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Some Relationships for the Generalized Integral Transform on Function Space

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Abstract: In this paper, we recall a more generalized integral transform, a generalized convolution product and a generalized first variation on function space. The Gaussian process and the bounded linear operators on function space are used to define them. We then establish the existence and various relationships between the generalized integral transform and the generalized convolution product. Furthermore, we obtain some relationships between the generalized integral transform and the generalized first variation with the generalized Cameron–Storvick theorem. Finally, some applications are demonstrated as examples.

Keywords: generalized integral transform; generalized convolution product; bounded linear operator; Gaussian process; Cameron–Storvick theorem; translation theorem

MSC: 47A60; 60J65; 28C20

1. Introduction

For $T > 0$, let $C_0[0,T]$ be the one-parameter Wiener space and let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$. Let m denote Wiener measure. Then, the space $(C_0[0,T], \mathcal{M}, m)$ is complete, and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)dm(x).$$

Let $K \equiv K_0[0,T]$ be the space of all complex-valued continuous functions defined on $[0,T]$ which vanishes at $t = 0$ and whose real and imaginary parts are elements of $C_0[0,T]$.

In [1], Lee studied an integral transform of analytic functionals on abstract Wiener spaces

$$\mathcal{F}_{\gamma,\beta}(F)(y) = \int_{C_0[0,T]} F(\gamma x + \beta y)dm(x), \qquad y \in K. \tag{1}$$

For some parameters γ and β and for certain classes of functionals, the Fourier–Wiener transform, the modified Fourier–Wiener transform, the analytic Fourier–Feynman transform and the Gauss transform are popular examples of the integral transform defined by (1) above (see [1–12]). Researchers have studied some theories of integral transform for functionals on function space. Recently, the integral transform is generalized by some methods in various papers. One of them uses the concept of Gaussian process instead of the ordinary process. For a function h on $[0,T]$, the Gaussian process is defined by the formula

$$Z_h(x,t) = \int_0^t h(s)\tilde{d}x(s)$$

where $\int_0^t h(s)\tilde{d}x(s)$ the Paley–Wiener–Zygmund (PWZ) stochastic integral. Many mathematician use this process to generalize the integral. As representative examples, the generalized integral transforms

$$\mathcal{F}^{h}_{\gamma,\beta}(F)(y) = \int_{C_0[0,T]} F(\gamma Z_h(x,\cdot) + \beta y)dm(x) \tag{2}$$

and

$$\mathcal{F}^{h_1,h_2}_{\gamma,\beta}(F)(y) = \int_{C_0[0,T]} F(\gamma Z_{h_1}(x,\cdot) + \beta Z_{h_2}(y,\cdot))dm(x) \tag{3}$$

are studied in [13–15]. In fact, if h, h_1 and h_2 are identically 1 on $[0, T]$, then Equations (2) and (3) reduce to Equation (1).

Another method is using the operators on K. Let S and R be bounded linear operators on K. In [6,16], the authors used this operators to generalize the integral transforms. A more generalized form is given by

$$\mathcal{G}_{S,R}(F)(y) = \int_{C_0[0,T]} F(Sx + Ry)dm(x). \tag{4}$$

If R is a constant operator and $Sx = Z_h(x,\cdot)$ for some function h, then Equation (4) reduces to Equation (2), and hence it reduces to Equation (1) again. In previous studies, many relationships among the integral transform, the convolution and the first variation have been obtained. However, most of the results consist of fixed parameters.

In this paper, we use the both concepts, the Gaussian process and the operator, to define a more generalized integral transform, a generalized convolution product and a generalized first variation of functionals on function space. We then give some necessary and sufficiently conditions for holding some relationships between the generalized integral transforms and the generalized convolution products, and between the generalized integral transforms and the generalized first variations. In addition, some examples are given to illustrate usefulness for our formulas and results. By choosing the kernel functions and operators, all results and formulas in previous papers are corollaries of our results and formulas in this paper.

2. Definitions and Preliminaries

We first list some definitions and properties needed to understand this paper.

A subset B of $C_0[0,T]$ is called scale-invariant measurable if ρB is $\mathcal{M}$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is called a scale-invariant null set provided $m(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.) [17]. For $v \in L_2[0,T]$ and $x \in C_0[0,T]$, let $\langle v, x\rangle$ denote the Paley–Wiener–Zygmund (PWZ) stochastic integral. Then, we have the following assertions.

(i) For each $v \in L_2[0,T]$, $\langle v, x\rangle$ exists for a.e. $x \in C_0[0,T]$.
(ii) If $v \in L_2[0,T]$ is a function of bounded variation on $[0,T]$, $\langle v, x\rangle$ equals the Riemann–Stieltjes integral $\int_0^T v(t)dx(t)$ for s-a.e. $x \in C_0[0,T]$.
(iii) The PWZ stochastic integral $\langle v, x\rangle$ has the expected linearity property.
(iv) The PWZ stochastic integral $\langle v, x\rangle$ is a Gaussian process with mean 0 and variance $\|v\|_2^2$.

For a more detailed study of the PWZ stochastic integral, see [4,5,7–9,11–15,18].

