_L D_x^{-1} D_L D_x^{-1} D_L = DxDxDxD\,, \ldots\,, \end{aligned} \tag{26}$$

and so on.

We can conclude that the L-exponentials (and the relevant L-circular and L-hyperbolic functions) are determined by an iterative application of the considered differential isomorphism.

4. Examples of Laguerre-Type Problems

4.1. L-Diffusion Equations

Theorem 6. *For any fixed integer n, consider the problem* (see [4] (Theorem 5.1)):

$$\begin{cases} D_{nL}\, S(x,t) = \frac{\partial}{\partial t}\, S(x,t), & \text{in the half plane } t > 0, \\ S(0,t) = s(t), \end{cases} \tag{27}$$

with analytic boundary condition $s(t)$.

The operational solution of problem (27) *is given by:*

$$S(x,t) = e_n\left(x\frac{\partial}{\partial t}\right) s(t) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}} \frac{d^k}{dt^k} s(t) \tag{28}$$

Representing $s(t) = \sum_{k=0}^{\infty} a_k t^k$, from Equation (28) we find, in particular:

$$S(x,0) = \sum_{k=0}^{\infty} a_k \frac{x^k}{(k!)^n}. \tag{29}$$

Please note that the operational solution becomes an effective solution whenever the series in Equation (28) is convergent. The validity of this condition depends on the growth of the coefficients a_k of the boundary data $s(t)$, but it is usually satisfied in physical problems.

More general problems are shown in [4,10], where evolution problems related to an operator of the type

$$D^{p_1} x^{q_1} D^{p_2} x^{q_2} \cdots D^{p_r} x^{q_r} D^{p_{r+1}}\,, \tag{30}$$

where $p_1, p_2, \ldots, p_{r+1}; q_1, q_2, \ldots q_r$ are fixed integers, have been considered.

An operational solution of the problem

$$D^{p_1} x^{q_1} D^{p_2} x^{q_2} \cdots D^{p_r} x^{q_r} D^{p_{r+1}} S(x,t) = D_t S(x,t), \quad \text{in the half plane } t > 0\,,$$

with suitable initial conditions have been determined, in terms of the eigenfunctions of the same operator.

Remark 2. *Please note that the above operators generalize the subsequent Laguerre-type derivatives, since they are written as:*

$$D_{nL}^r = \underbrace{(DxDx\cdots DxD)^r}_{(n+1)\ Derivatives} = D^r x^r D^r x^r \cdots D^r x^r D^r\,, \tag{31}$$

which is an equation extending (6).

The operator (30) closely recalls the general case of hyper-Bessel B operators, in [17], since integers $q_1, q_2, \ldots q_r$ could be replaced by arbitrary real numbers, as are parameters $\alpha_0, \alpha_1, \ldots, \alpha_n$, considered in [17,30]. The solutions of the general differential equation $By(x) + \lambda y(x) = f(x)$ are given by Kiryakova et al. in [31].

4.2. L-Hyperbolic-Type Problems

Theorem 7. *Let $\hat{\Omega}_x$ be a 2nd order differential operator with respect to the x variable, $D_{nL} := (D_{nL})_t$ the nL-derivative with respect to the t variable, and denote by $\psi(t)$ and $\chi(t)$ two functions such that:*

$$\begin{aligned} D_{nL}\,\psi(t) &= \chi(t), \qquad D_{nL}\,\chi(t) = \psi(t) \\ \psi(0) &= 1, \qquad \chi(0) = 0 \end{aligned} \tag{32}$$

then the abstract L-hyperbolic-type problem:

$$\begin{cases} \hat{\Omega}_x^2\, S(x,t) = D_{nL}^2\, S(x,t), & \text{in the half plane } t > 0, \\ S(x,0) = q(x), \\ D_{nL}\, S(x,t)|_{t=0} = v(x) \end{cases} \tag{33}$$

with analytic initial condition $q(x), v(x)$, admits the operational solution (see [4], Theorem 5.3):

$$S(x,t) = \psi\left(t\hat{\Omega}_x\right) q(x) + \chi\left(t\hat{\Omega}_x\right) w(x), \tag{34}$$

where $w(x) := \hat{\Omega}_x^{-1} v(x)$.

