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Controllability of Nonlinear Fractional Dynamical Systems with a Mittag–Leffler Kernel

Jiale Sheng [1,*], Wei Jiang [1,*], Denghao Pang [2] and Sen Wang [1]

[1] School of Mathematical Sciences, Anhui University, Hefei 230601, China; wangsahu@163.com
[2] School of Internet, Anhui University, Hefei 230601, China; pangdh197@163.com
* Correspondence: shengjlny@163.com (J.S.); jiangwei@ahu.edu.cn (W.J.)

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Abstract: This paper is concerned with controllability of nonlinear fractional dynamical systems with a Mittag–Leffler kernel. First, the solution of fractional dynamical systems with a Mittag–Leffler kernel is given by Laplace transform. In addition, one necessary and sufficient condition for controllability of linear fractional dynamical systems with Mittag–Leffler kernel is established. On this basis, we obtain one sufficient condition to guarantee controllability of nonlinear fractional dynamical systems with a Mittag–Leffler kernel by fixed point theorem. Finally, an example is given to illustrate the applicability of our results.

Keywords: controllability; Mittag–Leffler kernel; nonlinear; fixed point theorem

1. Introduction

Fractional calculus is a very popular topic that has a history of more than 300 years. In recent decades, fractional calculus and its applications have grown rapidly. It turns out that fractional calculus is a useful tool for studying many phenomena in physics, engineering, chemistry, economics, and other fields. Moreover, different types of fractional differential equations are beginning to be studied by more and more scholars and have become very popular topics. For more details, please refer to [1–10] etc.

Recently, many authors have attempted to find new fractional operators with different kernels in order to better describe these phenomena. Up to now, except for the classical Riemann–Liouville fractional calculus and the Caputo fractional calculus, there are various novel fractional operators, such as the Hadamard fractional calculus [11], the conformable fractional calculus [12], the Caputo–Fabrizio fractional calculus [13], the Yang–Srivastava–Machado fractional derivative [14]. In 2016, based on the Caputo–Fabrizio fractional calculus, Atangana and Baleanu [15] introduced new definition of fractional derivatives called AB fractional derivatives by replacing the kernel $exp(-\frac{\alpha}{1-\alpha}(t-s))$ with $E_\alpha(-\frac{\alpha}{1-\alpha}(t-s)^\alpha)$ and derived the fractional integral associate to AB fractional derivatives by inverse Laplace transform and convolution theorem. More details about AB fractional calculus are given in Section 2. It is worth noting that Riemann–Liouville, Caputo, Yang–Srivastava–Machado and AB fractional derivatives are nonlocal operators, which are useful to discuss some complex dynamics in physical phenomena. However, unlike the Riemann–Liouville and Caputo fractional derivatives, the AB fractional derivatives have non-singular kernels which can model practical physical phenomena well such as the heat transfer model [15], the diffusion equation [16], electromagnetic waves in dielectric media [17], chaos [18], circuit model [19], etc. In addition, there are plenty of basic properties of AB fractional calculus have been studied, such as integration by parts [20], mean value theorem and Taylor's theorem [21], semigroup property, product rule and chain rule [22]. All kinds of problems about different types of fractional dynamical systems in the sense of AB fractional derivative are also studied by several researchers [23–26].

Controllability means that the dynamical system can be transformed from any initial state to any final state through a set of controls, which is one of the most basic and important concepts in control theory. Therefore, many scholars have done a lot of work and research on control theory and its application. In [27], controllability of linear systems was established perfectly. In [28,29], the paper established a set of sufficient conditions for the controllability of nonlinear fractional dynamical system of order $0 < \alpha < 1$ and $1 < \alpha < 2$ in finite-dimensional by using the Mittag–Leffler matrix function and Schauder fixed point theorem. In [30], the paper studied the controllability of linear and nonlinear fractional damped dynamical systems by using the Mittag–Leffler matrix function and the iterative technique. In [31–34], the controllability of various fractional evolution equations were discussed based on operator semigroup theory and fixed point theorem. However, all the above papers are about the traditional Caputo fractional derivatives. There are few papers investigating the controllability results in the sense of AB fractional derivatives. Inspired by the above-mentioned works, this paper will deal with the controllability of linear and nonlinear fractional dynamical systems in the sense of AB fractional derivatives by using the Mittag–Leffler matrix function and Schauder fixed point theorem.