Let

$$C_0' \equiv C_0'[0,T] = \left\{ v \in C_0[0,T] : v(t) = \int_0^t z_v(s)ds,\ z_v \in L_2[0,T] \right\}.$$

Then, C_0' is the Hilbert space with the inner product

$$(v_1, v_2)_{C_0'} = \int_0^T z_{v_1}(t) z_{v_2}(t)dt,$$

where $v_j(t) = \int_0^t z_{v_j}(s)ds$ for $j = 1,2$. Furthermore, we note that $C_0'[0,T] \subset C_0[0,T]$ and $(C_0'[0,T], C_0[0,T], m)$ is one example of the abstract Wiener space [1,16,19,20]. For $x \in C_0[0,T]$ and $v \in C_0'[0,T]$ with $v(t) = \int_0^t z_v(s)ds, z_v \in L_2[0,T]$, $(v,x)^\sim \equiv \langle z_v, x\rangle$ is a well-defined Gaussian random variable with mean 0 and variance $\|v\|_{C_0'}^2 = \|z_v\|_2^2$, where $(\cdot,\cdot)^\sim$ is the complex bilinear form on $K^* \times K$.

The following is a well-known integration formula which is used several times in this paper. For each $v \in C_0'$ with $v(t) = \int_0^t z_v(s)ds$,

$$\int_{C_0[0,T]} \exp\{(v,x)^\sim\}dm(x) = \exp\left\{\frac{1}{2}\|v\|_{C_0'}^2\right\} = \exp\left\{\frac{1}{2}\|z_v\|_2^2\right\}. \tag{5}$$

For each $v \in C_0'[0,T]$, let

$$\Phi_v(x) = \exp\{(v,x)^\sim\}. \tag{6}$$

These functionals are called the exponential functionals on $C_0[0,T]$. It is a well-known fact that the class

$$\mathcal{A} \equiv \{\Phi_v : v \in C_0'[0,T]\} \tag{7}$$

is a fundamental set in $L_2(C_0[0,T])$. Thus, there is a countable dense $\mathcal{S}(C_0[0,T]) = \{\Phi_{v_n}\}_{n=1}^\infty \equiv \{\Phi_n\}_{n=1}^\infty$ which is dense in $L_2(C_0[0,T])$. Thus, we have that, for each $F \in L_2(C_0[0,T])$,

$$F(x) = \lim_{n\to\infty} \sum_{j=1}^n a_j \Phi_{v_j}(x)$$

in the L_2-sense, where $\{a_j\}_{j=1}^\infty$ is a sequence of constants.

Let $\mathcal{L} \equiv \mathcal{L}(K)$ be the class of all bounded linear operators on K. Then, for each $v \in C_0'[0,T]$ and $S \in \mathcal{L}$,

$$(v, Sx)^\sim = (S^*v, x)^\sim$$

where S^* is the adjoint operator of S, see [16,19,21]. We state the conditions for the function h to obtain mathematically consistency as follows:

(i) For each $h \in L_\infty[0,T] \subset L_2[0,T]$,

$$\langle z_v, Z_h(x,\cdot)\rangle = \langle z_v h, x\rangle$$

where $v(t) = \int_0^t z_v(s)ds$ for some $z_v \in L_2[0,T]$ because, although $z_v \in L_2[0,T]$, $z_v h$ may not be an element of $L_2[0,T]$ for $h \in L_2[0,T]$.

(ii) Let

$$h(t) = \begin{cases} 0, & 0 \le t < T/2 \\ t+2, & T/2 \le t \le T \end{cases}.$$

Then, h is in $L_\infty[0,T]$ (and hence $h \in L_2[0,T]$). However, $Z_h(x,t)$ may not be a Gaussian process. A condition for h is needed. Let h be an element of $L_\infty[0,T]$ such that $m_L(supp\,(h)) = m_L(\{t \in [0,T] : h(t) \neq 0\}) = T$, where m_L is the Lebesgue measure. Then, we have $\|h\|_2 > 0$ and $Z_h(x,t)$ is a Gaussian process.

(iii) For each $h \in L_\infty[0,T]$ and $x \in C_0[0,T]$, $Z_h(x,t)$ is stochastically continuous but it is not continuous, namely $Z_h(x,t)$ may not element of $C_0[0,T]$. However, if h is a function of bounded variation on $[0,T]$, the Gaussian process $Z_h(x,t)$ is continuous and hence $SZ_h(x,\cdot)$ is well-defined for all $S \in \mathcal{L}$. Since for $v \in C_0'$ with $v(t) = \int_0^t z_v(s)ds$, $(v,x)^\sim = \langle z_v, x\rangle$, we have that

$$(v, SZ_h(x,\cdot))^\sim = (S^*v, Z_h(x,\cdot))^\sim = \langle z_{S^*v}, Z_h(x,\cdot)\rangle = \langle hz_{S^*v}, x\rangle. \tag{8}$$

(iv) Let $\mathcal{H} = \{h : [0,T] \to \mathbb{R} : h \in BV[0,T], m_L(supp\,(h)) = T\}$.

3. Generalization of the Integral Transform with Related Topics

We start this section by giving definition of generalized integral transform, generalized convolution product and the generalized first variation of functionals on K.