Please note that conditions in (32) are satisfied, for any fixed integer n, assuming:

$$\psi(x) := \cosh_{nL}(x), \qquad \chi(x) := \sinh_{nL}(x)\,.$$

4.3. L-Elliptic-Type Problems

Theorem 8. *Let $\hat{\Omega}_x$ be a 2nd order differential operator with respect to the x variable, $D_{nL} := (D_{nL})_y$ the nL-derivative with respect to the y variable, and denote by $\varphi(y)$ and $\tau(y)$ two functions such that:*

$$\begin{aligned} D_{nL}\,\varphi(y) &= -\tau(y), \qquad D_{nL}\,\tau(y) = \varphi(y) \\ \varphi(0) &= 1, \qquad \tau(0) = 0 \end{aligned} \tag{35}$$

then the abstract L-elliptic-type problem:

$$\begin{cases} \hat{\Omega}_x^2\, S(x,y) + D_{nL}^2\, S(x,y) = 0, & \text{in the half plane } t > 0, \\ S(x,0) = q(x), \end{cases} \tag{36}$$

with analytic boundary condition $q(x)$, admits the operational solution (see [4], Theorem 5.4):

$$S(x,y) = \varphi\left(y\hat{\Omega}_x\right) q(x). \tag{37}$$

Please note that conditions in (35) are satisfied, for any fixed integer n, assuming:

$$\varphi(x) := \cos_{nL}(x), \qquad \tau(x) := \sin_{nL}(x)\,.$$

Further examples of PDE's problems involving the Laguerre derivatives can be found in [10,11].

5. Laguerre-Type Special Functions

5.1. Laguerre-Type Bessel Functions

The Laguerre-type Bessel functions, of order 1, (shortly *L-Bessel functions*), denoted by ${}_LJ_n(x)$, are obtained substituting the exponential with the L-exponential $e_1(x)$ in the classic generating function, i.e., by putting

$$e_1\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} {}_LJ_n(x)\, t^n\,.$$

We can derive the explicit expression by applying the isomorphism $\mathcal{T}_x$ to both sides of the explicit expression of the Bessel functions, so that we find:

$${}_LJ_n(x) := \sum_{n=0}^{\infty} \frac{(-1)^h x^{n+2h}}{2^{n+2h}\, h!(n+h)!(n+2h)!}\,.$$

We proved the results:

Theorem 9. *The L-Bessel functions ${}_LJ_n(x)$ satisfy the recurrence relation* (see [8], Theorem 2.3):

$$\begin{cases} \hat{D}_x^{-1}\left[{}_LJ_{n-1}(x) + {}_LJ_{n+1}(x)\right] = 2n\, {}_LJ_n(x), \\ {}_LJ_{n-1}(x) - {}_LJ_{n+1}(x) = 2\hat{D}_L\, {}_LJ_n(x). \end{cases}$$

Theorem 10. *The differential equation satisfied by the L-Bessel functions ${}_LJ_n(x)$ is* (see [8], Theorem 2.5):

$$\left(\hat{D}_L^2 + \hat{D}_x\hat{D}_L - n^2\hat{D}_x^2 + \hat{I}\right)\, {}_LJ_n(x) = 0\,,$$

where $\hat{I}$ denotes the identity operator. This equation can be derived by applying the isomorphism $\mathcal{T}_x$ to both sides of the differential equation of the ordinary first kind Bessel functions.

5.2. Laguerre-Type Hypergeometric Functions

By using the isomorphism technique it is possible to define in general Laguerre-type special functions, and in particular, the 1st order Laguerre-type hypergeometric functions.

In fact, starting from the Gauss' hypergeometric equation:

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$

and applying the isomorphism $\mathcal{T}_x$, we find the equation

$$x(1-x)D_L^2 y + [c-(a+b+1)x]D_L y - aby = 0\,, \tag{38}$$

that is:

$$[x(1-x)](x^2y^{iv} + 4xy''' + 2y'') + [c-(a+b+1)x](y' + xy'') - aby = 0\,. \tag{39}$$

The solution of Equation (38), corresponding to the Gauss' hypergeometric equation $F(a,b,c;x)$, is given by

$$ {}_L F(a,b,c;x) = 1 + \sum_{n=1}^{\infty} \frac{a^{(n)} b^{(n)}}{c^{(n)}} \frac{x^n}{(n!)^2}, \tag{40} $$

where the symbol $a^{(n)}$ denotes the rising factorial.