In this paper, we will concern with the following linear fractional dynamical systems in the sense of AB fractional derivatives

$$\begin{cases} {}^{ABC}D_t^{\alpha} x(t) = Ax(t) + Bu(t), \ \ t \in I := [0, b], \\ x(0) = x_0, \end{cases} \tag{1}$$

and nonlinear fractional dynamical systems

$$\begin{cases} {}^{ABC}D_t^{\alpha} x(t) = Ax(t) + Bu(t) + f(t, x(t), u(t)), \ \ t \in I, \\ x(0) = x_0, \end{cases} \tag{2}$$

where $0 < \alpha < 1$, $x(\cdot) \in R^n$, $u(\cdot) \in R^m$, $A \in R^{n\times n}$ and $B \in R^{n\times m}$ are constant matrices, $f : I \times R^n \times R^m \to R^n$ is continuous function. Through out this paper, R^n is the n-dimensional Euclidean space, $R^{n\times m}$ is the set of all $n \times m$ real-value matrices. A^* is the transpose of matrix A, A^{-1} is the inverse of matrix A.

This article is organized as follows. In Section 2, we will introduce some basic definitions and useful lemmas and important properties of AB fractional calculus and Mittag–Leffler function. In Section 3, we solve the solution representation of fractional dynamical systems with a Mittag–Leffler kernel. In Section 4, we first consider controllability of linear fractional dynamical systems with a Mittag–Leffler kernel and give one necessary and sufficient condition of controllability for linear such systems. Furthermore, we establish one sufficient condition to ensure that nonlinear fractional dynamical systems with a Mittag–Leffler kernel are controllable. In Section 5, an example is given to illustrate the applicability of our results.

2. Preliminaries

In this section, we present the definition of AB fractional derivatives and integral. In addition, some lemmas and properties of AB fractional calculus and the Mittag–Leffler function are introduced.

Definition 1 ([8])**.** *Let $0 < \alpha < 1$, $f \in L^1(0, b)$, $b > 0$. Then the AB fractional integral of order α is defined by*

$${}^{AB}I_t^{\alpha} f(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)} (I_t^{\alpha} f)(t),$$

where $(I_t^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$ is the classical Riemann-Liouville fractional integral, $M(\alpha)$ denotes a real-valued normalization function satisfying $M(\alpha) > 0$, $M(0) = M(1) = 1$.

Definition 2 ([8]). *Let $0 < \alpha < 1$, $f \in L^1(0,b)$, $b > 0$. Then the ABR fractional derivative of order α is defined by*

$$^{ABR}D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t E_\alpha(-\frac{\alpha}{1-\alpha}(t-s)^\alpha)f(s)ds,$$

where E_α is one parameter Mittag–Leffler function denoted by

$$E_\alpha(t) = \sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\alpha n+1)}.$$

Definition 3 ([8]). *Let $0 < \alpha < 1$, $f \in L^1(0,b)$, $b > 0$. Then the ABC fractional derivative of order α is defined by*

$$^{ABC}D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t E_\alpha(-\frac{\alpha}{1-\alpha}(t-s)^\alpha)f'(s)ds.$$

Lemma 1 ([8]). *For $0 < \alpha < 1$, the Laplace transform of AB fractional derivative is*

$$L\{^{ABC}D_t^\alpha f(t);s\} = \frac{M(\alpha)}{1-\alpha}\cdot\frac{s^\alpha F(s) - s^{\alpha-1}f(0)}{s^\alpha + \frac{\alpha}{1-\alpha}},$$

where $F(s) = L\{f(t);s\}$.

