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Fractional Diffusion–Wave Equation with Application in Electrodynamics

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Abstract: We consider a diffusion–wave equation with fractional derivative with respect to the time variable, defined on infinite interval, and with the starting point at minus infinity. For this equation, we solve an asympotic boundary value problem without initial conditions, construct a representation of its solution, find out sufficient conditions providing solvability and solution uniqueness, and give some applications in fractional electrodynamics.

Keywords: diffusion–wave equation; fundamental solution; fractional derivative on infinite interval; asympotic boundary value problem; problem without initial conditions; Gerasimov–Caputo fractional derivative; Kirchhoff formula; retarded potential

MSC: 35R11; 35Q60

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1. Introduction

Consider the equation

$$\left(\frac{\partial^\alpha}{\partial t^\alpha} - \Delta_x\right) u(x,t) = f(x,t), \tag{1}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes a fractional derivative with respect to t of order $\alpha \in (0,2)$, and

$$\Delta_x = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2},$$

is the Laplace operator with respect to $x = (x_1, x_2, ..., x_n) \in S \subset \mathbb{R}^n$.

If $\alpha = 1$, then Equation (1) coincides with the diffusion equation, and when α tends to 2, this equation turns to the wave equation. Therefore, in the case under consideration ($0 < \alpha < 2$), this equation is usually called the diffusion–wave equation.

In recent decades, fractional diffusion–wave equations are studied very intensively. The first works in this direction include [1–4]. Any close-to-complete analysis of the multitude of works devoted to the diffusion-wave equation would require a separate special study. To give an idea of the variety of problems considered for this type of equations, as well as the multiplicity of approaches to their solution, we mention [5–30]. A brief overview is provided in [29]. A more detailed survey can be found in the article [31] and monographs [32–34].

Interest in the study of this equation is caused by numerous applications fractional calculus in modeling and various fields of natural science. In this regard, we recall the works [35–40].

The overwhelming majority of works devoted to fractional differential equations consider fractional derivatives that are defined on finite intervals. Starting points of these derivatives, at which initial conditions are specified, are finite. Equations with fractional derivatives on infinite intervals, with starting points at plus or minus infinity (usually associated with the names of Liouville, Weyl, or Gerasimov), have been studied much less. A feature of such equations is that problems for them do not require initial conditions. Instead, conditions can be imposed on the asymptotics of the sought solutions at infinity. For parabolic equations, the study of problems without initial conditions began after the publication of [41], and to this day, there is a large list of works in this direction. As for fractional order equations, among works devoted to equations close to (1), we emphasize [42], in which a fundamental solution of an evolution equation with the Liouville fractional derivative was constructed, and a boundary value problem in the right half-plane was solved.

In this work, we consider Equation (1) with the Caputo-type fractional derivative with the starting point at minus infinity. We solve an asympotic boundary value problem for this equation, construct a representation of its solution, find out sufficient conditions providing solvability and solution uniqueness, and give some applications in fractional electrodynamics.

2. Fractional Differentiation

The fractional derivatives of order ζ ($0 < \zeta \leq p,\ p \in \mathbb{N}$) with respect to t, having a starting point at $t = s$ ($-\infty \leq s \leq \infty$), in the Riemann–Liouville and Caputo senses, are defined by ([35] (p. 11), [33] (§2.1))

$$D^{\zeta}_{st} g(t) = \operatorname{sign}^p(t-s) \frac{\partial^p}{\partial t^p} D^{\zeta-p}_{st} g(t) \qquad \text{and} \qquad \partial^{\zeta}_{st} g(t) = \operatorname{sign}^p(t-s) D^{\zeta-p}_{st} \frac{\partial^p}{\partial t^p} g(t),$$

respectively. Here, for $\zeta \leq 0$, D^{ζ}_{st} denotes the Riemann–Liouville fractional integral:

$$D^{\zeta}_{st} g(t) = \operatorname{sign}(t-s) \int_s^t g(\eta) \frac{|t-\eta|^{-\zeta-1}}{\Gamma(-\zeta)}\, d\eta \qquad (\zeta < 0), \qquad \text{and} \qquad D^{0}_{st} g(t) = g(t). \tag{2}$$

In (1), the fractional differentiation is given by the Caputo-type fractional derivative defined on infinite interval with the starting point at minus infinity, i.e.

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} u(x,t) = \partial^{\alpha}_{-\infty t} u(x,t) = \int_{-\infty}^{t} \frac{(t-s)^{m-\alpha-1}}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial s^m} u(x,s)\, ds \qquad (m-1 < \alpha \leq m, \quad m \in \{1,2\}). \tag{3}$$

As was noted in [31], partial differential equations with fractional derivatives of the form (3), apparently for the first time, were studied by A.N. Gerasimov in [43]. Nowadays, they are increasingly called Gerasimov–Caputo derivatives.

3. Domain, Regular Solutions, and Problem

We consider the equation

$$(\partial^{\alpha}_{-\infty t} - \Delta_x)\, u(x,t) = f(x,t), \tag{4}$$

in the domain

$$\Omega_T = \mathbb{R}^n \times (-\infty, T) = \{(x,t) :\ x \in \mathbb{R}^n,\ t \in (-\infty, T)\}.$$

In what follows, m denotes an integer number equal to 1 or 2, chosen so that $m - 1 < \alpha \leq m$.