Of course the rth order Laguerre-type hypergeometric functions are obtained by applying to both sides of the hypergeometric equation the iterated isomorphism of order r, but the corresponding differential equation becomes more and more complicated as r increases.

The generalized hypergeometric functions have their 1st order Laguerre-type counterpart, which are given by:

$$ {}_L {}_p F_q(a_1, \ldots, a_p; b_1, \ldots, b_q, ; x) = \sum_{n=0}^{\infty} \frac{a_1^{(n)} \cdots a_p^{(n)}}{b_1^{(n)} \cdots b_q^{(n)}} \frac{x^n}{(n!)^2}, \tag{41} $$

and those of higher order immediately follow.

Please note that the function in (41) can be viewed as a generalized hypergeometric function of the form ${}_p F_{q+1}$, by moving one of the $n!$ in the first fraction under the sum and considering $\Gamma(n+1)/\Gamma(1) = 1^{(n)} = n! = b_{q+1}^{(n)}$ as the $(q+1)$-th term.

5.3. Laguerre-Type Bell Polynomials

We first note that for the Laguerre derivative, the chain rule

$$ \frac{d}{dt} = \frac{d}{dx} \frac{dx}{dt} $$

becomes:

$$ \frac{d}{dt} t \frac{d}{dt} = \frac{d}{dx} \frac{d}{dt} t \frac{dx}{dt}, \qquad \text{that is :} \qquad (D_L)_t = \frac{d}{dx} (D_L)_t \, x \tag{42} $$

and in general:

$$ (D_{nL})_t = \frac{d}{dx} (D_{nL})_t \, x. $$

The problem of constructing Bell polynomials can be extended in the natural way to the case of the Laguerre-type derivatives.

To this aim, we introduce the definition:

Definition 2. *The* n*th Laguerre-type Bell polynomial, denoted by* ${}_{rL}Y_n(x; [f,g]_n)$, *represents the* n*th* r*Laguerre-type derivative of the composite function* $f(g(t))$.

In [12] we showed that ${}_{rL}Y_n$ can be expressed as a polynomial in the independent variable x, depending on $f_1, g_1; f_2, g_2; \ldots; f_n, g_n$, in terms of the classical Bell polynomials.

According to Equation (6), the Leibniz rule, gives:

$$ \begin{aligned} (DxD)^n = D^n (x^n D^n) = \sum_{k=0}^{n} \binom{n}{k} D^{n-k} x^n D^{n+k} = \\ = \sum_{k=0}^{n} \left[\binom{n}{k} \right]^2 (n-k)! \, x^k D^{n+k} = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n}{k} x^k D^{n+k}. \end{aligned} \tag{43} $$

Therefore, the following representation formula for the Laguerre-type Bell polynomials, denoted by ${}_LY_n$, holds:

Theorem 11. *The ${}_LY_n$ polynomials are expressed in terms of the ordinary Bell polynomials according to the equation* (see [12], Theorem 4.1):

$$ {}_LY_n\left(x;[f,g]_n\right) = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n}{k} x^k\, Y_{n+k}\left([f,g]_{n+k}\right) . \tag{44}$$

The above results can be easily generalized, since

$$\begin{aligned}(D_{2L})^n &= (DxDxD)^n = D^n\left(x^n D^n x^n D^n\right) = \\ &= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n} \frac{n!}{k_1!}\frac{(n+k_1)!}{(k_1+k_2)!}\binom{n}{k_1}\binom{n}{k_2} x^{k_1+k_2}\, D^{n+k_1+k_2} .\end{aligned} \tag{45}$$

In [12] even the general case of polynomials ${}_{rL}Y_n$ Bell is considered, but we do not report here the equation which is a little more complicated.

6. The Multivariate Case

6.1. Laguerre-Type Appell Polynomials

In a preceding article [32] multivariate extensions of the Appell polynomials (including the Bernoulli and Euler cases) have been introduced, by means of the generating function [23]:

$$A(t)\exp(xt+y^j) = \sum_{n=0}^{\infty} R_n^{(j)}(x,y)\frac{t^n}{n!},$$

where j is a fixed integer.

The application of the isomorphism $\mathcal{T}_x$, and its iterations allows defining new classes of multivariate special polynomials, the Laguerre-type Appell polynomials, and to build their main properties (recurrence relations, shift operators, differential equations, etc), in an easy and uniform way.

