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A Remark on Quadrics in Projective Klingenberg Spaces over a Certain Local Algebra

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Abstract: This article is devoted to some polar properties of quadrics in the projective Klingenberg spaces over a local ring which is a linear algebra generated by one nilpotent element. In this case, polar subspaces are described; the notion "degree of neighborhood" is used for the geometric description of polar subspaces of quadrics. The polarity induced by a quadric is also studied.

Keywords: projective Klingenberg space; point; hyperplane; local ring; plural algebra; A-space; neighbor points; order of the quadratic form; quadric; polarity; characteristic of the quadratic form

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1. Introduction

Projective Klingenberg spaces (PKS) are incidence structures whose homomorphic image is a projective space over a field. W. Klingenberg [1] started studying these structures (originally *projective planes with homomorphisms*) as a special case of ring geometry in the mid-20th Century; PKS of a general dimension $n, n \geq 2$ was introduced by H.H. Lück [2]. In the 1980s, F. Machala [3] introduced projective Klingenberg spaces over local rings. The arithmetical fundament of such spaces is a free finite dimensional A-module over a local ring A (A-space in the sense of B.R. McDonald [4]). Projective geometry is also related to the theory of geodesic mappings (see, e.g., in [5]).

In the case of PKS over certain local rings (*plural algebras* [6]), we may study in more detail the structure of PKS and we can find some special properties—some of these are presented in [7], where "linear subsets" of KPS were described, while this article is devoted to some polar properties of quadrics in KPS over plural algebra. We present some geometric interpretation of certain "algebraic" properties of quadrics and quadratic forms in such case.

Now, to make the paper self-contained, we remind some properties of KPS over the following local algebra A.

Definition 1. [6] *A plural algebra of order m over a field T is every linear algebra A on T having as a vector space over T a basis:*

$$\{1, \eta, \eta^2, \ldots, \eta^{m-1}\} \text{ with } \eta^m = 0. \tag{1}$$

Remark 1. *It follows from Definition 1 that any element α of A may be uniquely expressed in the form*

$$\alpha = \sum_{j=0}^{m-1} a_j \eta^j.$$

A is a local ring with the maximal ideal $\mathfrak{a}=\eta A$ and all proper ideals of A are just $\eta^j A$, $1 \leq j \leq m$. Evidently, $\{0\} = \eta^m A \subset \eta^{m-1} A \subset \cdots \subset \eta^1 A \subset \eta^0 A = A$.

Furthermore, A has following properties,

1. *the ring A is isomorphic to the factor ring of polynomials $T[x]/(x^m)$,*
2. *the ring A is isomorphic to the ring of all $m \times m$ matrices over T of the following form,*

$$\begin{pmatrix} b_0 & b_1 & \dots & b_{m-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & b_0 & b_1 \\ 0 & \dots & 0 & b_0 \end{pmatrix}.$$

Definition 2. *An incidence structure (plane, space) is understood to be any triple (P, H, I), where P, $H \neq \emptyset$, $I \subseteq P \times H$. Elements of the set P are called points, elements of the set H are called hyperplanes, and I is called the incidence relation; instead of $(X, \mathcal{H}) \in I$ we will also write $X \, I \, \mathcal{H}$.*

If $\mathcal{P}=(P, H, I)$ and $\mathcal{P}'=(P', H', I')$ are incidence structures then a homomorphism of $\mathcal{P}$ to $\mathcal{P}'$ is understood to be any mapping $\mu\colon P \cup H \to P' \cup H'$ such that $\operatorname{Im} \mu|P \subseteq P'$, $\operatorname{Im} \mu|H \subseteq H'$ and $\forall X \in P, \forall \mathcal{H} \in H\colon (X \, I \, \mathcal{H}) \Rightarrow (\mu(X) \, I' \, \mu(\mathcal{H}))$.

According to [8], let us define the following.