In addition, the Mittag–Leffler function which plays an important role in AB fractional derivative will appear frequently in this paper. The generalized Mittag–Leffler functions(two parameters) is defined by $E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\alpha n+\beta)}$, $\alpha,\beta > 0$, $t \in R$. For $0 < \alpha < 1$, the functions E_α and $E_{\alpha,\alpha}$ are nonnegative and for $t = 0$, we have $E_\alpha(0) = 1$, $E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}$. The Laplace transform of the Mittag–Leffler function is given as

$$L\{E_\alpha(-\lambda t^\alpha);s\} = \frac{s^{\alpha-1}}{s^\alpha+\lambda}.$$

$$L\{t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha);s\} = \frac{s^{\alpha-\beta}}{s^\alpha+\lambda}.$$

Lemma 2 ([35]). *Assume that continuous function $f : K \times R^n \to R^m$ satisfies $\lim_{|v|\to\infty}\frac{|f(w,v)|}{|v|} = 0$ uniformly in $w \in K$, where K is a bounded subset of R^p, then for every pair of constants c and d, there exists a positive constant r such that if $|v| \le r$, then $c|f(w,v)| + d \le r$, for all $w \in K$.*

3. Solution Representation

In this section, we give the solution representation of fractional dynamical systems with Mittag–Leffler kernel. Consider the following system:

$$\begin{cases} ^{ABC}D_t^\alpha x(t) = Ax(t) + f(t), \quad t \in I, \\ x(0) = x_0, \end{cases} \tag{3}$$

where $0 < \alpha < 1$, $x(\cdot) \in R^n$, $A \in R^{n\times n}$, $f : I \to R^n$ is continuous function.

Theorem 1. *Assume that matrix $M(\alpha)I-(1-\alpha)A$ is nonsingular and denote $A_\alpha=[M(\alpha)I-(1-\alpha)A]^{-1}$. Then the solution of system (3) is*

$$\begin{aligned} x(t) =& M(\alpha)A_\alpha E_\alpha(\alpha A A_\alpha t^\alpha)x_0+(1-\alpha)A_\alpha f(t) \\ &+\alpha M(\alpha)A_\alpha^2\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\alpha A A_\alpha(t-\tau)^\alpha)f(\tau)d\tau. \end{aligned} \tag{4}$$

Proof. Denote $X(s)=L\{x(t);s\}$ and $F(s)=L\{f(t);s\}$. Taking Laplace transform of system (3), we have

$$\frac{M(\alpha)}{1-\alpha}\cdot\frac{s^\alpha X(s)-s^{\alpha-1}x_0}{s^\alpha+\frac{\alpha}{1-\alpha}}=AX(s)+F(s).$$

The above equation can be rewritten as

$$[(M(\alpha)I-(1-\alpha)A)s^\alpha-\alpha A]X(s)=M(\alpha)s^{\alpha-1}x_0+[(1-\alpha)s^\alpha+\alpha]F(s).$$

This is equivalent to

$$[s^\alpha A_\alpha^{-1}-\alpha A]X(s)=M(\alpha)s^{\alpha-1}x_0+[(1-\alpha)s^\alpha+\alpha]F(s).$$

Then

$$(s^\alpha I-\alpha A A_\alpha)A_\alpha^{-1}X(s)=M(\alpha)s^{\alpha-1}x_0+[(1-\alpha)s^\alpha+\alpha]F(s).$$

It follows that

$$\begin{aligned} X(s) =& M(\alpha)A_\alpha s^{\alpha-1}(s^\alpha I-\alpha A A_\alpha)^{-1}x_0+(1-\alpha)A_\alpha s^\alpha(s^\alpha I-\alpha A A_\alpha)^{-1}F(s) \\ &+\alpha A_\alpha(s^\alpha I-\alpha A A_\alpha)^{-1}F(s). \end{aligned} \tag{5}$$

