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Valuation of Exchange Option with Credit Risk in a Hybrid Model

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Abstract: In this paper, the valuation of the exchange option with credit risk under a hybrid credit risk model is investigated. In order to build the hybrid model, we consider both the reduced-form model and the structural model. We adopt the probabilistic approach to derive the closed-form formula of an exchange option price with credit risk under the proposed model. Specifically, the change of measure technique is used repeatedly, and the pricing formula is provided as the standard normal cumulative distribution functions.

Keywords: exchange option; credit risk; hybrid model; option pricing

1. Introduction

Under the Black–Scholes model [1], Margrabe [2] first derived the closed-form pricing formula of the European exchange option which provides the option holder the right to exchange one risky asset for another. Since its introduction by Margrabe, the option has become one of the most popular exotic options in the over-the-counter (OTC) market. Thus, many researchers have studied the valuation of an exchange option with various extensions of the Black–Scholes model. Geman, Karoui, and Rochet [3] adopted the change of numeraire approach to derive the pricing formula of an exchange option. Antonelli, Ramponi, and Scarlatti [4] provided the price of the exchange option under the stochastic volatility model using a correlation expansion. Kim and Park [5] used a stochastic volatility model with fast mean reversion for a more precise price of the exchange option. In this paper, the exchange option pricing model is extended with the credit risk.

Options with credit risk have been called "vulnerable options" in general. For valuing the vulnerable options, most of the researchers have used one model between the reduced-form model and the structural model. The default of the firm in the reduced-form model is triggered by the counting process with some intensity. Recently, there have been various studies for vulnerable options in the reduced-form model such as the generalized jump model [6], catastrophe put option [7], and Generalized Autoregressive Conditionally Heteroscedastic (GARCH) model [8]. Because default is determined by the jump of the counting process, there is no relation between default and the firm value of the option issuer. In contrast to the reduced-form model, the structural model considers the dependence of them. Under the structural model, Johnson and Stulz [9] first provided the pricing formula of the vulnerable European option. Klein [10] extended the result of [9] by considering the correlation between the firm value process of the option issuer and the underlying asset process of the option. Based on the work of [10], there have been many extended results on vulnerable options with models such as the stochastic interest rate model [11], the early counterparty risk model [12,13], the stochastic volatility model [14–17], and the jump-diffusion model [18–21]. Furthermore, many researchers have studied vulnerable exotic options such as the American option [22], the Asian option [23], the exchange option [24], and the path-dependent option [25] under the structural model. Recently, several researchers proposed the hybrid credit risk models, incorporating the structural model and reduced-form model and provided the pricing formula of vulnerable European option [26,27]. In this paper, we deal with the option valuation based on a hybrid credit risk model. Specifically, we derive the closed-form solution for vulnerable exchange option pricing under a hybrid credit risk model.

In recent years, there have been studies on the pricing of exchange options with credit risk which is called the vulnerable exchange option. Kim and Koo [24] provided the pricing formula of an exchange option with credit risk based on the Mellin transform approach. Kim [28] used a probabilistic approach to obtain the closed-form solution of the vulnerable exchange option price under the structure model of Klein [10]. In addition, some researchers considered the generalization of the vulnerable exchange option as a power exchange option. They have developed the approaches to price the vulnerable power exchange options under the extensions of the Klein's credit risk model, such as the jump risk [29,30], the possible default prior to the maturity [31], jumps under the double risk [32] and the intensity based approach [33]. In this sense, we also develop the pricing of exchange option with credit risk. Our main contribution is to propose a hybrid credit risk model and to provide the closed-form pricing formula of vulnerable exchange option under the proposed model based on the probabilistic approach.

The rest of this paper is organized as follows. In Section 2, the hybrid credit model used in this paper and formulation for the exchange option with credit risk is is presented In Section 3, a closed-form formula for valuing of an exchange option with credit risk under the hybrid credit risk model is presented. In Section 4, concluding remarks are presented.