Definition 1. *Let h, h_1, h_2 be an element of $\mathcal{H}$ and let F and G be functionals on K. Let $S, R, A, B, C, D, S_1, S_2 \in \mathcal{L}$. Then, the generalized integral transform $\mathcal{T}^h_{S,R}(F)$ of F, a generalized convolution product $(F*G)^{h_1,h_2}_{A,B,C,D}$ of F and G, and a generalized first variation $\delta^{h_1,h_2}_{S_1,S_2}F$ of F with respect to h_1, h_2, S_1 and S_2 are defined by the formulas*

$$\mathcal{T}^h_{S,R}(F)(y) = \int_{C_0[0,T]} F(SZ_h(x,\cdot) + Ry)dm(x), \tag{9}$$

$$(F*G)^{h_1,h_2}_{A,B,C,D}(y) = \int_{C_0[0,T]} F(AZ_{h_1}(x,\cdot) + By)G(CZ_{h_2}(x,\cdot) + Dy)dm(x) \tag{10}$$

and

$$\begin{aligned}\delta^{h_1,h_2}_{S_1,S_2}F(x|u) &\equiv \delta F(S_1Z_{h_1}(x,\cdot)|S_2Z_{h_2}(u,\cdot))\\ &= \frac{\partial}{\partial\alpha}F(S_1Z_{h_1}(x,\cdot) + \alpha S_2Z_{h_2}(u,\cdot))\Big|_{\alpha=0}\end{aligned} \tag{11}$$

for $x, u, y \in K$ if they exist.

Remark 1.

(1) When $h(t) \equiv 1$ on $[0,T]$, the generalized integral transform $\mathcal{T}^1_{S,R}$ is the Fourier–Gauss transform $\mathcal{G}_{S,R}$ [16].
(2) When S and R are the constant operators, the generalized integral transform $\mathcal{T}^h_{\gamma,\beta}$ is a generalized integral transform $\mathcal{F}^h_{\gamma,\beta}$ used in [14,15]. In particular, if $h(t) \equiv 1$ on $[0,T]$, then $\mathcal{T}^1_{\gamma,\beta}$ is the integral transform used in [5,6,8,10,11,13,22].
(3) When $h_1(t) \equiv 1$ and $h_2(t) \equiv 1$ on $[0,T]$, $(F*G)^{1,1}_{A,B,C,D}$ is the convolution product used in [11].

We next state some notations used in this paper. For $v \in L_2[0,T], h_1, h_2, \cdots, h_n \in \mathcal{H}$ and $R_1, \cdots, R_n \in \mathcal{L}$, let

$$M(R_1,\cdots,R_n : h_1,\cdots,h_n : v) \equiv \exp\left\{\frac{1}{2}\sum_{j=1}^n \|h_j z_{R_j^*v}\|_2^2\right\}, \tag{12}$$

where $R_j^*v(t) = \int_0^t z_{R_j^*v}(s)ds$ for each $j = 1, 2, \cdots, n$. Furthermore, we have the symmetric property for $M(\cdot : \cdot : v)$.

In Theorem 1, we obtain the existence of generalized integral transform, generalized convolution product and generalized first variation of functionals in $\mathcal{S}(C_0[0,T])$. In addition, we show that they are elements of $\mathcal{S}(C_0[0,T])$.

Theorem 1. *Let h, h_1, h_2 be elements of $\mathcal{H}$ and let $S, R, A, B, C, D, S_1, S_2 \in \mathcal{L}$. Let Φ_v and Φ_w be elements of $\mathcal{S}(C_0[0,T])$ and let $u(t) = \int_0^t z_u(s)ds \in C_0'$. In addition, let $k_{h_j}(t) = \int_0^t h_j(s)ds$ for $j = 1, 2$. Then, the generalized integral transform $\mathcal{T}^h_{S,R}(\Phi_v)$ of Φ_v, the generalized convolution product $(\Phi_v * \Phi_w)^{h_1,h_2}_{A,B,C,D}$ of Φ_v and Φ_w and the generalized first variation $\delta^{h_1,h_2}_{S_1,S_2}\Phi_v(x|u)$ with respect to h_1, h_2, S_1 and S_2 exist, belong to $\mathcal{S}(C_0[0,T])$ and are given by the formulas*

$$\mathcal{T}^h_{S,R}(\Phi_v)(y) = M(S : h : v)\Phi_{R^*v}(y), \tag{13}$$

$$\begin{aligned}&(\Phi_v * \Phi_w)^{h_1,h_2}_{A,B,C,D}(y)\\ &= M(A : h_1 : v)M(C : h_2 : w)\exp\{(h_1z_{A^*v}, h_2z_{C^*w})_2\}\Phi_{B^*v+D^*w}(y)\end{aligned} \tag{14}$$

and

$$\delta_{S_1,S_2}^{h_1,h_2}\Phi_v(x|u) = (h_2 z_{S_2^*v}, z_u)_2 \Phi_{h_1 z_{S_1^*v}}(x). \tag{15}$$

for $x, y, u \in K$.