For the second term of Equation (5), we have

$$\begin{aligned} s^\alpha(s^\alpha I-\alpha A A_\alpha)^{-1} &= (s^\alpha I-\alpha A A_\alpha+\alpha A A_\alpha)(s^\alpha I-\alpha A A_\alpha)^{-1} \\ &= I+\alpha A A_\alpha(s^\alpha I-\alpha A A_\alpha)^{-1}. \end{aligned}$$

Thus, Equation (5) can be rewritten as

$$\begin{aligned} X(s) =& M(\alpha)A_\alpha s^{\alpha-1}(s^\alpha I-\alpha A A_\alpha)^{-1}x_0+(1-\alpha)A_\alpha F(s) \\ &+(1-\alpha)\alpha A A_\alpha^2(s^\alpha I-\alpha A A_\alpha)^{-1}F(s)+\alpha A_\alpha(s^\alpha I-\alpha A A_\alpha)^{-1}F(s) \\ =& M(\alpha)A_\alpha s^{\alpha-1}(s^\alpha I-\alpha A A_\alpha)^{-1}x_0+(1-\alpha)A_\alpha F(s) \\ &+[(1-\alpha)\alpha A+\alpha A_\alpha^{-1}]A_\alpha^2(s^\alpha I-\alpha A A_\alpha)^{-1}F(s) \\ =& M(\alpha)A_\alpha s^{\alpha-1}(s^\alpha I-\alpha A A_\alpha)^{-1}x_0+(1-\alpha)A_\alpha F(s) \\ &+\alpha M(\alpha)A_\alpha^2(s^\alpha I-\alpha A A_\alpha)^{-1}F(s). \end{aligned} \tag{6}$$

Using inverse Laplace transform and Convolution theorem of Equation (6), we get

$$\begin{aligned} x(t) =& M(\alpha)A_\alpha L^{-1}\{s^{\alpha-1}(s^\alpha I-\alpha A A_\alpha)^{-1};t\}x_0+(1-\alpha)A_\alpha L^{-1}\{F(s);t\} \\ &+\alpha M(\alpha)A_\alpha^2 L^{-1}\{(s^\alpha I-\alpha A A_\alpha)^{-1};t\}*L^{-1}\{F(s);t\}. \end{aligned}$$

Due to the fact that

$$
\begin{aligned}
&L^{-1}\{s^{\alpha-1}(s^{\alpha}I-\alpha AA_{\alpha})^{-1};t\}=E_{\alpha}(\alpha AA_{\alpha}t^{\alpha}),\\
&L^{-1}\{(s^{\alpha}I-\alpha AA_{\alpha})^{-1};t\}=t^{\alpha-1}E_{\alpha,\alpha}(\alpha AA_{\alpha}t^{\alpha}),
\end{aligned}
$$

then

$$
\begin{aligned}
x(t)=&M(\alpha)A_{\alpha}E_{\alpha}(\alpha AA_{\alpha}t^{\alpha})x_0+(1-\alpha)A_{\alpha}f(t)\\
&+\alpha M(\alpha)A_{\alpha}^2\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\alpha AA_{\alpha}(t-\tau)^{\alpha})f(\tau)d\tau.
\end{aligned}
$$

The proof is complete. □

4. Controllability for Linear and Nonlinear Systems

In this section, we discuss the controllability of linear and nonlinear fractional dynamical systems in the sense of AB fractional derivative. Firstly, we give the definition of controllability of system (1) and (2).