2. The Model

We assume that there is a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions, where P is a risk-neutral probability measure. Then, under the measure P, the dynamics of two risky underlying assets $S_1(t)$ and $S_2(t)$ are given by

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dW_1(t), \tag{1}$$

$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dW_2(t), \tag{2}$$

where σ_i, $i = 1, 2$ are the volatilities of asset i, r is a constant interest rate, $W_1(t)$ and $W_2(t)$ are the standard Brownian motions under the measure P and ρ_{12} is a correlation between asset $S_1(t)$ and asset $S_2(t)$ satisfying $dW_1(t)dW_2(t) = \rho_{12}dt$. We adopt the model of Klein to describe the credit risk under the structure model. Then, the asset value process $V(t)$ of option issuer is governed by the geometric Brownian motions (GBM) as

$$dV(t) = rV(t)dt + \sigma_3 V(t)dW_3(t), \tag{3}$$

where σ_3 is the volatility of asset $V(t)$, $W_3(t)$ is a standard Brownian motion satisfying $dW_1(t)dW_3(t) = \rho_{13}dt$ and $dW_2(t)dW_3(t) = \rho_{23}dt$. As mentioned in [10], we assume that if a default of the option issuer happens, the option issuer's asset is liquidated and provides only the scrap value at the maturity T. The scrap value is defined by

$$(1-\alpha)\frac{V(T)}{D}(S_1(T) - S_2(T))^+,$$

where α is a deadweight cost related with the default, T is a time to maturity and D is a critical value (or a value of the option issuer's liability) that a default occurs if the value of the option issuer asset is lower than D.

We now define the default intensity process for the reduced-form model. As in Fard [6], under the risk-neutral measure P, we assume that the default intensity process is given by

$$d\lambda(t) = a(b - \lambda(t))dt + \sigma_4 dW_4(t),$$

where σ_4 is a positive constant and $W_4(t)$ is a standard Brownian motion satisfying $dW_1(t)dW_4(t) = \rho_{14}dt$, $dW_2(t)dW_4(t) = \rho_{24}dt$ and $dW_3(t)dW_4(t) = \rho_{34}dt$. We also define the filtration $\mathcal{F}_t$ generated by $\mathcal{F}_t = \mathcal{F}_t^{S_1} \vee \mathcal{F}_t^{S_2} \vee \mathcal{F}_t^{\lambda} \vee \mathcal{H}_t$, where $\mathcal{F}_t^{S_1} = \sigma(S_1(t), s \leq t)$,$\mathcal{F}_t^{S_2} = \sigma(S_2(t), s \leq t)$, $\mathcal{F}_t^{\lambda} = \sigma(\lambda(t), s \leq t)$ and $\mathcal{H}_t = \sigma(\mathbf{1}_{\{\tau \leq t\}}, s \leq t)$, where τ is the default time defined by $\mathrm{P}(\tau > t) = \mathrm{E}\left[e^{-\int_0^t \lambda(s)ds}\right]$. Then, with the underlying assets $S_1(t)$, $S_2(t)$ and $V(t)$, the price of the exchange option C with credit risk at time 0 under the measure P is given by

$$\begin{aligned} C &= e^{-rT}E^P\left[(S_1(T) - S_2(T))^+ \left(\mathbf{1}_{\{\tau > T, V(T) > D\}} + \frac{(1-\alpha)V(T)}{D}(1 - \mathbf{1}_{\{\tau > T, V(T) > D\}})\right) |\mathcal{F}_0\right] \\ &= e^{-rT}E^P\left[(S_1(T) - S_2(T))^+ \mathbf{1}_{\{\tau > T, V(T) > D\}} |\mathcal{F}_0\right] \\ &\quad + \frac{(1-\alpha)}{D} e^{-rT}E^P\left[V(T)(S_1(T) - S_2(T))^+ |\mathcal{F}_0\right] \\ &\quad - \frac{(1-\alpha)}{D} e^{-rT}E^P\left[V(T)(S_1(T) - S_2(T))^+ \mathbf{1}_{\{\tau > T, V(T) > D\}} |\mathcal{F}_0\right]. \end{aligned} \tag{4}$$