Proof. First, using Equations (5), (1) and (8), it follows that, for all $y \in K$, we have

$$\begin{aligned}
\mathcal{T}_{S,R}^{h}(\Phi_v)(y) &= \int_{C_0[0,T]} \exp\left\{(v, SZ_h(x,\cdot))^\sim + (v, Ry)^\sim\right\} dm(x) \\
&= \int_{C_0[0,T]} \exp\left\{\langle hz_{S^*v}, x\rangle + (R^*v, y)^\sim\right\} dm(x) \\
&= \exp\left\{\frac{1}{2}\|hz_{S^*v}\|_2^2\right\} \Phi_{R^*v}(y).
\end{aligned}$$

Finally, by using Equations (12) and (13) is obtained. We next use Equations (5), (8) and (14) to obtain the following calculation

$$\begin{aligned}
(\Phi_v * \Phi_w)_{A,B,C,D}^{h_1,h_2}(y) &= \int_{C_0[0,T]} \Phi_v(AZ_{h_1}(x,\cdot) + By)\Phi_w(CZ_{h_2}(x,\cdot) + Dy) dm(x) \\
&= \int_{C_0[0,T]} \exp\left\{\langle h_1 z_{A^*v} + h_2 z_{C^*w}, x\rangle + (B^*v + D^*w, y)^\sim\right\} dm(x) \\
&= \exp\left\{\frac{1}{2}\|h_1 z_{A^*v} + h_2 z_{C^*w}\|_2^2\right\} \Phi_{B^*v}(y)\Phi_{D^*w}(y).
\end{aligned}$$

Since $\|h\|_2^2 = (h, h)_2$ for all $h \in L_2[0, T]$, we now note that

$$\begin{aligned}
&\frac{1}{2}\|h_1 z_{A^*v} + h_2 z_{C^*w}\|_2^2 = \frac{1}{2}\Big(h_1 z_{A^*v} + h_2 z_{C^*w}, h_1 z_{A^*v} + h_2 z_{C^*w}\Big)_2 \\
&= \frac{1}{2}\Big[(h_1 z_{A^*v}, h_1 z_{A^*v})_2 + (h_2 z_{C^*w}, h_2 z_{C^*w})_2 + 2(h_1 z_{A^*v}, h_2 z_{C^*w})_2\Big]
\end{aligned}$$

and $\Phi_v(y) + \Phi_w(y) = \Phi_{v+w}(y)$ for all $v, w \in C_0'[0, T]$. Hence, we can obtain Equation (14) as desired. Finally, we use Equations (8) and (11) to establish Equation (15) as follows:

$$\begin{aligned}
\delta_{S_1,S_2}^{h_1,h_2}\Phi_v(x|u) &= \frac{\partial}{\partial\alpha}\Big[\Phi_v(S_1 Z_{h_1}(x,\cdot) + \alpha S_2 Z_{h_2}(u,\cdot))\Big]\Big|_{\alpha=0} \\
&= \frac{\partial}{\partial\alpha}\Big[\exp\{\langle h_1 z_{S_1^*v}, x\rangle + \alpha\langle h_2 z_{S_2^*v}, u\rangle\}\Big]\Big|_{\alpha=0} \\
&= \langle h_2 z_{S_2^*v}, u\rangle \exp\{\langle h_1 z_{S_1^*v}, x\rangle\}.
\end{aligned}$$

We now note that

$$\langle h_2 z_{S_2^*v}, u\rangle = \int_0^T h_2(t) z_{S_2^*v}(t) z_u(t) dt = (h_2 z_{S_2^*v}, z_u)_2,$$

which establishes Equation (15) as desired. □

4. Some Relationships with the Generalized Convolution Products.

In this section, we obtain some relationships between the generalized integral transform and the generalized convolution product of functionals in $\mathcal{S}(C_0[0,T])$. In the first theorem in Section 4, we give a formula for the generalized integral transforms of functionals in $\mathcal{S}(C_0[0,T])$. To establish some relationships, the following lemma is needed.

Lemma 1. *Let $h_1, h_2 \in \mathcal{H}$ and let $S_1, S_2, R \in \mathcal{L}$. Then, for each $v \in C_0'$,*

$$M(S_1 : h_1 : R^*v)M(S_2 : h_2 : v) = M(RS_1, S_2 : h_1, h_2 : v). \tag{16}$$

Proof. Using the following fact $S_1^*R^* = (RS_1)^*$ and Equation (12) repeatedly, we have

$$\begin{aligned} M(S_1 : h_1 : R^*v)M(S_2 : h_2 : v) &= \exp\left\{\frac{1}{2}\|h_1 z_{S_1^*R^*v}\|_2^2\right\}\exp\left\{\frac{1}{2}\|h_2 z_{S_2^*v}\|_2^2\right\} \\ &= \exp\left\{\frac{1}{2}\|h_1 z_{S_1^*R^*v}\|_2^2 + \frac{1}{2}\|h_2 z_{S_2^*v}\|_2^2\right\} \\ &= \exp\left\{\frac{1}{2}\|h_1 z_{(RS_1)^*v}\|_2^2 + \frac{1}{2}\|h_2 z_{S_2^*v}\|_2^2\right\} \\ &= M(RS_1, S_2 : h_1, h_2 : v), \end{aligned}$$

which complete the proof of Lemma 1. □

Theorem 2. *Let S_1, S_2, R_1 and R_2 be elements of $\mathcal{L}$ and let h_1 and h_2 be elements of $\mathcal{H}$. In addition, let Φ_v be an element of $\mathcal{S}(C_0[0,T])$. Then,*

$$\mathcal{T}^{h_1}_{S_1,R_1}(\mathcal{T}^{h_2}_{S_2,R_2}(\Phi_v))(y) = M(R_2S_1, S_2 : h_1, h_2 : v)\Phi_{(R_1R_2)^*v}(y) \tag{17}$$

for $y \in K$.