This has been achieved in [6] starting from generating functions of the type

$$A(t)e_s(xt)e_\sigma(yt^j) = \sum_{n=0}^{\infty} R_n^{(j)}(\mathcal{T}_x^s(x),\mathcal{T}_y^\sigma(y))\frac{t^n}{n!}$$

where $e_s(\cdot)$ and $e_\sigma(\cdot)$ are Laguerre-type exponentials. Many properties of these functions have been derived, including recursions and differental equations.

The results obtained in this case are easily extended to the functions of r variables, since the technique works regardless of the number r.

6.2. Laguerre-Type Appell Series

We limit ourselves to the case of series in two variables, but the equations trivially extend to the general case. For $|x| < 1$, $|y| < 1$ the double series

$${}_LF_1(a,b_1,b_2;c;x,y) = \sum_{m,n=0}^{\infty} \frac{a^{(m+n)}b_1^{(m)}b_2^{(n)}}{c^{(m+n)}}\frac{x^m}{(m!)^2}\frac{y^n}{(n!)^2} \tag{46}$$

is the Laguerre-type Appell series, obtained by the classical one acting on it with the two isomorphisms $\mathcal{T}_x$ and $\mathcal{T}_y$.

We avoid to consider further extension to the case of multivariate functions with several parameters, since they are trivially obtained.

7. Applications to Population Dynamics

7.1. Exponential and L-Exponential Models

In this section a possible application of the Laguerre derivative is recalled [9,13]. Since the L-exponentials for every $x \geq 0$ are convex increasing functions, with a graph lower with respect to $\exp(x)$, it is possible to use these function in the framework of population dynamics, as it seems that in some cases the growth of the exponential is too fast.

Consider the number $N(t)$ of population individuals at time t and let $N(0) = N_0$ the initial number at time $t = 0$.

In the Malthus model, the variation is assumed to be proportional to $N(t)$, i.e.,

$$\begin{cases} \dfrac{d}{dt}N(t) = rN(t), \\ \\ N(0) = N_0\,, \end{cases}$$

where the *growth rate* r is a suitable constant.

The solution is given by the exponential function

$$N(t) = N_0\, e^{rt}\,.$$

Using the Laguerre derivative, the Laguerre-type Malthus reads:

$$\frac{d}{dt}t\frac{dN}{dt} = rN(t) \qquad \text{i.e.} \qquad \frac{dN}{dt} + t\frac{d^2N}{dt^2} = rN(t)\,,$$

where r is a positive constant. Assuming the initial conditions

$$\begin{cases} N(0) = N_0, \\ \\ N'(0) = N_1 = N_0 r\,, \end{cases}$$

we find the solution

$$N(t) = N_0 e_1(rt) = N_0 \sum_{k=0}^{+\infty} r^k \frac{t^k}{(k!)^2}\,.$$

In this case the population growth increases according to the Laguerre exponential function $e_1(x)$, so that the relevant increasing is slower with respect to the classical Malthus model.

In [9] it has been shown, with tables of data taken from real population dynamics, that the Laguerre-type Malthus model produces data closer to real population growth.

7.2. Logistic vs. L-logistic Model

Taking into account that the growth rate cannot be constant, since it depends on the environmental resources, Pierre Verhulst considered the so-called *logistic model*

$$\begin{cases} \dfrac{dN}{dt} = r\left[1 - \dfrac{1}{K}N(t)\right]N(t), \\ \\ N(0) = N_0\,, \end{cases}$$

where r is called the *intrinsic growth rate*, and K denotes the *environmental capacity*.

The exact solution of this problem is given by

$$N(t) = \frac{N_0 K}{N_0 + (K - N_0)e^{-rt}},$$

so that, if $N_0 < K$ the solution is a function monotonically increasing to K, whereas, if $N_0 > K$, the solution is monotonically decreasing to K. In any case,

$$\lim_{t \to \infty} N(t) = K,$$

and the value $N(t) = K$ is a stable equilibrium point for the logistic equation.

The Laguerre-logistic (shortly L-logistic) model is expressed by

$$\begin{cases} N'(t) + tN''(t) = rN(t)\left(1 - \dfrac{N(t)}{K}\right), \\ N(0) = N_0\,, \\ N'(0) = N_1\,. \end{cases} \tag{47}$$

Please note that if in the above equation N is small with respect to K, then N/K is close to 0 and consequently $D_t t D_t N \approx rN(t)$.