3. Valuation of the Exchange Option with Credit Risk under the Hybrid Model

We propose a valuation of exchange option with credit risk exchange option with credit risk under the hybrid model in this section. By the law of iterated conditional expectations, the price C in the Equation (4) is given by

$$\begin{aligned} C &= \frac{(1-\alpha)}{D} e^{-rT}E^P\left[V(T)(S_1(T) - S_2(T))^+ |\mathcal{F}_0\right] \\ &\quad + e^{-rT}E^P\left[e^{-\int_0^T \lambda(s)ds}(S_1(T) - S_2(T))^+ \mathbf{1}_{\{V(T) > D\}} |\mathcal{F}_0\right] \\ &\quad - \frac{(1-\alpha)}{D} e^{-rT}E^P\left[e^{-\int_0^T \lambda(s)ds} V(T)(S_1(T) - S_2(T))^+ \mathbf{1}_{\{V(T) > D\}} |\mathcal{F}_0\right]. \end{aligned} \tag{5}$$

In order to simplify the notations, we denote that

$$\begin{aligned} J_1 &= \frac{(1-\alpha)}{D} e^{-rT}E^P\left[V(T)(S_1(T) - S_2(T))^+ |\mathcal{F}_0\right], \\ J_2 &= e^{-rT}E^P\left[e^{-\int_0^T \lambda(s)ds}(S_1(T) - S_2(T))^+ \mathbf{1}_{\{V(T) > D\}} |\mathcal{F}_0\right], \\ J_3 &= \frac{(1-\alpha)}{D} e^{-rT}E^P\left[e^{-\int_0^T \lambda(s)ds} V(T)(S_1(T) - S_2(T))^+ \mathbf{1}_{\{V(T) > D\}} |\mathcal{F}_0\right]. \end{aligned}$$

Then, the price C can be written as

$$C = J_1 + J_2 - J_3. \tag{6}$$

We calculate J_1, J_2 and J_3 in the following Lemmas, respectively.

Lemma 1. *Let us consider J_1 in the Equation (6), then J_1 is given by*

$$J_1 = \frac{(1-\alpha)}{D} S_1(0)V(0)e^{rT+\sigma_1\sigma_3\rho_{13}T} N(a_1) - \frac{(1-\alpha)}{D} S_2(0)V(0)e^{rT+\sigma_1\sigma_3\rho_{13}T} N(a_2), \tag{7}$$

where

$$a_1 = \frac{1}{\sigma\sqrt{T}} \ln\frac{S_1(0)}{S_2(0)} + \left(\frac{\rho_{13}\sigma_1\sigma_3 - \rho_{23}\sigma_2\sigma_3}{\sigma} + \frac{\sigma}{2}\right)\sqrt{T},$$
$$a_2 = \frac{1}{\sigma\sqrt{T}} \ln\frac{S_1(0)}{S_2(0)} + \left(\frac{\rho_{13}\sigma_1\sigma_3 - \rho_{23}\sigma_2\sigma_3}{\sigma} - \frac{\sigma}{2}\right)\sqrt{T},$$

with $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}$ and $N(a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-\frac{1}{2}x^2}dx$.

Proof. We can write J_1 as

$$\begin{aligned} J_1 &= \frac{(1-\alpha)}{D} e^{-rT} E^P\left[V(T)S_1(T)\mathbf{1}_{\{S_1(T)>S_2(T)\}}|\mathcal{F}_0\right] \\ &\quad - \frac{(1-\alpha)}{D} e^{-rT} E^P\left[V(T)S_2(T)\mathbf{1}_{\{S_1(T)>S_2(T)\}}|\mathcal{F}_0\right] \\ &:= I_1 - I_2. \end{aligned} \tag{8}$$

To calculate I_1, we define a new measure Q_1 as

$$\frac{dQ_1}{dP} = \exp\left[\sigma_1 W_1(T) + \sigma_3 W_3(T) - \frac{1}{2}(\sigma_1^2 + \sigma_3^2 + 2\rho_{13}\sigma_1\sigma_3)T\right].$$