Definition 1. *We call a function $u(x,t)$ a regular solution of the Equation (4) if: $u(x,t)$ has continuous derivatives with respect to $t \in (-\infty, T)$ up to m-th order for any $x \in \mathbb{R}^n$; $(R-t)^{m-\alpha-1}(\partial^m/\partial t^m)u(x,t)$, as a function of t, is integrable on $(-\infty, R)$ for any $x \in \mathbb{R}^n$ and $R < T$; in Ω_T, $u(x,t)$ has continuous first- and second-order derivatives with respect to x_j ($j = \overline{1,n}$), and satisfies the Equation (4).*

The problem we are going to solve is

Problem 1. *Find a regular solution $u(x,t)$ of the Equation (4) in the domain Ω_T satisfying*

$$\lim_{t\to-\infty} t^k \frac{\partial^k}{\partial t^k} u(x,t) = 0 \qquad (x \in \mathbb{R}^n, \qquad k = \overline{0, m-1}). \tag{5}$$

4. Preliminaries

Consider the function [16]

$$\Gamma_{\alpha,n}(x,s) = C_n s^{\beta(2-n)-1} f_\beta \left(|x| s^{-\beta}; n-1, \beta(2-n) \right) \qquad (x \in \mathbb{R}^n, \quad s > 0). \tag{6}$$

From now on

$$\beta = \frac{\alpha}{2}, \qquad C_n = 2^{-n} \pi^{\frac{1-n}{2}},$$

and

$$f_\beta(z;\mu,\delta) = \begin{cases} \dfrac{2}{\Gamma\left(\frac{\mu}{2}\right)} \displaystyle\int_1^\infty \phi\left(-\beta,\delta;-z\xi\right) \left(\xi^2-1\right)^{\frac{\mu}{2}-1} d\xi, & \mu > 0, \\ \phi\left(-\beta,\delta;-z\right), & \mu = 0, \end{cases}$$

where

$$\phi\left(a,b;z\right) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(ak+b)} \qquad (a > -1)$$

is the Wright function [44,45].

It was proven in [16] that the Function (6) satisfies the inequalities

$$\left| D_{0s}^{\zeta} \Gamma_{\alpha,n}(x,s) \right| \le C s^{\beta(2-n)-\zeta-1} g_p\left(|x| s^{-\beta} \right) E\left(|x| s^{-\beta}, \rho \right), \tag{7}$$

$$\left| \frac{\partial}{\partial x_j} D_{0s}^{\zeta} \Gamma_{\alpha,n}(x,s) \right| \le C |x_j| s^{-\beta n-\zeta-1} g_{p+2}\left(|x| s^{-\beta} \right) E\left(|x| s^{-\beta}, \rho \right), \tag{8}$$

and

$$\left| \frac{\partial^2}{\partial x_j^2} D_{0s}^{\zeta} \Gamma_{\alpha,n}(x,s) \right| \le C s^{-\beta n-\zeta-1} g_q\left(|x| s^{-\beta} \right) E\left(|x| s^{-\beta}, \rho \right), \tag{9}$$

where

$$p = \begin{cases} n, & \text{for} \quad \zeta \in \mathbb{N} \cup \{0\}, \\ n+2, & \text{for} \quad \zeta \notin \mathbb{N} \cup \{0\}, \end{cases} \qquad q = \begin{cases} n+2, & \text{for} \quad \zeta \in \mathbb{N} \cup \{0\} \quad \text{or} \quad n = 1, \\ n+4 & \text{for} \quad \zeta \notin \mathbb{N} \cup \{0\} \quad \text{and} \quad n \ge 2, \end{cases}$$

and

$$E(z,\rho) = \exp\left(-\rho z^{\frac{1}{1-\beta}} \right), \qquad g_n(z) = \begin{cases} 1 & \text{for} \quad n \le 3, \\ |\ln z| + 1 & \text{for} \quad n = 4, \\ z^{4-n} & \text{for} \quad n \ge 5, \end{cases}$$

$C = C(n,\alpha,\rho)$, $\rho < (1-\beta)\beta^{\frac{\beta}{1-\beta}}$, and (by choosing C) ρ can be taken arbitrarily close to $(1-\beta)\beta^{\frac{\beta}{1-\beta}}$.

Here and subsequently, the letter C stands for positive constants, different in different cases and, if necessary, the parameters on which they depend are indicated in brackets: $C = C(a,b,...)$.

Moreover, assuming $s < t$, $|x-y| > 0$, and $\zeta \in \mathbb{R}$, we can assert (see [16] (§5)) that $\Gamma_{\alpha,n}(x-y,t-s)$, as a function of x and t, is a solution of the equation

$$\left(D_{st}^{\alpha} - \Delta_x \right) D_{st}^{\zeta} \Gamma_{\alpha,n}(x-y,t-s) = 0; \tag{10}$$

and a solution of the equation

$$\left(D^{\alpha}_{ts} - \Delta_y\right) D^{\zeta}_{ts}\Gamma_{\alpha,n}(x-y,t-s) = 0,$$

as a function of y and s. In addition, it is known that

$$\int_{\mathbb{R}^n} D^{\zeta}_{st}\Gamma_{\alpha,n}(x-y,t-s)\,dy = \frac{(t-s)^{\alpha-\zeta-1}}{\Gamma(\alpha-\zeta)}. \tag{11}$$