Definition 4. *System (1) (or (2)) is said to be controllable on I if for every x_0, x_1 there exists a control $u(t)$ such that the solution $x(t)$ of such a system satisfies the conditions $x(0)=x_0$ and $x(b)=x_1$.*

According to Equation (4), the solution representation of system (1) can be expressed as

$$
\begin{aligned}
x(t)=&M(\alpha)A_{\alpha}E_{\alpha}(\alpha AA_{\alpha}t^{\alpha})x_0+(1-\alpha)A_{\alpha}Bu(t)\\
&+\alpha M(\alpha)A_{\alpha}^2\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\alpha AA_{\alpha}(t-\tau)^{\alpha})Bu(\tau)d\tau.
\end{aligned}
\tag{7}
$$

Then, we will give one controllability criteria of of system (1).

Theorem 2. *Assume that $M(\alpha)I-(1-\alpha)A$ is nonsingular and denote $A_{\alpha}=[M(\alpha)I-(1-\alpha)A]^{-1}$. Define the Gramian matrix*

$$
W=\int_0^b(b-\tau)^{\alpha-1}E_{\alpha,\alpha}(\alpha AA_{\alpha}(b-\tau)^{\alpha})BB^*E_{\alpha,\alpha}^*(\alpha AA_{\alpha}(b-\tau)^{\alpha})d\tau.
$$

Then system (1) is controllable on I, if and only if the matrix $W_1:=\frac{1-\alpha}{\Gamma(\alpha)}BB^*+\alpha M(\alpha)A_{\alpha}W$ is nonsingular.

Proof. Firstly, we prove the sufficiency. Since matrices A_{α} and W_1 are nonsingular, then for any vectors $x_0, x_1\in R^n$, we can take the control input function $u(t)$ as

$$
u(t)=B^*E_{\alpha,\alpha}^*(\alpha AA_{\alpha}(b-t)^{\alpha})W_1^{-1}A_{\alpha}^{-1}y_1,
\tag{8}
$$

where

$$
y_1=x_1-M(\alpha)A_{\alpha}E_{\alpha}(\alpha AA_{\alpha}b^{\alpha})x_0.
$$

Through Equation (7), we can get the value of $x(t)$ at final time $t = b$ as

$$\begin{aligned} x(b) =& M(\alpha)A_\alpha E_\alpha(\alpha A A_\alpha b^\alpha)x_0 + (1-\alpha)A_\alpha Bu(b) \\ &+ \alpha M(\alpha)A_\alpha^2 \int_0^b (b-\tau)^{\alpha-1} E_{\alpha,\alpha}(\alpha A A_\alpha (b-\tau)^\alpha) Bu(\tau)d\tau \\ =& M(\alpha)A_\alpha E_\alpha(\alpha A A_\alpha b^\alpha)x_0 + \frac{1-\alpha}{\Gamma(\alpha)} A_\alpha BB^* W_1^{-1} A_\alpha^{-1} y_1 + \alpha M(\alpha) A_\alpha^2 \times \\ & \int_0^b (b-\tau)^{\alpha-1} E_{\alpha,\alpha}(\alpha A A_\alpha (b-\tau)^\alpha) BB^* E^*_{\alpha,\alpha}(\alpha A A_\alpha (b-\tau)^\alpha) W_1^{-1} A_\alpha^{-1} y_1 d\tau \\ =& M(\alpha)A_\alpha E_\alpha(\alpha A_\alpha A b^\alpha)x_0 + \frac{1-\alpha}{\Gamma(\alpha)} A_\alpha BB^* W_1^{-1} A_\alpha^{-1} y_1 + \alpha M(\alpha) A_\alpha^2 W W_1^{-1} A_\alpha^{-1} y_1 \\ =& M(\alpha)A_\alpha E_\alpha(\alpha A_\alpha A b^\alpha)x_0 + A_\alpha (\frac{1-\alpha}{\Gamma(\alpha)} BB^* + \alpha M(\alpha) A_\alpha W) W_1^{-1} A_\alpha^{-1} y_1 \\ =& M(\alpha)A_\alpha E_\alpha(\alpha A_\alpha A b^\alpha)x_0 + y_1 \\ =& x_1. \end{aligned}$$

Hence, system (1) is controllable on I.