Definition 3. *A projective Klingenberg space of dimension n, $n \geq 2$, is an incidence structure $\mathcal{P} = (P, H, I)$ with a homomorphism μ of $\mathcal{P}$ onto an n-dimensional projective space $\mathcal{P}_0 = (P_0, H_0, I_0)$ such that*

1. *If $X_1, \dots, X_k$, $1 \leq k \leq n$, are points in P such that $\mu(X_1), \dots, \mu(X_k)$ are independent in P_0, then there exists a hyperplane $\mathcal{H}$ in H such that $X_1, \dots, X_k \, I \, \mathcal{H}$. This hyperplane is unique if $k = n$.*
2. *This condition is dual of the condition 1.*
3. *If $X_1, \dots, X_{n-1} \in P$ and $\mathcal{H}_1, \mathcal{H}_2 \in H$ are such that $\mu(X_1), \dots, \mu(X_{n-1})$ are independent, $\mu(H_1), \mu(H_2)$ are independent and $X_1, \dots, X_{n-1} \, I \, \mathcal{H}_1, \mathcal{H}_2$, then $Y \, I \, \mathcal{H}_1, \mathcal{H}_2$ and $X_1, \dots, X_{n-1} \, I \, \mathcal{H}$ imply $Y \, I \, \mathcal{H}$.*

Definition 4. *Points $X, Y \in \mathcal{P}$ are called neighbors, if $\mu(X) = \mu(Y)$. Otherwise, we speak of non-neighbor points.*

Let us remind the reader of a definition of a *(coordinate) projective Klingenberg space over the ring* A (according to Machala [3]). For $n \geq 3$, any projective Klingenberg space is isomorphic to a certain projective Klingenberg space over a local ring. In the case of planes, it is true only for Desarguesian ones.

Through this paper, by the symbol $[G]$ we will denote the linear span of a set G; the symbol $\langle \boldsymbol{x} \rangle$ will denote the coset determined by an element $\boldsymbol{x}$.

Definition 5. *Let A be a local ring and $\mathfrak{a}$ be its maximal ideal. Let us denote $M = A^{n+1}$, $n \geq 2$, $\bar{M} = M/\mathfrak{a}M$. Then, an incidence structure $\mathcal{P}_A$ such that*

1. *points in $\mathcal{P}_A$ are just all submodules $[\boldsymbol{x}] \subseteq M$ such that $\langle \boldsymbol{x} \rangle$ is a nonzero element of $\bar{M}$,*
2. *hyperplanes in $\mathcal{P}_A$ are just all submodules $[\boldsymbol{u}_1, \boldsymbol{u}_2 \dots \boldsymbol{u}_n] \subseteq M$ such that $\langle \boldsymbol{u}_1 \rangle, \langle \boldsymbol{u}_2 \rangle, \dots, \langle \boldsymbol{u}_n \rangle$ are linearly independent elements of $\bar{M}$,*
3. *the incidence relation is an inclusion,*

is called an n-dimensional projective Klingenberg space over the ring A.

For any point $X = [\boldsymbol{x}]$ of $\mathcal{P}_A$, an element $\boldsymbol{x}$ is called an arithmetical representative of the point X. The module M is called the arithmetical fundament of the space $\mathcal{P}_A$, any of basis of M is called an arithmetical basis of $\mathcal{P}_A$.

Let us remark that the homomorphic image of $\mathcal{P}_A$ is defined to be the n-dimensional projective space $\mathcal{P}_0$ over the field $A/\mathfrak{a}$ (with an arithmetical fundament $\bar{M}$). The respective homomorphism μ is defined by

$$\forall X, X = [\boldsymbol{x}],\ \boldsymbol{x} \in M\colon\ \mu(X) = [\langle \boldsymbol{x} \rangle].$$

The following definition is natural.

Definition 6. *Let $\mathcal{P}_A$ be an n-dimensional projective Klingenberg space and M be its arithmetical fundament. Let a submodule K of M be given. A set*

$$\mathcal{K} = \{X \in \mathcal{P}_A,\ X{=}[\boldsymbol{x}]:\ \boldsymbol{x} \in K\} \tag{2}$$

is called a k-dimensional subspace in $\mathcal{P}_A$, $0 \leq k \leq n{-}1$, if $K = [\boldsymbol{u}_1, \boldsymbol{u}_2 \dots \boldsymbol{u}_{k+1}]$, where $\langle \boldsymbol{u}_1 \rangle$, $\langle \boldsymbol{u}_2 \rangle, \dots, \langle \boldsymbol{u}_{k+1} \rangle$ are linearly independent elements of $\bar{M}$.