If $N \to K$, then $N/K \to 1$, and $D_t t D_t N \to 0$.

The L-logistic equation cannot be solved explicitly, but numerically, using a Runge-Kutta method.

The behavior of the approximate solutions for the L-logistic model is shown in Figure 4. It is worth noting that the solution tends to the environmental capacity K by an oscillating behavior.

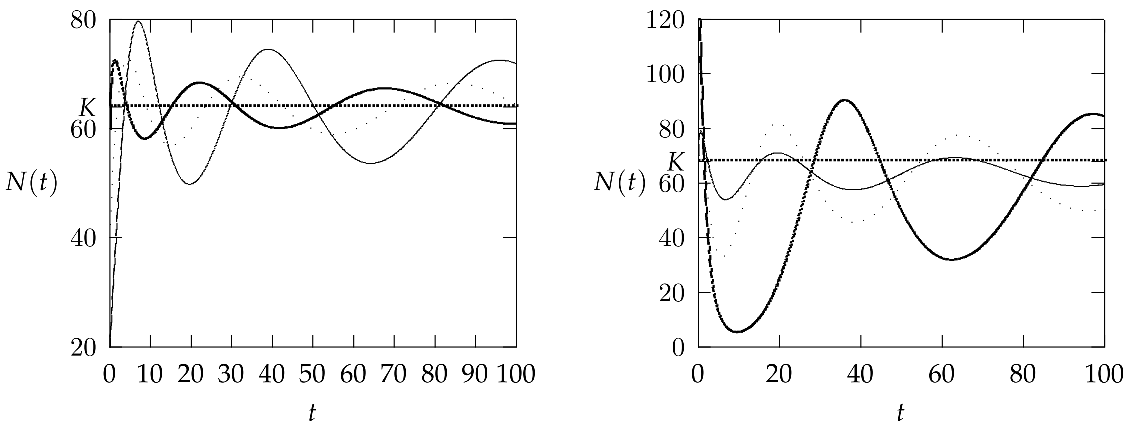

Figure 4. Solutions to the L-logistic model with $N(0) = N'(0) < K$ (on the left), $N(0) = N'(0) > K$ (on the right), $K = 64$, $r = 0.8$, $T = 100$, $\Delta t = 0.1$.

This is the main difference with respect to the ordinary logistic model, since in that case the solution was monotonically increasing or decreasing to K.

Similar results could be obtained by using the nL-derivatives, introducing suitable initial conditions which can be easily derived from the initial observations data.

Please note that as the order n increases, for $x > 0$, the Laguerrian exponential attenuates its growth and for $n \to \infty$ it tends to assume the linear value $1 + x$, so it can be used to model a growth as slow as it is needed.

Remark 3. *We recall that the oscillating asymptotic trend of solutions occurs in reality. For example, the classical experiment of G.F. Gause, relative to the protozoon paramecium shows such a typical behavior, represented in Figure 5. In this figure the true values, represented by a dotted line, are compared with the exponential trend of Malthus and with the logistic curve.*

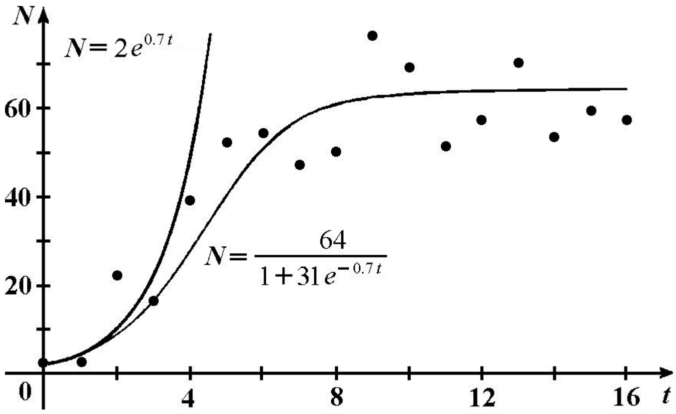

Figure 5. The behavior of growth in the Gause experiment.

7.3. Modified L-Logistic Models

Many different models modifying the basic logistic model appeared in the literature: the Bernoulli, the modified logistic, the Gompertz, the Alee, and the Beverton-Holt models.