Proof. From Theorem 1, we have

$$\mathcal{T}^{h_2}_{S_2,R_2}(\Phi_v)(y) = M(S_2 : h_2 : v)\Phi_{R_2^*v}(y).$$

Applying Theorem 1 once more,

$$\mathcal{T}^{h_1}_{S_1,R_1}(\mathcal{T}^{h_2}_{S_2,R_2}(\Phi_v))(y) = M(S_2 : h_2 : v)M(S_1 : h_1 : R_2^*v)\Phi_{(R_1R_2)^*v}(y).$$

Finally, using Equation (16) in Lemma 1, we complete the proof of Theorem 2 as desired. □

Equations (18) and (19) in Theorem 3 are the commutative of the generalized integral transform and the Fubini theorem with respect to the generalized integral transform, respectively.

Theorem 3. *Let S_1, S_2, R_1 and R_2 be elements of $\mathcal{L}$ and let h_1 and h_2 be elements of $\mathcal{H}$. In addition, let Φ_v be an element of $\mathcal{S}(C_0[0,T])$. Then,*

$$\mathcal{T}^{h_1}_{S_1,R_1}(\mathcal{T}^{h_2}_{S_2,R_2}(\Phi_v))(y) = \mathcal{T}^{h_2}_{S_2,R_2}(\mathcal{T}^{h_1}_{S_1,R_1}(\Phi_v))(y) \tag{18}$$

if and only if

$$R_1R_2 = R_2R_1, \quad \textit{and} \quad M(R_2S_1, S_2 : h_1, h_2 : v) = M(S_1, R_1S_2 : h_1, h_2 : v).$$

Furthermore,

$$\mathcal{T}^{h_1}_{S_1,R_1}(\mathcal{T}^{h_2}_{S_2,R_2}(\Phi_v))(y) = \mathcal{T}^{h_3}_{S_3,R_3}(\Phi_v)(y) \tag{19}$$

if and only if

$$R_1R_2 = R_3, \quad \textit{and} \quad M(R_2S_1, S_2 : h_1, h_2 : v) = M(S_3 : h_3 : v).$$

Proof. Using Equation (17) twice, we have

$$\mathcal{T}^{h_1}_{S_1,R_1}(\mathcal{T}^{h_2}_{S_2,R_2}(\Phi_v))(y) = M(R_2S_1, S_2 : h_1, h_2 : v)\Phi_{(R_1R_2)^*v}(y)$$

and

$$\mathcal{T}^{h_2}_{S_2,R_2}(\mathcal{T}^{h_1}_{S_1,R_1}(\Phi_v))(y) = M(S_1, R_1S_2 : h_1, h_2 : v)\Phi_{(R_2R_1)^*v}(y).$$

Using these facts and Equation (13), we can establish Equations (18) and (19). □

From Theorems 2 and 3, we can establish the n-dimensional version for the generalized integral transform.

Corollary 1. *Let $S_1, \cdots, S_n, R_1, \cdots, R_{n-1}$ and R_n be elements of $\mathcal{L}$ and let h_j be an element of $\mathcal{H}$, $j = 1, 2, \cdots$. In addition, let Φ_v be an element of $\mathcal{S}(C_0[0, T])$. Then,*

$$\begin{aligned}&\mathcal{T}^{h_n}_{S_n,R_n}(\cdots(\mathcal{T}^{h_1}_{S_1,R_1}(\Phi_v)\cdots))(y)\\&= M(S_1, R_1S_2, R_1R_2, S_3, \cdots, R_1R_2\cdots R_{n-1}S_n : h_1, \cdots, h_n : v)\Phi_{(R_1\cdots R_n)^*v}(y).\end{aligned}$$

In our next theorem, we show that our generalized convolution product is commutative.

Theorem 4. *Let A, B, C and D be elements of $\mathcal{L}$ and let $h_1, h_2 \in \mathcal{H}$. Let Φ_v and Φ_w be elements of $\mathcal{S}(C_0[0, T])$. Then,*

$$(\Phi_v * \Phi_w)^{h_1,h_2}_{A,B,C,D}(y) = (\Phi_w * \Phi_v)^{h_1,h_2}_{A,B,C,D}(y) \tag{20}$$

if and only if

$$M(A : h_1 : v) = M(C : h_2 : v) \text{ and } M(A : h_1 : w) = M(C : h_2 : w).$$

Proof. The proof of Theorem 4 is a straightforward application of Theorem 1. □

In Theorem 5, we give a necessary and sufficient condition for holding a relationship between the generalized integral transform and the generalized convolution product.