The submodule K is called an arithmetical fundament of the subspace $\mathcal{K}$.

Using this definition, we have that points and hyperplanes (according to Definition 5) correspond to the cases $k = 0$ and $k = n - 1$, respectively.

Through the the rest of the paper, we study an n-dimensional PKS $\mathcal{P}_A$ over a plural algebra A (the generator of A is denoted by η) with arithmetical fundament denoted by M (Definition 5). Obviously, the module M is an A-space.

Now, subspaces of $\mathcal{P}_A$ may be characterized as follows:

Theorem 1. [9] *Let $\mathcal{P}_A$ be a projective Klingenberg space. Then k-dimensional subspaces of $\mathcal{P}_A$, $0 \leq k \leq n-1$, are just all subsets (2) such that K is a $(k{+}1)$-dimensional A-subspace in M.*

(Let us remark, that Theorem 1 holds not only for KPSs over the plural algebra A, it follows from [9] (cf. the proof of Lemma 1) that it holds true also in cases, when in the respective A-module M (arithmetical fundament of $\mathcal{P}_A$), any linearly independent system of elements of M can be completed to a basis of M.)

Definition 7. [9] *Let $X{=}[\boldsymbol{x}]$ and $Y{=}[\boldsymbol{y}]$ be points of a projective Klingenberg space $\mathcal{P}_A$ and let r be a non-negative integer satisfying:*

$$(\eta^{m-r}\boldsymbol{x} \in [\boldsymbol{y}])\ \wedge\ (\eta^{m-r-1}\boldsymbol{x} \notin [\boldsymbol{y}] \vee r = m).$$

The number r is called the degree of neighborhood of the points X and Y.

Remark 2. *For a couple of non-neighbor points we have $r = 0$, for neighbor but distinct points $1 \leq r \leq m - 1$ and for identical points $r = m$.*

Definition 8. [7] *Let X be a point and $\mathcal{K}$ be a subset of points of a projective Klingenberg space. We say that r is a degree of neighborhood of X and $\mathcal{K}$ if there exists at least one point $Y \in \mathcal{K}$ such that the degree of neighborhood of points X, Y is equal to r and the degree of neighborhood of X and any point of $\mathcal{K}$ is not greater than r.*

Remark 3. *If $\mathcal{K}$ is a subspace of $\mathcal{P}_A$ and K is an arithmetical fundament of $\mathcal{K}$, then the degree of neighborhood of a point $X{=}[\boldsymbol{x}]$ and subspace $\mathcal{K}$ is equal to r if and only if*

$$(\eta^{m-r}\boldsymbol{x} \in K)\ \wedge\ (\eta^{m-r-1}\boldsymbol{x} \notin K \vee r = m).$$

Let us recall (see [10]) that any linear form $\varphi\colon M \to A$ may be written in the form $\varphi = \eta^h \varphi_0$, where φ_0 is a linear form with $\mathrm{Im}\varphi_0 \not\subset \mathfrak{a}$ and h, $0 \leq h \leq m$, is uniquely determined integer (called the *order of the linear form* φ).

Theorem 2. [7] *Let $\mathcal{P}_A$ be a projective Klingenberg space. Let φ be an arbitrary linear form on M of order k. Then, the set*

$$\mathcal{H} = \{X \in \mathcal{P}_A, X = [\boldsymbol{x}] \colon \boldsymbol{x} \in \operatorname{Ker}\varphi\}$$

is formed by all points with the degree of neighborhood at least $m-k$ to a certain hyperplane $\mathcal{H}_0$ of $\mathcal{P}_A$. If φ_0 is a form of zero order such that $\varphi = \eta^k\varphi_0$, then

$$\mathcal{H}_0 = \{X \in \mathcal{P}_A, X = [\boldsymbol{x}] \colon \boldsymbol{x} \in \operatorname{Ker}\varphi_0\}.$$

2. Quadrics in Projective Klingenberg Spaces

The notions of bilinear and quadratic forms will be used in the usual sense.

A quadratic form determines a quadric. Any quadratic form on an A-space over the algebra A has two important algebraic characteristics—an *order* (Definition 9) and a *characteristic* (Definition 11). Naturally, these notions may be assigned to a quadric (determined by a given quadratic form). Then, there is a question how these algebraic properties may be described from a geometric point of view. Geometric interpretations of them will be found in this section.