In [13] we considered the Laguerre-type version of all of them, showing that in all cases the oscillating asymptotic behavior of solutions takes the place of the monotonic one.

Instead, it was found that the model of Volterra-Lotka model is invariant under the action of the isomorphism $\mathcal{T}_x$, since the Laguerre derivative satisfies again the chain rule, according to Equation (42).

8. Laguerre-Type Linear Dynamical Systems

Let $\mathcal{A}$ be a $r \times r$ matrix and denote by u_k, $(k = 1, 2, \ldots, r)$ the invariants of $\mathcal{A}$, i.e., the sum of principal minors (i.e., the elementary symmetric functions of the eigenvalues). The invariants of the matrix $t\mathcal{A}$ are given by $u_k(t) = t^k u_k$, $(k = 1, 2, \ldots, r)$.

Consider the vectors

$$\begin{cases} Z(t) = (Z_1(t), \ldots, Z_r(t))^T \\ Z_0 = (Z_1(0), \ldots, Z_r(0))^T . \end{cases}$$

Then the solution of the linear dynamical system

$$\begin{cases} Z'(t) = \mathcal{A}\, Z(t) \, , \\ Z(0) = Z_0 \end{cases}$$

writes [33]:

$$Z(t) = e^{t\mathcal{A}} Z_0 = \sum_{h=0}^{r-1} \left[\frac{1}{2\pi i} \sum_{j=0}^{r-h-1} (-1)^j u_j(t) \oint_\gamma \frac{e^\lambda \lambda^{r-h-j-1}}{P(\lambda, t)} \, d\lambda \right] \cdot t^h Z_0^h \, ,$$

where $P(\lambda, t)$ is the characteristic polynomial of the matrix $t\mathcal{A}$ and γ denotes a simple Jordan curve encircling all the eigenvalues of $\mathcal{A}$. The choice of γ, without computing the eigenvalues, can be done by using the Gershgorin theorem.

In [15], a Laguerre-type version of the above classic result has been shown.

Consider the above $r \times r$ matrix, and the vectors

$$\begin{cases} Z(t) = (Z_1(t), \ldots, Z_r(t))^T \\ Z_0 = (Z_1(0), \ldots, Z_r(0))^T \\ Z_0' = (Z_1'(0), \ldots, Z_r'(0))^T = \mathcal{A} \cdot Z_0 \\ \vdots \\ Z_0^{r-1} = (Z_1^{r-1}(0), \ldots, Z_r^{r-1}(0))^T = \mathcal{A} \cdot Z_0^{r-2} . \end{cases}$$

The following result holds (see [15] (Theorem 10)):

Theorem 12. *The Laguerre-type Cauchy problem for a homogeneous linear differential system*

$$\begin{cases} D_L Z(t) = Z'(t) + tZ''(t) = \mathcal{A} \cdot Z(t) \\ Z(0) = Z_0 \\ Z_0' = \mathcal{A} \cdot Z_0 , \end{cases}$$

has the solution:

$$Z(t) = e_1(t\mathcal{A})\, Z_0 = \sum_{h=0}^{r-1} \left[\frac{1}{2\pi i} \sum_{j=0}^{r-h-1} (-1)^j u_j(t) \oint_\gamma \frac{e_1(\lambda)\, \lambda^{r-h-j-1}}{P(\lambda, t)}\, d\lambda \right] \cdot t^h Z_0^h ,$$

where $P(\lambda, t)$ and γ have been defined above.

The proof of this result is a straightforward application of the isomorphism $\mathcal{T}_t$. In [15] worked examples are reported.

9. Conclusions

The Laguerre derivative and the relevant Laguerre-type exponentials allow to associate, to any given integer n, a new class of special functions. This fact is obtained by exploiting the properties of an isomorphism, within the space of analytic functions, which acts in such a way as to preserve the differentiation properties. The successive iterations of this isomorphism produce a cyclic construction within the space that repeats the same structure at a higher level of the order of derivation.

Infinite many special functions can be defined in this way. A few of them have been presented explicitly, and the general technique to produce the others has been indicated.

This Survey has shown even possible applications of the Laguerrian derivative in the context of population dynamics and in the solution of Cauchy problems related to particular linear dynamical systems.

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