5. Solution Representation

For a function $g(x,t)$, defined on Ω_T, we set

$$\left(\mathcal{T}_\rho g\right)(t) = \sup_{x\in\mathbb{R}^n}\left\{|g(x,t)|\cdot\exp\left(\rho\frac{|x|^{\frac{2}{2-\alpha}}}{(T-t)^{\frac{\alpha}{2-\alpha}}}\right)\right\}.$$

Definition 2. *We say that a function $g(x,t)$, defined on Ω_T, belongs to the class $\mathbf{T}_\alpha$ if*

$$\left(\mathcal{T}_\rho g\right)(t) < \infty$$

for some $\rho < (1-\beta)\beta^{\frac{\beta}{1-\beta}}$, the same for all $t < T$. (Here, as elsewhere, $\beta = \frac{\alpha}{2}$.)

Theorem 1. *Let $\alpha \in (0,2)$, $m \in \{1,2\}$, $\alpha \in (m-1,m]$, $f(x,t)$ be locally integrable on Ω_T,*

$$f(x,t) \in \mathbf{T}_\alpha, \qquad \left(\mathcal{T}_\rho f\right)(t) \in L(-\infty, T-\varepsilon) \qquad \textit{for}\quad \textit{any} \qquad \varepsilon > 0,$$

and

$$\frac{\partial^k}{\partial t^k}u(x,t) \in \mathbf{T}_\alpha \qquad \textit{and} \qquad \lim_{t\to-\infty} t^k\left(\mathcal{T}_\rho\frac{\partial^k}{\partial t^k}u\right)(t) = 0 \qquad (k = \overline{0,m-1}, \quad t < T) \tag{12}$$

If $u(x,t)$ is a regular solution of the problem (4) *and* (5)*, then*

$$u(x,t) = \int_{-\infty}^{t}\int_{\mathbb{R}^n} f(\xi,\eta)\,\Gamma_{\alpha,n}(x-\xi,t-\eta)\,d\xi d\eta, \qquad (x,t) \in \Omega_T. \tag{13}$$

Proof. Consider the function

$$v(x,t;\xi,\eta) \equiv \Gamma_{\alpha,n}(x-\xi,t-\eta)h_\varepsilon(|x-\xi|)h^r(|x-\xi|) \qquad (\varepsilon > 0, \quad r > 1),$$

where

$$h_\varepsilon(z) = \begin{cases} 30\varepsilon^{-5}\int_0^z s^2(\varepsilon-s)^2 ds & \text{if} \quad z \in [0,\varepsilon], \\ 1 & \text{else}, \end{cases}$$

and

$$h^r(z) = \begin{cases} 1 & \text{if} \quad z < r-1, \\ 30\int_z^r (s-r+1)^2(r-s)^2 ds & \text{if} \quad t \in [r-1,r], \\ 0 & \text{else}. \end{cases}$$

It is easy to check that

$$h_\varepsilon(z),\ h^r(z) \in C^2[0,\infty); \quad 0 \le h_\varepsilon(z),\ h^r(z) \le 1; \tag{14}$$

$$h'_\varepsilon(z) = h''_\varepsilon(z) = 0 \quad \text{for} \quad z \ge \varepsilon; \quad \text{and} \quad {h^r}'(z) = {h^r}''(z) = 0 \quad \text{if} \quad z \notin (r-1,r). \tag{15}$$

In what follows, we use the notations

$$\mathbf{L}_{\xi,\eta} = \left(\partial^\alpha_{-\infty\eta} - \Delta_\xi\right), \qquad \mathbf{L}^R_{\xi,\eta} = \left(\partial^\alpha_{R\eta} - \Delta_\xi\right), \qquad \text{and} \qquad \mathbf{L}^*_{\xi,\eta} = \left(D^\alpha_{t\eta} - \Delta_\xi\right); \tag{16}$$

and B_x^r denotes an open ball in $\mathbb{R}^n$ with center at point x and radius r,

$$B_x^r = \{\xi \in \mathbb{R}^n : |x-\xi| < r\}.$$

By the notation (16), we can write

$$\mathbf{L}_{\xi,\eta} u(\xi,\eta) = \left[\mathbf{L}^R_{\xi,\eta} + J^R\right] u(\xi,\eta), \tag{17}$$

where

$$J^R u(\xi,\eta) = \frac{1}{\Gamma(m-\alpha)} \int_{-\infty}^{R} (\eta - s)^{m-\alpha-1} \frac{\partial^m}{\partial s^m} u(\xi,s)\, ds \qquad (\eta > R).$$

For $r > 0$ and $R < 0$, both sufficiently large in absolute value, the formula of fractional integration by parts (see, for example, [33] (p. 76)), (7), and (17) give

$$\int_R^t \int_{B_x^r} v(x,t,\xi,\eta) \left[f(\xi,\eta) - J^R u(\xi,\eta)\right] d\xi d\eta = \int_R^t \int_{B_x^r} v(x,t,\xi,\eta) \mathbf{L}^R_{\xi,\eta} u(\xi,\eta)\, d\xi d\eta =$$

$$= \int_R^t \int_{B_x^r} u(\xi,\eta) \mathbf{L}^*_{\xi,\eta} v(x,t,\xi,\eta) d\xi d\eta - \sum_{k=0}^{m-1} \int_{B_x^r} \left[\frac{\partial^k}{\partial \eta^k} u(\xi,\eta) \cdot D_{t\eta}^{\alpha-k-1} v(x,t,\xi,\eta)\right]_{\eta=R} d\xi. \tag{18}$$