By Girsanov's thoerem,

$$\begin{aligned} W_1^{Q_1} &= W_1(T) - \sigma_1 T - \sigma_3\rho_{13}T, \\ W_2^{Q_1} &= W_2(T) - \sigma_1\rho_{12}T - \sigma_3\rho_{23}T, \\ W_3^{Q_1} &= W_3(T) - \sigma_3 T - \sigma_1\rho_{13}T \end{aligned}$$

are the standard Brownian motions under the measure Q_1. Then we have

$$\begin{aligned} I_1 &= \frac{(1-\alpha)}{D} S_1(0)V(0)e^{rT+\sigma_1\sigma_3\rho_{13}T} \mathrm{E}^{Q_1}\left[\frac{dP}{dQ_1}\mathbf{1}_{\{S_1(T)>S_2(T)\}}|\mathcal{F}_0\right] \\ &= \frac{(1-\alpha)}{D} S_1(0)V(0)e^{rT+\sigma_1\sigma_3\rho_{13}T} \mathrm{P}^{Q_1}\left(S_1(T) > S_2(T)\right) \\ &= \frac{(1-\alpha)}{D} S_1(0)V(0)e^{rT+\sigma_1\sigma_3\rho_{13}T} \\ &\quad \times \mathrm{P}^{Q_1}\left(\sigma_2 W_2^{Q_1}(T) - \sigma_1 W_1^{Q_1}(T) < \ln\left(\frac{S_1(0)}{S_2(0)}\right) + \frac{1}{2}\sigma^2 T + \sigma_1^2 T + \sigma_1\sigma_3\rho_{13}T - \sigma_2\sigma_3\rho_{23}T\right) \\ &= \frac{(1-\alpha)}{D} S_1(0)V(0)e^{rT+\sigma_1\sigma_3\rho_{13}T} \mathrm{N}(a_1). \end{aligned} \tag{9}$$

To calculate I_2, we define a new measure Q_2 as

$$\frac{dQ_2}{dP} = \exp\left[\sigma_2 W_1(T) + \sigma_3 W_3(T) - \frac{1}{2}(\sigma_2^2 + \sigma_3^2 + 2\rho_{23}\sigma_2\sigma_3)T\right].$$

Since

$$W_1^{Q_2} = W_1(T) - \sigma_2\rho_{12}T - \sigma_3\rho_{13}T,$$
$$W_2^{Q_2} = W_2(T) - \sigma_2 T - \sigma_3\rho_{23}T,$$
$$W_3^{Q_2} = W_3(T) - \sigma_3 T - \sigma_2\rho_{23}T$$

are the standard Brownian motions under the measure Q_2, I_2 can be calculated as

$$\begin{aligned} I_2 &= \frac{(1-\alpha)}{D} S_2(0)V(0)e^{rT+\sigma_2\sigma_3\rho_{23}T}\mathrm{E}^{Q_2}\left[\frac{dP}{dQ_2}\mathbf{1}_{\{S_1(T)>S_2(T)\}}|\mathcal{F}_0\right] \\ &= \frac{(1-\alpha)}{D} S_2(0)V(0)e^{rT+\sigma_2\sigma_3\rho_{23}T}\mathrm{P}^{Q_2}\left(S_1(T)>S_2(T)\right) \\ &= \frac{(1-\alpha)}{D} S_2(0)V(0)e^{rT+\sigma_2\sigma_3\rho_{23}T} \\ &\quad\times \mathrm{P}^{Q_2}\left(\sigma_2 W_2^{Q_2}(T) - \sigma_1 W_1^{Q_2}(T) < \ln\left(\frac{S_1(0)}{S_2(0)}\right) - \frac{1}{2}\sigma^2 T + \sigma_1^2 T + \sigma_1\sigma_3\rho_{13}T - \sigma_2\sigma_3\rho_{23}T\right) \\ &= \frac{(1-\alpha)}{D} S_2(0)V(0)e^{rT+\sigma_1\sigma_3\rho_{13}T}\mathrm{N}(a_2). \end{aligned} \tag{10}$$