Let $\Phi : M \times M \to A$ be a symmetric bilinear form on the A-space M. Then, Φ_q denotes the quadratic form $M \to A$ determined by the form Φ (*polar bilinear form* of the quadratic form Φ_q), i.e., $\forall \boldsymbol{x} \in M \colon \Phi_q(\boldsymbol{x}) = \Phi(\boldsymbol{x}, \boldsymbol{x})$.

The image of any bilinear and quadratic form has the following algebraic characterization (cf. [11]).

Definition 9. *A nonzero bilinear form $\Phi : M \times M \to A$ is called a bilinear form of order k, $0 \le k \le m-1$, if*

$$(\operatorname{Im}\Phi \subseteq \eta^k A) \wedge (\operatorname{Im}\Phi \not\subseteq \eta^{k+1} A); \tag{3}$$

the order of a zero bilinear form is defined to be equal to m.

By the order of a quadratic form we mean the order of its polar bilinear form.

Theorem 3. [11] *If Φ is a bilinear form of order k, then there exists at least one bilinear form Φ_0 of zero order such that $\Phi = \eta^k\Phi_0$.*

The notion of a quadric in KPS $\mathcal{P}_A$ will be defined in the natural way.

Definition 10. *Let a quadratic form Φ_q on M be given. Then, the set Q_{Φ_q} defined by*

$$Q_{\Phi_q} = \{X \in \mathcal{P}_A, X = [\boldsymbol{x}] \colon \; \Phi_q(\boldsymbol{x}) = 0\}$$

is called a quadric in $\mathcal{P}_A$ (determined by the quadratic form Φ_q).

In [11], the existence of a basis of M polar with respect to arbitrary quadratic form is proved (the notion of *polar basis* is used in the usual sense, i.e., it is any basis of M so that the matrix of given quadratic form with respect to this basis is diagonal). If a quadric Q_{Φ_q} is given, then a basis of M polar with respect to Φ_q is called an *arithmetical basis of $\mathcal{P}_A$ polar with respect to the quadric Q_{Φ_q}*.

In what follows, we will consider that A is a complex plural algebra, i.e., $T = \mathbb{C}$ (Definition 1). As for every unit $\alpha \in A$ there exists a unit $\beta \in A$ with $\alpha = \beta^2$, any polar basis may be "normalized" and the following theorem holds.

Theorem 4. *Let Φ_q be a quadratic form on M. Then, there exists at least one basis $\mathcal{U}$ of M such that the matrix of Φ with respect to $\mathcal{U}$ is equal to* $\operatorname{Diag}(a_{00}, a_{11}, \ldots a_{nn})$ *with*

$$\forall i = 0, \ldots, n : a_{ii} \in \{1, \eta, \eta^2, \ldots, \eta^{m-1}, 0\}.$$

Definition 11. [11] *Let Φ_q be a quadratic form on M and let $\mathcal{U}$ be a basis of M polar with respect to Φ. Let us define a set of integers $p_0, \ldots, p_{m-1}$ as follows,*

$$p_k = \mathrm{card}(\{\boldsymbol{u} \in \mathcal{U} : \ \Phi_q(\boldsymbol{u}) \in \eta^k A \backslash \eta^{k+1} A\}), \ 0 \leq k \leq m-1.$$

Then $\mathfrak{Ch}(\Phi, \mathcal{U}) = (p_0, ..., p_{m-1})$ is called a characteristic of the quadratic form Φ_q with respect to the basis $\mathcal{U}$.

In [11], it is proved that for arbitrary bases $\mathcal{U}$, $\mathcal{V}$ is polar with respect to the same quadratic form Φ_q it holds: $\mathfrak{Ch}(\Phi, \mathcal{U}) = \mathfrak{Ch}(\Phi, \mathcal{V})$ *(invariance of the characteristic)*. Further, it may be shown that two quadratic forms Φ_q, Ψ_q determine the same quadric in $\mathcal{P}_A$ if and only if there exists a unit $\alpha \in A \backslash \mathfrak{a}$ such that $\Phi_q = \alpha \Psi_q$. Therefore, the following definition is correct.