For necessity, we prove by contradiction. Suppose that system (1) is controllable. If W_1 is singular, then there exists a vector $z^* \neq 0$ such that $z^* W_1 z = 0$, i.e.,

$$\begin{aligned} \frac{1-\alpha}{\Gamma(\alpha)} z^* BB^* z + \alpha M(\alpha) A_\alpha \int_0^b (b-\tau)^{\alpha-1} z^* E_{\alpha,\alpha}(\alpha A A_\alpha (b-\tau)^\alpha) B \times \\ B^* E^*_{\alpha,\alpha}(\alpha A A_\alpha (b-\tau)^\alpha) z d\tau = 0. \end{aligned}$$

It follows that

$$z^* B = 0, \;\; z^* E_{\alpha,\alpha}(\alpha A A_\alpha (b-\tau)^\alpha) B = 0.$$

Let $x_0 = (M(\alpha)A_\alpha E_\alpha(\alpha A A_\alpha b^\alpha))^{-1} z$. By the assumption that system (1) is controllable, there exists an input u such that it steers x_0 to the origin in the interval I. It follows that

$$\begin{aligned} 0 = x(b) =& M(\alpha)A_\alpha E_\alpha(\alpha A A_\alpha b^\alpha)x_0 + (1-\alpha)A_\alpha Bu(b) \\ &+ \alpha M(\alpha)A_\alpha^2 \int_0^b (b-\tau)^{\alpha-1} E_{\alpha,\alpha}(\alpha A A_\alpha (b-\tau)^\alpha) Bu(\tau)d\tau. \end{aligned}$$

This implies that

$$0 = z + (1-\alpha)A_\alpha Bu(b) + \alpha M(\alpha)A_\alpha^2 \int_0^b (b-\tau)^{\alpha-1} E_{\alpha,\alpha}(\alpha A A_\alpha (b-\tau)^\alpha) Bu(\tau)d\tau. \quad (9)$$

Multiplying by z^* both sides of Equation (9), it yields $z^* z = 0$. This is in contradiction with $z \neq 0$. Thus, the matrix W_1 is invertible. The proof is completed. □

Now, we consider controllability of nonlinear fractional dynamical systems with the Mittag–Leffler kernel. For the corresponding nonlinear system (2), according to Equation (4), we know that if $x(t)$ is the solution of system (2), then $x(t)$ satisfies the following equation:

$$\begin{aligned} x(t) =& M(\alpha)A_\alpha E_\alpha(\alpha A A_\alpha t^\alpha)x_0 + (1-\alpha)A_\alpha Bu(t) + (1-\alpha)A_\alpha f(t, x(t), u(t)) \\ &+ \alpha M(\alpha)A_\alpha^2 \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\alpha A A_\alpha (t-\tau)^\alpha) Bu(\tau)d\tau \\ &+ \alpha M(\alpha)A_\alpha^2 \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\alpha A A_\alpha (t-\tau)^\alpha) f(\tau, x(\tau), u(\tau))d\tau. \end{aligned} \quad (10)$$

Let G be the Banach space of all continuous functions $(x,u) : I \times I \rightarrow R^n \times R^m$ with the norm $\|(x,u)\| = \|x\| + \|u\|$, where $\|x\| = sup\{|x(t)| : t \in I\}$ and $\|u\| = sup\{|u(t)| : t \in I\}$. For brevity, we denote