Theorem 5. *For $j = 1, 2, 3$, let $S_j, R_j \in \mathcal{L}$, and, for $= 1, 2$, let $A_i, B_i, C_i, D_i \in \mathcal{L}$. In addition, for $k = 1, 2, \cdots, 7$, let $h_k \in \mathcal{H}$. Then,*

$$\mathcal{T}^{h_1}_{S_1,R_1}(\Phi_v * \Phi_w)^{h_2,h_3}_{A_1,B_1,C_1,D_1}(y) = (\mathcal{T}^{h_4}_{S_2,R_2}\Phi_v * \mathcal{T}^{h_5}_{S_3,R_3}\Phi_w)^{h_6,h_7}_{A_2,B_2,C_2,D_2}(y) \tag{21}$$

if and only if the following equations hold

$$\begin{cases} B_1R_1 = R_2B_2 \text{ and } D_1R_1 = R_3D_2 \\ M(B_1S_1, A_1 : h_1, h_2 : v) = M(S_2, A_2 : h_4, h_6 : v) \\ M(D_1S_1, C_1 : h_1, h_3 : w) = M(S_3, C_2 : h_5, h_7 : w) \\ (h_2A_1^*v, h_3C_1^*w)_2 = (h_6A_2^*v, h_7C_2^*w)_2 \end{cases}.$$

Proof. To complete the proof of Theorem 5, we first calculate the left hand side of Equation (21). From Equation (14) in Theorem 1, we have

$$\begin{aligned}&(\Phi_v * \Phi_w)^{h_2,h_3}_{A_1,B_1,C_1,D_1}(y)\\&= M(A_1 : h_2 : v)M(C_1 : h_3 : w)\exp\{(h_2A_1^*v, h_3C_1^*w)_2\}\Phi_{B_1^*v+D_1^*w}(y).\end{aligned} \tag{22}$$

Using Equations (13), (12), (16) and (22), we have

$$\begin{aligned}\mathcal{T}^{h_1}_{S_1,R_1}(\Phi_v * \Phi_w)^{h_2,h_3}_{A_1,B_1,C_1,D_1}(y) &= M(B_1S_1, A_1 : h_1, h_2 : v)M(D_1S_1, C_1 : h_1, h_3 : w)\\&\quad\cdot \exp\{(h_2A_1^*v, h_3C_1^*w)_2\}\Phi_{R_1^*B_1^*v+R_1^*D_1^*w}(y).\end{aligned}$$

We next calculate the left hand side of Equation (21). From Equations (12) and (13) twice, we have

$$\mathcal{T}^{h_4}_{S_2,R_2}(\Phi_v)(y) = M(S_2 : h_4 : v)\Phi_{R_2^* v}(y) \tag{23}$$

and

$$\mathcal{T}^{h_5}_{S_3,R_3}(\Phi_w)(y) = M(S_3 : h_5 : w)\Phi_{R_3^* w}(y). \tag{24}$$

We now use Equations (14), (16), (23) and (24) repeatedly to obtain the following calculation

$$\begin{aligned}(\mathcal{T}^{h_4}_{S_2,R_2}\Phi_v * \mathcal{T}^{h_5}_{S_3,R_3}\Phi_w)^{h_6,h_7}_{A_2,B_2,C_2,D_2}(y) &= M(S_2, A_2 : h_4, h_6 : v)M(S_3, C_2 : h_5, h_7 : w) \\ &\quad \cdot \exp\{(h_6 A_2^* v, h_7 C_2^* w)_2\}\Phi_{B_2^* R_2^* v + D_2^* R_3^* w}(y).\end{aligned}$$

Hence, we complete the proof of Theorem 5 as desired. □

Corollary 2. *The following results and formulas stated bellow easily from Theorem 5.*

(1) Let S and R be elements of $\mathcal{L}$, and, for $= 1,2$, let $A_i, B_i, C_i, D_i \in \mathcal{L}$. In addition, for $k = 1,2,\cdots,5$, let $h_k \in \mathcal{H}$. Then,

$$\mathcal{T}^{h_1}_{S,R}(\Phi_v * \Phi_w)^{h_2,h_3}_{A_1,B_1,C_1,D_1}(y) = (\mathcal{T}^{h_1}_{S,R}\Phi_v * \mathcal{T}^{h_1}_{S,R}\Phi_w)^{h_4,h_5}_{A_2,B_2,C_2,D_2}(y)$$

if and only if the following equations hold

$$\begin{cases} B_1 R = R B_2 \textit{ and } D_1 R = R D_2 \\ M(B_1 S, A_1 : h_1, h_2 : v) = M(S, A_2 : h_1, h_4 : v) \\ M(D_1 S, C_1 : h_1, h_3 : w) = M(S, C_2 : h_1, h_5 : w) \\ (h_2 A_1^* v, h_3 C_1^* w)_2 = (h_4 A_2^* v, h_5 C_2^* w)_2 \end{cases}.$$

(2) For $j = 1,2,3$, let $S_j, R_j \in \mathcal{L}$ and $A, B, C, D \in \mathcal{L}$. In addition, for $k = 1,2,\cdots,7$, let $h_k \in \mathcal{H}$. Then,

$$\mathcal{T}^{h_1}_{S_1,R_1}(\Phi_v * \Phi_w)^{h_2,h_3}_{A,B,C,D}(y) = (\mathcal{T}^{h_4}_{S_2,R_2}\Phi_v * \mathcal{T}^{h_5}_{S_3,R_3}\Phi_w)^{h_2,h_3}_{A,B,C,D}(y)$$

if and only if the following equations hold

$$\begin{cases} B R_1 = R_2 B \textit{ and } D R_1 = R_3 D \\ M(B S_1, A : h_1, h_2 : v) = M(S_2, A : h_4, h_2 : v) \\ M(D S_1, C : h_1, h_3 : w) = M(S_3, C : h_5, h_{:} w) \\ (h_2 A^* v, h_3 C^* w)_2 = (h_6 A^* v, h_3 C^* w)_2 \end{cases}.$$

5. Some Relationships with the Generalized First Variations

In this section, we establish some formulas involving the generalized first variation. We next obtain a generalized Cameron–Storvick theorem for the generalized first variation and use this to apply for the generalized integral transform.