Definition 12. *Let a quadric Q in $\mathcal{P}_A$ be given. Let Φ_q be a quadratic form with $Q = Q_{\Phi_q}$ and $\mathfrak{Ch}(\Phi, \mathcal{U})$ be a characteristic of Φ_q with respect to an arbitrary arithmetical basis $\mathcal{U}$ of $\mathcal{P}_A$ polar with respect to Q. Then, the characteristic $\mathfrak{Ch}(\Phi, \mathcal{U})$ is called a characteristic of the quadric Q and it is denoted by $\mathfrak{Ch}(Q)$.*

The correctness of the following definition follows from the note before the Definition 12.

Definition 13. *Let a quadric $Q = Q_{\Phi_q}$ in $\mathcal{P}_A$ be given. Then, an order of the polar bilinear form Φ is called an order of the quadric Q.*

Remark 4. *Let $Q = Q_{\Phi_q}$ be a quadric in $\mathcal{P}_A$ and k be its order. Then, there exist elements $\boldsymbol{u}, \boldsymbol{v} \in M$ with $\Phi(\boldsymbol{u}, \boldsymbol{v}) \in \eta^k A \backslash \eta^{k+1} A$. Are they representatives of some points of $\mathcal{P}_A$? As any element from A-space M may be written by $\boldsymbol{x} = \sum_{i=0}^{m-1} \eta^j \boldsymbol{x}_j$, where $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{m-1} \in M_0$, $M_0 \cong M / \mathfrak{a}M$ (see [10]), we have that $\boldsymbol{u}, \boldsymbol{v} \in M \backslash \mathfrak{a}M$ (the opposite case yields, for example, $\boldsymbol{u} = \eta \boldsymbol{u}'$, $\boldsymbol{u}' \in M$, and we get $\Phi(\boldsymbol{u}', \boldsymbol{v}) \notin \eta^k A$, which contradicts (3)). It means that $U = [\boldsymbol{u}], V = [\boldsymbol{v}]$ are points of $\mathcal{P}_A$ (cf. Definition 5).*

Let $\boldsymbol{z}$ be an element from M. Then the mapping $\varphi_{\boldsymbol{z}} : M \to A$ defined for every $\boldsymbol{x} \in M$ by

$$\varphi_{\boldsymbol{z}}(\boldsymbol{x}) = \Phi(\boldsymbol{z}, \boldsymbol{x}) \tag{4}$$

is a linear form on M.

Let us construct to the given quadric Q the set of all linear forms defined by (4), i.e., $\{\varphi_{\boldsymbol{z}}\}_{\boldsymbol{z} \in M}$ (this set is determined by Q uniquely up to a multiplication by a unit of A). From the consideration in this remark and from Definition 9, it follows that all of these forms are of order $h, h \geq k$, and at least one of them—e.g., $\varphi_{\boldsymbol{u}}$—has order equal to k.

The following notion is a natural generalization of the notion *polar subspace of a quadric and a given point* (as it is known in projective geometry over field).

Definition 14. *Let a quadric $Q = Q_{\Phi_q}$ in $\mathcal{P}_A$ be given. Let $Y = [\boldsymbol{y}]$ be a point of $\mathcal{P}_A$. Then, the set*

$$\pi(Q, Y) = \{X \in \mathcal{P}_A, \ X = [\boldsymbol{x}] : \ \Phi(\boldsymbol{x}, \boldsymbol{y}) = 0\}$$

is called a polar submodule of a quadric Q and a point Y.

Remark 5. *Let us remind that the notion of submodule of $\mathcal{P}_A$ is in [7] defined as a set of points of $\mathcal{P}_A$ the arithmetical representatives of which belong to a submodule of the arithmetical fundament M of $\mathcal{P}_A$. Let $Y = [\boldsymbol{y}] \in \mathcal{P}_A$. We clearly see that*

$$\pi(Q, Y) = \{X \in \mathcal{P}_A, \ X = [\boldsymbol{x}] : \ \boldsymbol{x} \in \mathrm{Ker}\, \varphi_{\boldsymbol{y}}\}; \tag{5}$$

it means that $\pi(Q, Y)$ is a submodule of $\mathcal{P}_A$.