By (14) and (15), we obtain

$$\int_R^t \int_{B_x^r} u(\xi,\eta) \mathbf{L}^*_{\xi,\eta} v(x,t,\xi,\eta) d\xi d\eta =$$

$$= \int_R^t \int_{r-1<|x-\xi|<r} u(\xi,\eta) B(x-\xi, t-\eta)\, d\xi d\eta - \int_R^t \int_{|x-\xi|<\varepsilon} u(\xi,\eta) A(x-\xi,t-\eta)\, d\xi d\eta =$$

$$= \int_0^{t-R} \int_{|\xi|<\varepsilon} [u(x,t) - u(x+\xi, t-\eta)] A(\xi,\eta)\, d\xi d\eta - u(x,t) \int_0^{t-R} \int_{|\xi|<\varepsilon} A(\xi,\eta)\, d\xi d\eta +$$

$$+ \int_R^t \int_{r-1<|x-\xi|<r} u(\xi,\eta) B(x-\xi,t-\eta)\, d\xi d\eta, \tag{19}$$

where

$$A(\xi,\eta) = \sum_{j=1}^n \left(2 \frac{\partial}{\partial \xi_j} \Gamma_{\alpha,n}(\xi,\eta) \frac{\partial}{\partial \xi_j} h_\varepsilon(|\xi|) + \Gamma_{\alpha,n}(\xi,\eta) \frac{\partial^2}{\partial \xi_j^2} h_\varepsilon(|\xi|)\right), \tag{20}$$

$$B(\xi,\eta) = -\sum_{j=1}^n \left(2 \frac{\partial}{\partial \xi_j} \Gamma_{\alpha,n}(\xi,\eta) \frac{\partial}{\partial \xi_j} h^r(|\xi|) + \Gamma_{\alpha,n}(\xi,\eta) \frac{\partial^2}{\partial \xi_j^2} h^r(|\xi|)\right).$$

The estimates (7) and (8), and the condition (12) yields

$$\lim_{r\to\infty} \int_R^t \int_{r-1<|x-\xi|<r} u(\xi,\eta) B(x-\xi,t-\eta)\, d\xi d\eta = 0, \tag{21}$$

$$\lim_{\varepsilon\to 0} \int_\delta^{t-R} \int_{|\xi|<\varepsilon} [u(x+\xi,t-\eta) - u(x,t)] A(\xi,\eta)\, d\xi d\eta = 0,$$

and

$$\int_0^\delta \int_{|\xi|<\varepsilon} |A(\xi,\eta)|\, d\xi d\eta < \infty,$$

where δ is a sufficiently small positive number. Therefore

$$\left|\int_0^{t-R} \int_{|\xi|<\varepsilon} [u(x+\xi,t-\eta) - u(x,t)] A(\xi,\eta)\, d\xi d\eta\right| \le$$

$$\leq C \sup_{|\xi|<\varepsilon, \eta\in(0,\delta)} |u(x+\xi, t-\eta) - u(x,t)| + O(\varepsilon).$$

The continuity $u(x,t)$ in a neighborhood of (x,t) and an arbitrary choice of δ imply that

$$\lim_{\varepsilon\to 0} \int_0^{t-R} \int_{|\xi|<\varepsilon} [u(x+\xi, t-\eta) - u(x,t)] A(\xi,\eta)\, d\xi d\eta = 0. \tag{22}$$

Thus, (19), (21), and (22) give

$$\lim_{\varepsilon\to 0} \lim_{r\to\infty} \int_R^t \int_{B_x^r} u(\xi,\eta) \mathbf{L}^*_{\xi,\eta} v(x,t,\xi,\eta) d\xi d\eta = -u(x,t) \lim_{\varepsilon\to 0} J(\varepsilon) \qquad (x\in\mathbb{R}^n, \quad R<t<T), \tag{23}$$

where

$$J(\varepsilon) = \int_0^{t-R} \int_{|\xi|<\varepsilon} A(\xi,\eta)\, d\xi d\eta.$$

Let us compute $\lim_{\varepsilon\to 0} J(\varepsilon)$. For short, we take the notation

$$g_n(|\xi|) = \int_0^{t-R} \Gamma_{\alpha,n}(\xi,\eta)\, d\eta.$$

(Note that $\Gamma_{\alpha,n}(\xi,\eta)$ is a function of $|\xi|$ and η.) The formulas

$$\frac{\partial}{\partial\xi_j} h_\varepsilon(|\xi|) = \frac{\xi_j}{|\xi|} h'_\varepsilon(|\xi|), \qquad \Delta_\xi h_\varepsilon(|\xi|) = h''_\varepsilon(|\xi|) + \frac{n-1}{|\xi|} h'_\varepsilon(|\xi|),$$

and (see [16] (§5))

$$\frac{\partial}{\partial\xi_j} \Gamma_{\alpha,n}(\xi,\eta) = -2\pi\xi_j \Gamma_{\alpha,n+2}(\xi,\eta)$$

allow us rewrite $J(\varepsilon)$ as

$$J(\varepsilon) = \int_{|\xi|<\varepsilon} \left\{ \left[h''_\varepsilon(|\xi|) + \frac{n-1}{|\xi|} h'_\varepsilon(|\xi|) \right] g_n(|\xi|) - 4\pi|\xi|\, h'_\varepsilon(|\xi|)\, g_{n+2}(|\xi|) \right\} d\xi.$$