By the Equations (9) and (10), the proof of Lemma is completed. □

Lemma 2. *Let us consider J_2 in the Equation (6), then J_2 is given by*

$$J_2 = S_1(0)M_1(T)e^{-\frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds}N_2(b_1,b_2,\theta_1) - S_2(0)M_1(T)e^{-\frac{\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds}N_2(b_3,b_4,\theta_1), \tag{11}$$

where

$$\theta_1 = (\sigma_1\rho_{13} - \sigma_2\rho_{23})/\sigma, \sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}, f(s,t,u) = 1 - e^{-u(t-s)},$$
$$M_1(T) = \exp\left[-bT - \frac{\lambda(0)-b}{a}f(0,T,a) + \frac{\sigma_4^2}{a^2}\int_0^T f^2(s,T,a)ds\right],$$
$$b_1 = \frac{\ln(V(0)/D) - \frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds + \left(r - \frac{1}{2}\sigma_3^2 + \sigma_1\sigma_3\rho_{13}\right)T}{\sigma_3\sqrt{T}},$$
$$b_2 = \frac{\ln(S_1(0)/S_2(0)) - \frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds + \frac{\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds + \left(\frac{1}{2}(\sigma_1^2+\sigma_2^2) - \sigma_1\sigma_2\rho_{12}\right)T}{\sigma\sqrt{T}},$$
$$b_3 = \frac{\ln(V(0)/D) - \frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds + \left(r - \frac{1}{2}\sigma_3^2 + \sigma_2\sigma_3\rho_{23}\right)T}{\sigma_3\sqrt{T}},$$
$$b_4 = \frac{\ln(S_1(0)/S_2(0)) - \frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds + \frac{\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds + \left(\frac{1}{2}(\sigma_1^2+\sigma_2^2) + \sigma_1\sigma_2\rho_{12}\right)T}{\sigma\sqrt{T}},$$

and

$$N_2(n_1,n_2,\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{n_1}\int_{-\infty}^{n_2} e^{-\frac{1}{2(1-\rho^2)}(x^2-2xy\rho+y^2)}dydx.$$

Proof. We can write J_2 as

$$\begin{aligned} J_2 &= e^{-rT}E^P\left[e^{-\int_0^T \lambda(s)ds}S_1(T)\mathbf{1}_{\{S_1(T)>S_2(T),V(T)>D\}}|\mathcal{F}_0\right] \\ &\quad -e^{-rT}E^P\left[e^{-\int_0^T \lambda(s)ds}S_2(T)\mathbf{1}_{\{S_1(T)>S_2(T),V(T)>D\}}|\mathcal{F}_0\right] \\ &:= I_3 - I_4. \end{aligned} \tag{12}$$

To calculate I_4, we define a new measure Q_3 such that

$$\frac{dQ_3}{dP} = \frac{e^{-\int_0^T \lambda(s)ds}}{\mathrm{E}[e^{-\int_0^T \lambda(s)ds}|\mathcal{F}_0]}.$$

Then,

$$I_3 = e^{-rT}M_1(T)\mathrm{E}^{Q_3}\left[S_1(t)\mathbf{1}_{\{S_1(T)>S_2(T),V(T)>D\}}|\mathcal{F}_0\right]. \tag{13}$$

Under the measure Q_3,

$$\begin{aligned} W_1^{Q_3}(T) &= W_1(T) + \frac{\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds, \\ W_2^{Q_3}(T) &= W_2(T) + \frac{\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds, \\ W_3^{Q_3}(T) &= W_3(T) + \frac{\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds, \\ W_4^{Q_3}(T) &= W_4(T) + \frac{\sigma_4}{a}\int_0^T f(s,T,a)ds \end{aligned} \tag{14}$$

are the standard Brownian motions.

To calculate the Equation (13), we define a new measure $\widetilde{Q}_3$ such that

$$\frac{d\widetilde{Q}_3}{dP} = \exp\left[\sigma_1 W_1^{Q_3} - \frac{1}{2}\sigma_1^2\right].$$

Then,

$$I_3 = S_1(0)M_1(T)e^{-\frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds}\mathrm{E}^{\widetilde{Q}_3}\left[\mathbf{1}_{\{S_1(T)>S_2(T),V(T)>D\}}|\mathcal{F}_0\right], \tag{15}$$

where $M_1(T) = \mathrm{E}[e^{-\int_0^T \lambda(s)ds}|\mathcal{F}_0]$.