The following Theorems 5 and 6 bring a geometric interpretation of the order of a quadric.

Theorem 5. *Let Q be a quadric in $\mathcal{P}_A$. The order of Q is equal to zero if and only if there exists a point $Y \in \mathcal{P}_A$ such that the polar submodule $\pi(Q, Y)$ is a hyperplane of $\mathcal{P}_A$.*

Proof. Let $Q = Q_{\Phi_q}$. According to Remark 4, there exists at least one element $\boldsymbol{y} \in M \backslash \mathfrak{a}M$ such that the order of a linear form $\varphi_{\boldsymbol{y}}$ is equal to zero. Let us consider a point $Y = [\boldsymbol{y}]$. Using Theorem 2, Remark 2, and relation (5), we get $\pi(Q, Y) = \mathcal{H} = \mathcal{H}_0$, where $\mathcal{H}_0$ is a hyperplane of $\mathcal{P}_A$ and $Ker\varphi_{\boldsymbol{y}}$ is its arithmetical fundament. □

Theorem 6. *Let Q be a quadric in $\mathcal{P}_A$. The order of Q is equal to k, $1 \leq k \leq m-1$, if and only if*

1. *there exists a point $Y \in \mathcal{P}_A$ such that the polar submodule $\pi(Q, Y)$ is formed by all points with the degree of neighborhood at least $m-k$ to a certain hyperplane $\mathcal{H}_0$ of $\mathcal{P}_A$;*
   if $Y = [\boldsymbol{y}]$ and φ_0 is a linear form of zero order with $\varphi_{\boldsymbol{y}} = \eta^k \varphi_0$, then for the hyperplane $\mathcal{H}_0$ it holds $\mathcal{H}_0 = \{X \in \mathcal{P}_A, X = [\boldsymbol{x}] \colon \boldsymbol{x} \in \operatorname{Ker} \varphi_0\}$;
2. *there is no point $Z \in \mathcal{P}_A$ such that the polar submodule $\pi(Q, Z)$ is formed by all points with the degree of neighborhood at least $m-k+1$ to some hyperplane of $\mathcal{P}_A$.*

Proof. Let $Q = Q_{\Phi_q}$. Using Remark 4 and examining the system of linear forms $\{\varphi_{\boldsymbol{z}}\}_{\boldsymbol{z} \in M}$, we see that Q is of order k, $1 \leq k \leq m-1$, if and only if there exists at least one element $\boldsymbol{y} \in M \backslash \mathfrak{a}M$ such that the order of a linear form $\varphi_{\boldsymbol{y}}$ is equal to k and there is no element $\boldsymbol{z} \in M$ such that the order of $\varphi_{\boldsymbol{z}}$ is less than k.

Let us consider a point $Y = [\boldsymbol{y}]$. First, let us construct a linear form φ_0 of zero order such that $\varphi_{\boldsymbol{y}} = \eta^k \varphi_0$ (cf. a note before Theorem 2). Then, using the relation (5) and Theorem 2 we obtain a submodule $\mathcal{H} \subseteq \mathcal{P}_A$ with $\pi(Q, Y) = \mathcal{H}$ and a hyperplane $\mathcal{H}_0 = \{X \in \mathcal{P}_A, X = [\boldsymbol{x}] \colon \boldsymbol{x} \in \operatorname{Ker} \varphi_0\}$ such that $\pi(Q, Y)$ is formed by the set of points of $\mathcal{H}_0$ and all points of the degree of neighborhood at least $m-k$ to it.

As there is no linear form $\varphi_{\boldsymbol{z}}$ of order less than k, there is no point $Z \in \mathcal{P}_A$ and no hyperplane of $\mathcal{P}_A$ such that a submodule $\pi(Q, Z)$ is formed only by points of degree of neighborhood at least $m-k+1$ to a hyperplane. □

In the last part of the article we will find the link between the polarity of a KPS induced by a quadric and a characteristic of the quadric.

Definition 15. *Let a quadric $Q = Q_{\Phi_q}$ in $\mathcal{P}_A$ be given. Then, a mapping π_Q of the set of points of $\mathcal{P}_A$ to the set of submodules of a space $\mathcal{P}_A$, which assigns a polar submodule $\pi(Q, Y)$ to any point $Y = [\boldsymbol{y}] \in \mathcal{P}_A$ is called a polarity on $\mathcal{P}_A$ induced by a quadric Q.*

There is a natural question whether a polarity may be a bijection of the set of points onto the set of hyperplanes of $\mathcal{P}_A$.