It is easy to see that

$$h'_\varepsilon(\varepsilon|\omega|) = \varepsilon^{-1} h'_1(|\omega|) \qquad \text{and} \qquad h''_\varepsilon(\varepsilon|\omega|) = \varepsilon^{-2} h''_1(|\omega|).$$

After a change of variable $\xi = \varepsilon\omega$, we get

$$J(\varepsilon) = \varepsilon^n \int_{|\omega|<1} \left\{ \frac{1}{\varepsilon^2} \left[h''_1(|\omega|) + \frac{n-1}{|\omega|} h'_1(|\omega|) \right] g_n(\varepsilon|\omega|) - 4\pi|\omega|\, h'_1(|\omega|)\, g_{n+2}(\varepsilon|\omega|) \right\} d\omega.$$

The formula

$$\int_{|\omega|<1} f(|\omega|)\, d\omega = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^1 \sigma^{n-1} f(\sigma)\, d\sigma$$

yields

$$J(\varepsilon) = \frac{2\varepsilon^n \pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^1 \sigma^{n-1} \left\{ \frac{1}{\varepsilon^2} \left[h''_1(\sigma) + \frac{n-1}{\sigma} h'_1(\sigma) \right] g_n(\varepsilon\sigma) - 4\pi\sigma\, h'_1(\sigma) g_{n+2}(\varepsilon\sigma) \right\} d\sigma =$$

$$= \frac{2\varepsilon^n \pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^1 \left\{ \frac{1}{\varepsilon^2} \left[\sigma^{n-1} h'_1(\sigma) \right]' g_n(\varepsilon\sigma) - 4\pi\sigma^n\, h'_1(\sigma) g_{n+2}(\varepsilon\sigma) \right\} d\sigma.$$

Integrating by parts gives

$$\int_0^1 \left[\sigma^{n-1} h_1'(\sigma)\right]' g_n(\varepsilon\sigma)\, d\sigma = \left[\sigma^{n-1} h_1'(\sigma) g_n(\varepsilon\sigma)\right]_0^1 - \varepsilon \int_0^1 \sigma^{n-1} h_1'(\sigma) g_n'(\varepsilon\sigma)\, d\sigma.$$

Combining this with equality

$$g_n'(\varepsilon\sigma) = -2\pi\varepsilon^2\sigma\, g_{n+2}(\varepsilon\sigma),$$

we get

$$J(\varepsilon) = -\frac{4\varepsilon^n \pi^{1+\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^1 \sigma^n\, h_1'(\sigma) g_{n+2}(\varepsilon\sigma)\, d\sigma.$$

By

$$\lim_{z\to 0} z^n g_{n+2}(z) = \frac{2\, C_{n+2}\Gamma(n)}{\Gamma\left(\frac{n+1}{2}\right)},$$

we obtain

$$\lim_{\varepsilon\to 0} J(\varepsilon) = -\frac{2^{1-n}\Gamma(n)\sqrt{\pi}}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} = -1.$$

Combining this with (18) and (23) leads to

$$u(x,t) = \lim_{\varepsilon\to 0}\lim_{r\to\infty} \int_R^t \int_{B_x^r} v(x,t,\xi,\eta) \left[f(\xi,\eta) - J^R u(\xi,\eta)\right] d\xi d\eta +$$

$$+\lim_{\varepsilon\to 0}\lim_{r\to\infty} \sum_{k=0}^{m-1} \int_{B_x^r} \left[\frac{\partial^k}{\partial\eta^k} u(\xi,\eta)\cdot D_{t\eta}^{\alpha-k-1} v(x,t,\xi,\eta)\right]_{\eta=R} d\xi \qquad (x\in\mathbb{R}^n, \quad R<t<T).$$

We can rewrite $J^R u(\xi,\eta)$ in the form

$$J^R u(\xi,\eta) = \frac{(\eta-R)^{m-\alpha-1}}{\Gamma(m-\alpha)} \left[\frac{\partial^{m-1}}{\partial s^{m-1}} u(\xi,s)\right]_{s=R} + \int_{-\infty}^R \frac{(\eta-s)^{m-\alpha-2}}{\Gamma(m-\alpha-1)} \frac{\partial^{m-1}}{\partial s^{m-1}} u(\xi,s)\, ds.$$