Since

$$\begin{aligned} \mathrm{E}^{\widetilde{Q}_3}\left[\mathbf{1}_{\{V(T)>D,S_1(T)>S_2(T)\}}\right] &= \mathrm{P}^{\widetilde{Q}_3}\left(S_1(T) > S_2(T), V(T) > D\right) \\ &= \mathrm{P}^{\widetilde{Q}_3}\left(z_1 < b_1, z_2 < b_2\right) \end{aligned}$$

and z_1 and z_2 are the standard normal variables, we have

$$I_3 = S_1(0)M_1(T)e^{-\frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds}\mathrm{N}_2(b_1, b_2, \theta_1), \tag{16}$$

where θ_1 is the correlation between z_1 and z_2.

Under the measure Q_3, I_4 is represented by

$$I_4 = e^{-rT} M_1(T) \mathrm{E}^{Q_3} \left[S_2(t) \mathbf{1}_{\{S_1(T) > S_2(T), V(T) > D\}} | \mathcal{F}_0 \right]. \tag{17}$$

We define a new measure $\widehat{Q}_3$ such that

$$\frac{d\widehat{Q}_3}{dP} = \exp\left[\sigma_2 W_2^{Q_3} - \frac{1}{2}\sigma_2^2 \right].$$

Then, by using the standard Brownian motions under the measure $\widehat{Q}_3$, we have

$$\begin{aligned} I_4 &= S_2(0) M_1(T) e^{-\frac{\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds} \mathrm{E}^{\widehat{Q}_3} \left[\mathbf{1}_{\{S_1(T) > S_2(T), V(T) > D\}} | \mathcal{F}_0 \right] \\ &= S_2(0) M_1(T) e^{-\frac{\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds} \mathrm{P}^{\widehat{Q}_3} \left(S_1(T) > S_2(T), V(T) > D \right) \\ &= S_2(0) M_1(T) e^{-\frac{\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds} \mathrm{N}_2(b_3, b_4, \theta_1). \end{aligned} \tag{18}$$

□

Lemma 3. *Let us consider J_3 in the Equation (6), then J_3 is given by*

$$\begin{aligned} J_3 &= \frac{(1-\alpha)}{D} S_1(0) V(0) e^{(r+\sigma_1\sigma_3\rho_{13})T - \frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds - \frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds} M_1(T) N_2(c_1, c_2, \theta_1) \\ &\quad - \frac{(1-\alpha)}{D} S_2(0) V(0) e^{(r+\sigma_2\sigma_3\rho_{23})T - \frac{\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds - \frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds} \\ &\quad \times M_1(T) e^{-\frac{\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds} N_2(c_3, c_4, \theta_1), \end{aligned} \tag{19}$$

where

$$\begin{aligned} &\theta_1 = (\sigma_1\rho_{13} - \sigma_2\rho_{23})/\sigma, \sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}, f(s,t,u) = 1 - e^{-u(t-s)}, \\ &M_1(T) = \exp\left[-bT - \tfrac{\lambda(0)-b}{a} f(0,T,a) + \tfrac{\sigma_4^2}{a^2}\int_0^T f^2(s,T,a)ds \right], \\ &c_1 = \frac{\ln(V(0)/D) - \frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds + \left(r + \frac{1}{2}\sigma_3^2 + \sigma_1\sigma_3\rho_{13}\right)T}{\sigma_3\sqrt{T}}, \\ &c_2 = \frac{\ln(S_1(0)/S_2(0)) - \left(\frac{\sigma_1\sigma_4\rho_{14}}{a} - \frac{\sigma_2\sigma_4\rho_{24}}{a}\right)\int_0^T f(s,T,a)ds + \left(\sigma_1\sigma_3\rho_{13} - \sigma_1\sigma_2\rho_{12} - \sigma_2\sigma_3\rho_{23} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\right)T}{\sigma\sqrt{T}}, \\ &c_3 = \frac{\ln(V(0)/D) - \frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds + \left(r + \frac{1}{2}\sigma_3^2 + \sigma_2\sigma_3\rho_{23}\right)T}{\sigma_3\sqrt{T}}, \\ &c_4 = \frac{\ln(S_1(0)/S_2(0)) - \left(\frac{\sigma_1\sigma_4\rho_{14}}{a} - \frac{\sigma_2\sigma_4\rho_{24}}{a}\right)\int_0^T f(s,T,a)ds + \left(\sigma_1\sigma_3\rho_{13} + \sigma_1\sigma_2\rho_{12} - \sigma_2\sigma_3\rho_{23} - \frac{1}{2}(\sigma_1^2+\sigma_2^2)\right)T}{\sigma\sqrt{T}}, \end{aligned}$$

and θ_1, σ, f, $M_1(T)$ and N_2 are defined in Lemma 2.