$$a_1 = \sup_{t\in I} \|E_\alpha(\alpha A A_\alpha t^\alpha)x_0\|, \qquad a_2 = \sup_{t\in I} \|E^*_{\alpha,\alpha}(\alpha A A_\alpha (b-t)^\alpha)\|,$$
$$b_1 = (1-\alpha)\|A_\alpha\| + b^\alpha M(\alpha)\|A_\alpha^2\|a_2, \qquad b_2 = a_2\|B^*\|\|W^{-1}\|\|A_\alpha^{-1}\|b_1,$$
$$d_1 = a_2\|B^*\|\|W^{-1}\|\|A_\alpha^{-1}\|(|x_1| + M(\alpha)\|A_\alpha\|a_1), \qquad d_2 = 4M(\alpha)\|A_\alpha\|a_1,$$
$$a = \max\{b_1\|B\|, 1\}, \quad d_3 = 4ad_1, \qquad c_1 = 4ab_2, \quad c_2 = 4b_1,$$
$$c = \max\{c_1, c_2\}, \quad d = \max\{d_2, d_3\}, \qquad \sup\|f\| = \sup\{|f(s, z(s), v(s))| : s \in I\}.$$

Then, one sufficient condition of controllability for nonlinear system (2) will be given below.

Theorem 3. *Assume that linear system (1) is controllable on I and the continuous function f satisfies the following condition:*

$$\lim_{\|(x,u)\|\to\infty} \frac{\|f(t,x,u)\|}{\|(x,u)\|} = 0 \tag{11}$$

uniformly in I. Then system (2) is controllable on I.

Proof. Define the operator $T : G \rightarrow G$ by $T(x,u) = (z,v)$, where

$$\begin{aligned} z(t) =& M(\alpha)A_\alpha E_\alpha(\alpha A A_\alpha t^\alpha)x_0 + (1-\alpha)A_\alpha Bu(t) + (1-\alpha)A_\alpha f(t, x(t), u(t)) \\ &+ \alpha M(\alpha)A_\alpha^2 \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\alpha A A_\alpha (t-\tau)^\alpha) Bu(\tau) d\tau \\ &+ \alpha M(\alpha)A_\alpha^2 \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\alpha A A_\alpha (t-\tau)^\alpha) f(\tau, x(\tau), u(\tau)) d\tau \end{aligned} \tag{12}$$

and

$$\begin{aligned} v(t) =& B^* E^*_{\alpha,\alpha}(\alpha A A_\alpha (b-t)^\alpha) W_1^{-1} A_\alpha^{-1} [x_1 - M(\alpha)A_\alpha E_\alpha(\alpha A A_\alpha b^\alpha)x_0 - (1-\alpha)A_\alpha f(b, x(b), v(b)) \\ &- \alpha M(\alpha)A_\alpha^2 \int_0^b (b-\tau)^{\alpha-1} E_{\alpha,\alpha}(\alpha A A_\alpha (b-\tau)^\alpha) f(\tau, x(\tau), u(\tau)) d\tau]. \end{aligned} \tag{13}$$

According to Theorem 2, we know that if the operator T exists, a fixed point such that $(x,u) = (z,v)$, then there exists $u(t)$ such that $x(b) = x_1$, i.e., system (2) is controllable. Thus, the controllable problem is translated into proving that T have a fixed point.

Indeed, by Equations (12) and (13), we have

$$\begin{aligned} \|v(t)\| \leq& a_2\|B^*\|\|W^{-1}\|\|A_\alpha^{-1}\|(|x_1| + M(\alpha)\|A_\alpha\|a_1 + b_1 \sup\|f\|) \\ =& d_1 + b_2 \sup\|f\| \\ =& \frac{d_3}{4a} + \frac{c_1}{4a}\sup\|f\| \\ \leq& \frac{1}{4a}(d + c\sup\|f\|) \end{aligned} \tag{14}$$

and

$$\begin{aligned}\|z(t)\| \leq & \frac{d_2}{4} + b_1\|B\||v(t)| + b_1 \sup \|f\| \\ \leq & \frac{d_2}{4} + \frac{b_1\|B\|}{4a}(d + c\sup|f|) + \frac{c_2}{4}\sup\|f\| \\ \leq & \frac{d}{2} + \frac{c}{2}\sup\|f\|.\end{aligned} \tag{15}$$