By (12), we have

$$\left|\frac{\partial^{m-1}}{\partial s^{m-1}} u(\xi,s)\right| \le C \exp\left(\rho \frac{|\xi|^{\frac{2}{2-\alpha}}}{(T-R)^{\frac{\alpha}{2-\alpha}}}\right) \cdot \sup_{s<R} \left(\mathcal{T}_\rho \frac{\partial^{m-1}}{\partial s^{m-1}} u\right)(s) \qquad (s<R)$$

and consequently

$$\left|J^R u(\xi,\eta)\right| \le C(\eta-R)^{m-\alpha-1} \exp\left(\rho \frac{|\xi|^{\frac{2}{2-\alpha}}}{(T-R)^{\frac{\alpha}{2-\alpha}}}\right) \cdot \sup_{s<R} \left(\mathcal{T}_\rho \frac{\partial^{m-1}}{\partial s^{m-1}} u\right)(s). \tag{24}$$

This implies that

$$u(x,t) = \int_R^t \int_{\mathbb{R}^n} \Gamma_{\alpha,n}(x-\xi,t-\eta) \left[f(\xi,\eta) - J^R u(\xi,\eta)\right] d\xi d\eta +$$

$$+\sum_{k=0}^{m-1} \int_{\mathbb{R}^n} \left[\frac{\partial^k}{\partial\eta^k} u(\xi,\eta)\cdot D_{t\eta}^{\alpha-k-1} \Gamma_{\alpha,n}(x-\xi,t-\eta)\right]_{\eta=R} d\xi \qquad (x\in\mathbb{R}^n, \quad R<t<T).$$

The proof is completed by showing that

$$\lim_{R\to-\infty} \int_R^t \int_{\mathbb{R}^n} \Gamma_{\alpha,n}(x-\xi,t-\eta)\cdot J^R u(\xi,\eta)\, d\xi d\eta = 0 \tag{25}$$

and

$$\lim_{R\to-\infty}\int_{\mathbb{R}^n}\left[\frac{\partial^k}{\partial\eta^k}u(\xi,\eta)\cdot D_{t\eta}^{\alpha-k-1}\Gamma_{\alpha,n}(x-\xi,t-\eta)\right]_{\eta=R} d\xi = 0 \qquad (k=\overline{0,m-1}). \tag{26}$$

By (7) and (24) we get

$$\int_R^t\int_{\mathbb{R}^n}\left|\Gamma_{\alpha,n}(x-\xi,t-\eta)\cdot J^R u(\xi,\eta)\right| d\xi d\eta \le C(t-R)^{m-1}\sup_{s<R}\left(\mathcal{T}_\rho\,\frac{\partial^{m-1}}{\partial s^{m-1}}u\right)(s)$$

and

$$\int_{\mathbb{R}^n}\left|\frac{\partial^k}{\partial\eta^k}u(\xi,\eta)\cdot D_{t\eta}^{\alpha-k-1}\Gamma_{\alpha,n}(x-\xi,t-\eta)\right|_{\eta=R} d\xi \le C(t-R)^k\left(\mathcal{T}_\rho\,\frac{\partial^k}{\partial t^k}u\right)(R) \qquad (k=\overline{0,m-1}).$$

These two inequalities and (12) prove (25) and (26). □

Remark 1. *It should be noted that the conditions* (12) *combine* (5) *and the condition that restricts the growth of a sought solution as* $|x|\to\infty$*, which is analogous of Tychonoff's condition [41]. Thus, a function* $u(x,t)$ *satisfying* (12) *certainly satisfies* (5)*, but the converse is not true.*

6. Solution Uniqueness

Theorem 1 allows us to prove the uniqueness of the solution to the problem under consideration.

Theorem 2. *Let* $\alpha\in(0,2)$*. There is at most one regular solution of the problem* (4) *and* (5) *in the class of functions that satisfy* (12)*.*

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of the Equation (4) corresponding to the same $f(x,t)$, and satisfy (12) (as well as (5) consequently). Then, one can conclude that the function $v(x,t)=u_1(x,t)-u_2(x,t)$ satisfies (12) and the homogeneous equation

$$(\partial^\alpha_{-\infty t}-\Delta_x)\,v(x,t)=0.$$

By Theorem 1, this means that $v(x,t)\equiv 0$, i.e. $u_1(x,t)\equiv u_2(x,t)$. □

7. Existence Theorem

It is worth noting that Theorem 1 does not state that any function of the form (13) is an a priori solution to Problem 1. Here, we find out conditions for the right-hand side $f(x,t)$, ensuring that (13) is a solution to (4) and (5), and thereby proves the existence of the solution.

Theorem 3. *Let* $\alpha\in(0,2)$*,* $m\in\{1,2\}$*,* $\alpha\in(m-1,m]$*,* $f(x,t)$ *be presentable in the form*

$$f(x,t)=D^{-\delta}_{-\infty t}g(x,t) \qquad (\delta>m-\alpha), \tag{27}$$

where

$$g(x,t)\in\mathbf{T}_\alpha\cap C(\Omega_T), \qquad (\mathcal{T}_\rho g)\,(t)\le C(T-t)^{-\nu} \qquad (\nu>\delta+\alpha), \tag{28}$$

and $f(x,t)$ *be a locally Hölder continuous in* $x\in\mathbb{R}^n$ *for any fixed* $t<T$*, namely,* $f(x,t)$ *satisfy*

$$|f(x,t)-f(\xi,t)|\le C(T-t)^{\delta-\nu}|x-\xi|^\mu \qquad (\mu>0). \tag{29}$$

Then a function $u(x,t)$ *defined by* (13) *is a regular solution to the problem* (4) *and* (5)*.*