Proof. We rewrite J_3 as

$$\begin{aligned} J_3 &= \frac{(1-\alpha)}{D} e^{-rT} E^P \left[e^{-\int_0^T \lambda(s)ds} V(T) S_1(T) \mathbf{1}_{\{V(T)>D, S_1(T)>S_2(T)\}} | \mathcal{F}_0 \right] \\ &\quad - \frac{(1-\alpha)}{D} e^{-rT} E^P \left[e^{-\int_0^T \lambda(s)ds} V(T) S_2(T) \mathbf{1}_{\{V(T)>D, S_1(T)>S_2(T)\}} | \mathcal{F}_0 \right] \\ &:= \frac{(1-\alpha)}{D} I_5 - \frac{(1-\alpha)}{D} I_6. \end{aligned} \tag{20}$$

To calculate I_5, we use the measure Q_3 defined in Lemma 2. Under the measure Q_3, we have

$$I_5 = e^{-rT} M_1(T) E^{Q_3}\left[V(T)S_1(T)\mathbf{1}_{\{V(T)>D, S_1(T)>S_2(T)\}}|\mathcal{F}_0\right], \tag{21}$$

where $M_1(T)$ is defined in Lemma 2. With the standard Brownian motions under the measure Q_3 in (14), we define a new measure Q_4 such that

$$\frac{dQ_4}{dQ_3} = \exp\left[\sigma_1 W_1^{Q_3}(T) + \sigma_3 W_3^{Q_3}(T) - \frac{1}{2}(\sigma_1^2 + \sigma_3^2 + 2\rho_{13}\sigma_1\sigma_3)T\right].$$

By Girsanov's thoerem,

$$\begin{aligned} W_1^{Q_4} &= W_1^{Q_3}(T) - \sigma_1 T - \sigma_3\rho_{13}T, \\ W_2^{Q_4} &= W_2^{Q_3}(T) - \sigma_1\rho_{12}T - \sigma_3\rho_{23}T, \\ W_3^{Q_4} &= W_3^{Q_3}(T) - \sigma_3 T - \sigma_1\rho_{13}T \end{aligned}$$

are the standard Brownian motions under the measure Q_4. Then, we obtain

$$\begin{aligned} I_5 &= S_1(0)V(0)e^{rT+\sigma_1\sigma_3\rho_{13}T-\frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds-\frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds} M_1(T)\mathrm{E}^{Q_4}\left[\mathbf{1}_{\{V(T)>D,S_1(T)>S_2(T)\}}|\mathcal{F}_0\right] \\ &= S_1(0)V(0)e^{rT+\sigma_1\sigma_3\rho_{13}T-\frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds-\frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds} M_1(T)\mathrm{P}^{Q_4}\left(V(T)>D, S_1(T)>S_2(T)\right) \\ &= S_1(0)V(0)e^{rT+\sigma_1\sigma_3\rho_{13}T-\frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds-\frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds} M_1(T)\mathrm{N}_2(c_1, c_2, \theta_1). \end{aligned} \tag{22}$$

In a similar way, we can write I_6 under the measure Q_3 as

$$I_6 = e^{-rT} M_1(T) E^{Q_3}\left[V(T)S_2(T)\mathbf{1}_{\{V(T)>D, S_1(T)>S_2(T)\}}|\mathcal{F}_0\right]. \tag{23}$$

To calculate I_6, we define a new measure Q_5 equivalent to Q_3 by

$$\frac{dQ_5}{dQ_3} = \exp\left[\sigma_2 W_2^{Q_3}(T) + \sigma_3 W_3^{Q_3}(T) - \frac{1}{2}(\sigma_2^2 + \sigma_3^2 + 2\rho_{23}\sigma_2\sigma_3)T\right].$$