Theorem 7. *Let Q be a quadric in $\mathcal{P}_A$. A polarity on $\mathcal{P}_A$ induced by a quadric Q is a bijection of the set of points of $\mathcal{P}_A$ onto the set of hyperplanes of $\mathcal{P}_A$ if and only if $\mathfrak{Ch}(Q) = (n+1, 0, \ldots, 0)$.*

Proof. Let $Q = Q_{\Phi_q}$ be a quadric and $Y = [\boldsymbol{y}]$ be a point in $\mathcal{P}_A$. Let us choose an arithmetical basis $\mathcal{U}$ of $\mathcal{P}_A$. If a point Y has homogeneous coordinates $(y_0, y_1, \ldots, y_n)$ over $\mathcal{U}$ (a system of homogeneous coordinates in the space $\mathcal{P}_A$ is considered in the usual way: a point $X \in \mathcal{P}_A$ has over over $\mathcal{U}$ coordinates $(x_0, x_1, \ldots, x_n)$—which is denoted by $X = (x_0, x_1, \ldots, x_n)$—if and only if $X = [\boldsymbol{x}]$ and an element

$\boldsymbol{x} \in M$ has coordinates $(x_0, x_1, \ldots, x_n)$ over $\mathcal{U}$), then a polar submodule $\pi(Q, Y)$ is according to (5) given by the following relation:

$$\forall X{=}(x_0, \ldots, x_n){\in}\mathcal{P}_A : \; X \in \pi(Q, Y) \Leftrightarrow (y_0, \ldots, y_n)(Q,\mathcal{U})(x_0, \ldots, x_n)^T = 0, \tag{6}$$

where $(Q,\mathcal{U})$ is a matrix of Φ_q with respect to a basis $\mathcal{U}$.

Let $\mathcal{H}$ be an arbitrary hyperplane in $\mathcal{P}_A$. Then, there exists a linear form φ of zero order such that $\mathcal{H} = \{X \in \mathcal{P}_A, X = [\boldsymbol{x}] : \boldsymbol{x} \in \operatorname{Ker}\varphi\}$ (cf. 2). Thus, a point $X{=}(x_0, \ldots, x_n){\in}\mathcal{P}_A$ belongs to $\mathcal{H}$ if and only if

$$\sum_{i=0}^{n} a_i x_i = 0, \tag{7}$$

where $a_0, \ldots, a_n$ are coefficients of a linear form φ in a basis $\mathcal{U}$.

As φ is of zero order, at least one of $a_0, \ldots, a_n$ does not belong to the ideal $\mathfrak{a}$. Respecting the fact that an $(n+1)$-tuple $(y_0, y_1, \ldots, y_n)$ of elements of A may represent homogeneous coordinates of some point of $\mathcal{P}_A$ only if at least one $y_i \notin \mathfrak{a}$, from (6) and (7) we obtain that there exists a point Y with $\mathcal{H} = \pi(Q, Y)$ if and only if a matrix $(Q,\mathcal{U})$ is invertible (the unicity of a point Y is in this case evident).

A matrix over the ring A is invertible if and only if its determinant belongs to $A \backslash \mathfrak{a}$. Using Theorem 4 and Definition 11, we see that the matrix $(Q,\mathcal{U})$ is invertible if and only if $\mathfrak{Ch}(Q) = (\dim M, 0, \ldots, 0)$. $\square$

The notion of the "degree of neighborhood" has shown up as a key one for a pure geometric description of the set of points conjugated with respect to the given quadric in KPS $\mathcal{P}_A$ to a given point(*a polar submodule*). In general, such set does not have to be a hyperplane, but it is formed by points with a certain degree of neighborhood to the hyperplane. This "certain degree" is determined by an algebraic property of an image of the quadratic form associated with the given quadric (an *order of quadric*). The polarity of a KPS induced by a quadric does not have to be a bijection of the set of points of KPS onto the set of hyperplanes. We found a sufficient and necessary condition for the polarity to be such a bijection.

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