Proof. The formula of fractional integration by parts (see, e.g., [33] (p. 76)), (13) and (27) give

$$u(x,t) = \int_{-\infty}^{t} \int_{\mathbb{R}^n} g(\xi,\eta)\, D_{t\eta}^{-\delta} \Gamma_{\alpha,n}(x-\xi,t-\eta)\, d\xi d\eta.$$

By (7) and (28), we have

$$\int_{\mathbb{R}^n} \left| g(\xi,\eta)\, (\partial^k/\partial t^k) D_{t\eta}^{-\delta}\Gamma_{\alpha,n}(x-\xi,t-\eta) \right| d\xi \le C(T-\eta)^{-\nu}(t-\eta)^{\alpha+\delta-k-1} \qquad (k=\overline{0,m}).$$

Hence

$$\frac{\partial^k}{\partial t^k} u(x,t) = \int_{-\infty}^{t} \int_{\mathbb{R}^n} g(\xi,\eta)\, D_{t\eta}^{k-\delta} \Gamma_{\alpha,n}(x-\xi,t-\eta)\, d\xi\, d\eta, \tag{30}$$

$$(\partial^k/\partial t^k) u(x,t) \in C(\Omega_T) \qquad \text{and} \qquad \left| (\partial^k/\partial t^k) u(x,t) \right| \le C(T-t)^{\alpha+\delta-\nu-k} \qquad (k=\overline{0,m}).$$

In particular, this proves that $u(x,t)$ satisfies (5), and $(R-t)^{m-\alpha-1}(\partial^m/\partial t^m)u(x,t)$ is integrable on $(-\infty,R)$ as a function of t, $R<T$.

Thus, it remains to be proven that $u(x,t)$, given by (13), satisfies (4). Using (11) and (30), we can write

$$\partial_{-\infty t}^{\alpha} u(x,t) = D_{-\infty t}^{\alpha-m} \frac{\partial^m}{\partial t^m} u(x,t) = \int_{-\infty}^{t} \int_{\mathbb{R}^n} g(\xi,\eta)\, D_{t\eta}^{\alpha-\delta} \Gamma_{\alpha,n}(x-\xi,t-\eta)\, d\xi\, d\eta =$$

$$= \int_{-\infty}^{t} \int_{\mathbb{R}^n} [g(\xi,\eta)-g(x,\eta)]\, D_{t\eta}^{\alpha-\delta} \Gamma_{\alpha,n}(x-\xi,t-\eta)\, d\xi\, d\eta + \int_{-\infty}^{t} g(x,\eta) \frac{(t-\eta)^{\delta-1}}{\Gamma(\delta)}\, d\eta.$$

Combining this with (2), (7), (27), and (29), we obtain

$$\partial_{-\infty t}^{\alpha} u(x,t) = \int_{-\infty}^{t} \int_{\mathbb{R}^n} [f(\xi,\eta)-f(x,\eta)]\, D_{t\eta}^{\alpha} \Gamma_{\alpha,n}(x-\xi,t-\eta)\, d\xi\, d\eta + f(x,t). \tag{31}$$

Now, let us consider the function

$$u_\varepsilon(x,t) = \int_{-\infty}^{t-\varepsilon} \int_{\mathbb{R}^n} f(\xi,\eta)\, \Gamma_{\alpha,n}(x-\xi,t-\eta)\, d\xi\, d\eta \qquad (\varepsilon>0).$$

By (9) and (29), we have

$$\Delta_x u_\varepsilon(x,t) = \int_{-\infty}^{t-\varepsilon} \int_{\mathbb{R}^n} f(\xi,\eta)\, \Delta_x \Gamma_{\alpha,n}(x-\xi,t-\eta)\, d\xi\, d\eta =$$

$$= \int_{-\infty}^{t-\varepsilon} \int_{\mathbb{R}^n} [f(\xi,\eta)-f(x,\eta)]\, \Delta_x \Gamma_{\alpha,n}(x-\xi,t-\eta)\, d\xi\, d\eta + \int_{-\infty}^{t-\varepsilon} f(x,\eta) \int_{\mathbb{R}^n} \Delta_x \Gamma_{\alpha,n}(x-\xi,t-\eta)\, d\xi\, d\eta.$$

It follows from (10) and (11) that

$$\int_{\mathbb{R}^n} \Delta_x \Gamma_{\alpha,n}(x-\xi,t-\eta)\, d\xi = \int_{\mathbb{R}^n} D_{\eta t}^{\alpha} \Gamma_{\alpha,n}(x-\xi,t-\eta)\, d\xi = 0.$$

Inequlities (8) and (29) also yield

$$\int_{\mathbb{R}^n} \left| [f(\xi,\eta)-f(x,\eta)]\, (\partial^2/\partial x_j^2) \Gamma_{\alpha,n}(x-\xi,t-\eta) \right| d\xi \le C(T-\eta)^{\delta-\nu}(t-\eta)^{\beta\mu-1}.$$

This allows us to conclude that

$$\Delta_x u(x,t) = \lim_{\varepsilon\to 0} \Delta_x u_\varepsilon(x,t) = \int_{-\infty}^{t} \int_{\mathbb{R}^n} [f(\xi,\eta)-f(x,\eta)]\, \Delta_x \Gamma_{\alpha,n}(x-\xi,t-\eta)\, d\xi\, d\eta.$$

This and (31) prove that (13) satisfies (4). □

Remark 2. *It is easy to see that if $f(x,t) \equiv 0$ for $t < a$ ($a < T$), then $u(x,t)$, defined by (13), is also equal to 0 for $t < a$. In this case, $u(x,t)$ is a solution of the equation*

$$(\partial^{\alpha}_{at} - \Delta_x)\, u(x,t) = f(x,t)$$

in the layer $\mathbb{R}^n \times (a,T)$, and satisfies the zero initial conditions $(\partial^k/\partial t^k)u(x,a) = 0$ ($k = \overline{0, m-1}$).