By Girsanov's thoerem,

$$\begin{aligned} W_1^{Q_5} &= W_1^{Q_3}(T) - \sigma_1\rho_{12}T - \sigma_3\rho_{13}T, \\ W_2^{Q_5} &= W_2^{Q_3}(T) - \sigma_2 T - \sigma_3\rho_{23}T, \\ W_3^{Q_5} &= W_3^{Q_3}(T) - \sigma_3 T - \sigma_2\rho_{23}T \end{aligned}$$

are the standard Brownian motions under the measure Q_5. Then, we obtain

$$\begin{aligned} I_6 &= S_2(0)V(0)e^{rT+\sigma_1\sigma_3\rho_{13}T-\frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds-\frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds} M_1(T)\mathrm{E}^{Q_4}\left[\mathbf{1}_{\{V(T)>D,S_1(T)>S_2(T)\}}|\mathcal{F}_0\right] \\ &= S_1(0)V(0)e^{rT+\sigma_1\sigma_3\rho_{13}T-\frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds-\frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds} M_1(T)\mathrm{P}^{Q_4}\left(V(T)>D, S_1(T)>S_2(T)\right) \\ &= S_1(0)V(0)e^{rT+\sigma_1\sigma_3\rho_{13}T-\frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds-\frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds} M_1(T)\mathrm{N}_2(c_1, c_2, \theta_1). \end{aligned} \tag{24}$$

This completes the proof. □

From above Lemmas, we can obtain the following theorem.

Theorem 1. *The value of exchange option with credit risk at time 0 under the hybrid credit risk model is given by*

$$
\begin{aligned}
C = & \frac{(1-\alpha)}{D} S_1(0)V(0)e^{rT+\sigma_1\sigma_3\rho_{13}T} N(a_1) - \frac{(1-\alpha)}{D} S_2(0)V(0)e^{rT+\sigma_1\sigma_3\rho_{13}T} N(a_2) \\
& + S_1(0)M_1(T)e^{-\frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds} N_2(b_1,b_2,\theta_1) - S_2(0)M_1(T)e^{-\frac{\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds} N_2(b_3,b_4,\theta_1) \\
& - \frac{(1-\alpha)}{D} S_1(0)V(0)e^{(r+\sigma_1\sigma_3\rho_{13})T - \frac{\sigma_1\sigma_4\rho_{14}}{a}\int_0^T f(s,T,a)ds - \frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds} M_1(T) N_2(c_1,c_2,\theta_1) \\
& + \frac{(1-\alpha)}{D} S_2(0)V(0)e^{(r+\sigma_2\sigma_3\rho_{23})T - \frac{\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds - \frac{\sigma_3\sigma_4\rho_{34}}{a}\int_0^T f(s,T,a)ds} \\
& \times M_1(T)e^{-\frac{\sigma_2\sigma_4\rho_{24}}{a}\int_0^T f(s,T,a)ds} N_2(c_3,c_4,\theta_1),
\end{aligned}
$$

where all parameters and notations are defined in Lemmas 1–3.

4. Concluding Remarks

Exchange option is one of the popular exotic options with two underlying assets in the OTC markets, and credit risk is undoubtedly very important issue in the OTC market. In fact, there have been numerous studies on the valuation of exchange option with credit risk. However, to the best of my knowledge, all results adopted only one of the two models (the reduced-form model and the structural model) for modeling credit risk. In this paper, we first deal with the valuation of exchange option under the hybrid credit risk model combining the reduced-form model and the structural model. Specifically, we use the reduced-form model of Fard [6] and the structural model of Klein [10] to build the hybrid credit risk model. To derive the pricing formula, we adopt the probabilistic approach and use the change of measure technique as the important tool. Finally, we provide the closed-form pricing formula of vulnerable exchange option with the standard normal cumulative distribution functions. Even though our approach has the limitation that the underlying assets for exchange option follow Geometric Brownian Motion (GBM) process, our approach will enable the valuation of other type of multi-asset options under the hybrid credit risk model.

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