Theorem 6. *Let $h_1, h_2, h_3 \in \mathcal{H}$ and $S_1, S_2, S_3 \in \mathcal{L}$. Let $u \in C_0'$ with $u(t) = \int_0^t z_u(s)ds$. Then,*

$$\mathcal{T}^{h_1}_{S_1,R}(\delta^{h_2,h_3}_{S_2,S_3}\Phi_v(\cdot|u))(y) = \delta^{h_2,h_3}_{S_2,S_3}\mathcal{T}^{h_1}_{S_1,R}(\Phi_v)(y|u) \tag{25}$$

if and only if $R = I$ and $M(S_1 : h_1 : \overline{v}_{S_2,h_2}) = M(S_1 : h_1 : v)$, where $\overline{v}_{S_2,h_2} = h_2 z_{S_2^* v}$.

Proof. First, using Equations (5), (12), (13) and (29), we have

$$
\begin{aligned}
&\mathcal{T}^{h_1}_{S_1,R}(\delta^{h_2,h_3}_{S_2,S_3}\Phi_v(\cdot|u))(y) \\
&= (h_3 z_{S_3^* v}, z_u)_2 \int_{C_0[0,T]} \Phi_{h_2 z_{S_2^* v}}(S_1 Z_{h_1}(x,\cdot) + Ry)dm(x) \\
&= (h_3 z_{S_3^* v}, z_u)_2 \int_{C_0[0,T]} \exp\{(\overline{v}_{S_2,h_2}, S_1 Z_{h_1}(x,\cdot))^{\sim} + (\overline{v}_{S_2,h_2}, Ry)^{\sim}\}dm(x) \\
&= (h_3 z_{S_3^* v}, z_u)_2 \int_{C_0[0,T]} \exp\{\langle h_1 z_{S_1^* \overline{v}_{S_2,h_2}}, x\rangle + (R^* \overline{v}_{S_2,h_2}, y)^{\sim}\}dm(x) \\
&= (h_3 z_{S_3^* v}, z_u)_2 M(S_1 : h_1 : \overline{v}_{S_2,h_2}) \exp\{(R^* \overline{v}_{S_2,h_2}, y)^{\sim}\} \\
&= (h_3 z_{S_3^* v}, z_u)_2 M(S_1 : h_1 : \overline{v}_{S_2,h_2}) \Phi_{R^* \overline{v}_{S_2,h_2}}(y).
\end{aligned}
$$

On the other hands, using Equations (11)–(13), we have

$$
\begin{aligned}
&\delta^{h_2,h_3}_{S_2,S_3}\mathcal{T}^{h_1}_{S_1,R}(\Phi_v)(y|u) \\
&= \frac{\partial}{\partial\alpha}\Big[\mathcal{T}^{h_1}_{S_1,R}(\Phi_v)(S_2 Z_{h_2}(y,\cdot) + \alpha S_3 Z_{h_3}(u,\cdot))\}\Big]\Big|_{\alpha=0} \\
&= \frac{\partial}{\partial\alpha}\Big[\exp\Big\{\frac{1}{2}\|h_1 z_{S_1^* v}\|_2^2\Big\}\Phi_{R^* v}(S_2 Z_{h_2}(y,\cdot) + \alpha S_3 Z_{h_3}(u,\cdot))\Big]\Big|_{\alpha=0} \\
&= \exp\Big\{\frac{1}{2}\|h_1 z_{S_1^* v}\|_2^2\Big\}\frac{\partial}{\partial\alpha}\Big[\exp\Big\{(R^* v, S_2 Z_{h_2}(y,\cdot))^{\sim} + \alpha(R^* v, S_3 Z_{h_3}(u,\cdot))^{\sim}\Big\}\Big]\Big|_{\alpha=0} \\
&= \exp\Big\{\frac{1}{2}\|h_1 z_{S_1^* v}\|_2^2\Big\}\frac{\partial}{\partial\alpha}\Big[\exp\Big\{\langle h_2 z_{S_2^* R^* v}, y\rangle + \alpha\langle h_3 z_{S_3^* R^* v}, u\rangle\Big\}\Big]\Big|_{\alpha=0} \\
&= (h_3 z_{S_3^* R^* v}, z_u)_2 M(S_1 : h_1 : v)\exp\Big\{\langle h_2 z_{S_2^* R^* v}, y\rangle\Big\} \\
&= (h_3 z_{S_3^* R^* v}, z_u)_2 M(S_1 : h_1 : v)\Phi_{h_2 z_{S_2^* R^* v}}(y).
\end{aligned}
$$

Hence, Equation (25) holds if and only if $R = I$ and

$$M(S_1 : h_1 : \overline{v}_{S_2,h_2}) = M(S_1 : h_1 : v).$$

□

To establish a generalized Cameron–Storvick theorem for the generalized first variation, we need two lemmas with respect to the translation theorem on Wiener space.