According to Lemma 2, we know that for the above given positive constants c and d, there exists a positive constant r such that, if $\|(x,u)\| \leq r$, then $c\|f(t,x,u)\| + d \leq r$, for all $t \in I$. Define the subspace of G as $G_r = \{(x,u) \in G : \|x\| \leq \frac{r}{2}, \|u\| \leq \frac{r}{2}\}$. Obviously, $\|x\| + \|u\| \leq r$. It follows that $c\|f(t,x,u)\| + d \leq r$ for all $t \in I$. By Equations (14) and (15), we deduce that $\|v\| \leq \frac{r}{4a} \leq \frac{r}{2}$, $\|z\| \leq \frac{r}{2}$. Thus, we obtain that T maps $G(r)$ into itself. In addition, since f is continuous and f is uniformly bounded for all $t \in I$, it follows that T is continuous and uniformly bounded on I. It is easy to prove that T is equicontinuous. Thus, by the famous Ascoli's theorem, we know that T is compact. Due to G_r being closed, bounded and convex, one can come to the conclusion that that T has a fixed point $(x,u) \in G_r$ such that $T(x,u) = (x,u) = (z,v)$ by Schauder fixed point theorem. Therefore, system (2) is controllable on I. The proof is completed. □

5. Illustrated Examples

In this section, an example is given to illustrate the applicability of Theorem 2 and Theorem 3.

Example 1. *Consider the following linear fractional dynamical system with a Mittag–Leffler kernel*

$$\begin{cases} {}^{ABC}D_t^{\alpha}x(t) = Ax(t) + Bu(t), \quad t \in [0,1], \\ x(0) = x_0, \end{cases} \tag{16}$$

with $\alpha = \frac{1}{2}$, $M(\alpha) = 1$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. *By simple calculation we get*

$$A_\alpha = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

and

$$W_1 = \begin{pmatrix} \frac{1}{4\pi} + \frac{35}{16\sqrt{\pi}} + \frac{1}{24} & \frac{1}{4\pi} + \frac{9}{8\sqrt{\pi}} \\ \frac{1}{2\pi} + \frac{9}{8\sqrt{\pi}} & \frac{1}{2\pi} + \frac{1}{2\sqrt{\pi}} \end{pmatrix}.$$

It is obvious that A_α and W_1 are nonsingular, which satisfies the condition of Theorem 2. Thus, the linear system of (16) is controllable on $[0,1]$.

Now, consider the following nonlinear fractional dynamical system with a Mittag–Leffler kernel

$$\begin{cases} {}^{ABC}D_t^{\alpha}x(t) = Ax(t) + Bu(t) + t\sin x(t) + \cos u(t), \quad t \in [0,1], \\ x(0) = x_0, \end{cases} \tag{17}$$

with $\alpha = \frac{1}{2}$, $M(\alpha) = 1$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. *Let* $f(t,x(t),u(t)) = t\sin x(t) + \cos u(t)$ *which satisfies the hypothesis of Equation (11) in Theorem 3 obviously. Thus, the nonlinear system (17) is controllable on* $[0,1]$.

6. Conclusions

This paper deals with the controllability of linear and nonlinear fractional dynamical systems with a Mittag–Leffler kernel. Firstly, the solution representation of fractional dynamical systems with a Mittag–Leffler kernel is given by the Laplace transform method. Secondly, one necessary and sufficient condition for controllability of linear fractional dynamical systems with Mittag–Leffler kernel are established. Thirdly, one sufficient condition to guarantee controllability of nonlinear fractional dynamical systems with a Mittag–Leffler kernel by Schauder fixed point theorem. Finally, we provide an example to illustrate the effectiveness of our results. In addition, since the interesting non-singular kernel of this new fractional derivative, there is much work to be done on this type of fractional calculus that is worth discussing in the future, such as trying to use some other fixed point theorem to solve the controllability problem of solutions for dynamical systems and considering fractional dynamical systems with a Mittag–Leffler kernel and delay.

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