8. Application in Electrodynamics

It is known that solutions of wave equations encountered in classical electrodynamics are usually expressed in terms of retarded potentials (see, e.g., [46]). For diffusion-wave equation with fractional derivative defined on a finite interval, an analogue of retarded potential was constructed in [47]. Here, we give an approach based on an equation of the form (1).

Consider the Equation

$$\left(\partial^{\alpha}_{-\infty t} - v^2 \Delta_{\mathbf{r}}\right) u(\mathbf{r},t) = f(\mathbf{r},t), \tag{32}$$

where $\mathbf{r}$ is the position vector, $\mathbf{r} \in \mathbb{R}^3$, t denotes the dimensionless time, and v is a constant with the dimension of length. By $u(\mathbf{r},t)$, we mean a scalar or vector potential, and $f(\mathbf{r},t)$ is given by the volumetric charge or current density.

The Formula (13) and an easy computation give the solution of (32), which has the form

$$u(\mathbf{r},t) = \frac{1}{v^2} \int_{-\infty}^{t} \int_{\mathbb{R}^3} f(\mathbf{r}',t')\, \Gamma_{\alpha,3}\left(\frac{\mathbf{r}-\mathbf{r}'}{v}, t-t'\right) d\mathbf{r}' dt'.$$

One can check that

$$\Gamma_{\alpha,3}\,(\mathbf{r},t) = \frac{1}{4\pi|\mathbf{r}|\, t}\phi\left(-\frac{\alpha}{2}, 0; -|\mathbf{r}|t^{-\frac{\alpha}{2}}\right).$$

This gives

$$u(\mathbf{r},t) = \frac{1}{4\pi v^2} \int_{\mathbb{R}^3} F_\alpha(\mathbf{r},\mathbf{r}',t)\, \frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}, \tag{33}$$

where

$$F_\alpha(\mathbf{r},\mathbf{r}',t) = \int_{-\infty}^{t} \frac{f(\mathbf{r}',t')}{t-t'} \phi\left(-\frac{\alpha}{2}, 0; -\frac{1}{v}|\mathbf{r}-\mathbf{r}'|(t-t')^{-\frac{\alpha}{2}}\right) dt' =$$

$$= \int_0^{\infty} f(\mathbf{r}', t-s)\, \frac{1}{s}\phi\left(-\frac{\alpha}{2}, 0; -\frac{1}{v}|\mathbf{r}-\mathbf{r}'|s^{-\frac{\alpha}{2}}\right) ds$$

gives the distributed (non-local, blurred in time) delay.

The relation (33) is an analogue of the Kirchhoff formula for retarded potentials. It follows from the properties of the Wright function (see [16] (Lemma 27)) that

$$\lim_{\alpha\to 2} F_\alpha(\mathbf{r},\mathbf{r}',t) = f\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{v}\right),$$

and, consequently,

$$\lim_{\alpha\to 2} u(\mathbf{r},t) = \frac{1}{4\pi v^2} \int_{\mathbb{R}^3} f\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{v}\right) \frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}. \tag{34}$$

This means that the potential (33) takes the form of the classical retarded potential (see, e.g., [46] (§. 62)).

The Formula (33) gives a general form for retarded potentials in fractional electrodynamics based on the Equation (32). It should be noted that in the stationary case (when charge or current density does not depend on time), the potentials (33) and (34) coincide up to the factor

$$\int_0^\infty \frac{1}{s}\phi\left(-\frac{\alpha}{2}, 0; -\frac{1}{v}|\mathbf{r}-\mathbf{r}'|s^{-\frac{\alpha}{2}}\right) ds = \frac{1}{\Gamma(\alpha/2)}.$$

According to Remark 2, the Formula (33) is completely consistent with the results of [47]. Thus, we can conclude that the use of fractional time derivatives is equivalent to a special time averaging of the charge density or current, which allows us to take into account the influence of the external environment.

9. Conclusions

In this paper, we construct a representation of solutions to an asympotic boundary value problem for a diffusion-wave equation with fractional derivative with respect to the time variable. For fractional differentiation, we use the Gerasimov–Caputo type fractional derivative, which is defined on an infinite interval and has the starting point at minus infinity. The problems do not require initial conditions. Instead, conditions are imposed on the asymptotics of the sought solutions at minus infinity. We prove the uniqueness theorem and find out sufficient conditions ensuring the existence of solutions, including smoothness properties and asymptotic behavior of the right-hand side function. It is shown that for the uniqueness of the solution, additional conditions are required for the growth of the desired solution at infinity. As applications, we discuss some questions of fractional electrodynamics.

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