Lemma 2. (Translation Theorem 1) *Let F be a integrable functional on $C_0[0,T]$ and let $x_0 \in C_0'$. Then,*

$$\int_{C_0[0,T]} F(x + x_0)dm(x) = \exp\Big\{-\frac{1}{2}\|x_0\|^2_{C_0'}\Big\}\int_{C_0[0,T]} F(x)\exp\{(x_0, x)^{\sim}\}dm(x). \tag{26}$$

In [23], the authors used Equation (26) to establish Equation (28), which is a generalized translation theorem. The main key in their proof is the change of kernel for the Gaussian process, i.e.

$$
\begin{aligned}
Z_{h_1}(\theta_0, t) &= \int_0^t h_1(s)d\Big(\int_0^s h_2(\tau)z_{x_0}(\tau)d\tau\Big) \\
&= \int_0^t h_1(s)h_2(s)z_{x_0}(s)ds \\
&= \int_0^t h_2(s)d\Big(\int_0^s h_1(\tau)z_{x_0}(\tau)d\tau\Big) = Z_{h_2}(u, t)
\end{aligned} \tag{27}
$$

where $\theta_0(t) = \int_0^t h_2(t)z_{x_0}(t)dt$ and $u(t) = \int_0^t h_1(s)z_{x_0}(s)ds$ for given $x_0 \in C_0'$.

The following lemma is said to be the translation theorem via the Gaussian process on Wiener space.

Lemma 3 (Translation Theorem 2)**.** *Let $h_1, h_2 \in \mathcal{H}$. Let $x_0(t) = \int_0^t z_{x_0}(s)ds$ and let $F(Z_{h_1}(x,\cdot))$ be a integrable functional on $C_0[0,T]$. Let*

$$\theta_0(t) = \int_0^t h_1(s)z_{x_0}(s)ds.$$

Then,

$$\begin{aligned} &\int_{C_0[0,T]} F(Z_{h_1}(x,\cdot) + Z_{h_2}(\theta_0,\cdot))dm(x) \\ &= \exp\left\{-\frac{1}{2}\|z_{x_0}h_2\|_2^2\right\} \int_{C_0[0,T]} F(Z_{h_1}(x,\cdot)) \exp\{(\theta_0, Z_{h_2}(x,\cdot))^\sim\}dm(x). \end{aligned} \tag{28}$$

In our next theorem, we establish the generalized Cameron–Storvick theorem for the generalized first variation.

Theorem 7. *Let $x_0 \in C_0'$ be given. Let $h_1, h_2 \in \mathcal{H}$ and $S \in \mathcal{L}$. In addition, let $u(t) = \int_0^t h_1(s)z_{x_0}(s)ds$ and $\theta_0(t) = \int_0^t h_2(s)z_{x_0}(s)ds$. Then,*

$$\int_{C_0[0,T]} \delta_{S,S}^{h_1,h_2}\Phi_v(x|u)dm(x) = \int_{C_0[0,T]} (x_0, Z_{h_2}(x,\cdot))^\sim \Phi_v(SZ_{h_1}(x,\cdot))dm(x). \tag{29}$$

Proof. First, by using Equation (11) and the dominated convergence theorem, we have

$$\begin{aligned} &\int_{C_0[0,T]} \delta_{S,S}^{h_1,h_2}\Phi_v(x|u)dm(x) \\ &= \frac{\partial}{\partial\alpha}\left[\int_{C_0[0,T]} \Phi_v(SZ_{h_1}(x,\cdot) + \alpha SZ_{h_2}(u,\cdot))dm(x)\right]\Bigg|_{\alpha=0} \\ &= \frac{\partial}{\partial\alpha}\left[\int_{C_0[0,T]} \Phi_v(SZ_{h_1}(x,\cdot) + SZ_{h_2}(\alpha u,\cdot))dm(x)\right]\Bigg|_{\alpha=0}. \end{aligned}$$

Now, let $F_S^h(x) = \Phi_v(SZ_h(x,\cdot))$. Using the key (27) used in [23], we have

$$F_S^{h_1}(x + \alpha\theta_0) = \Phi_v(SZ_{h_1}(x,\cdot) + SZ_{h_2}(\alpha u,\cdot))$$

where $\theta_0(t) = \int_0^t h_2(s)z_{x_0}(s)ds$ and $u(t) = \int_0^t h_1(s)z_{x_0}(s)ds$. This means that

$$\int_{C_0[0,T]} \delta_{S,S}^{h_1,h_2}F(x|u)dm(x) = \frac{\partial}{\partial\alpha}\left[\int_{C_0[0,T]} F_S^{h_1}(x + \alpha\theta_0)dm(x)\right]\Bigg|_{\alpha=0}.$$

We next apply the translation theorem to the functional $F_S^{h_1}$ instead of F in Lemma 2 to proceed the following formula

$$\begin{aligned} &\int_{C_0[0,T]} \delta_{S,S}^{h_1,h_2}F(x|u)dm(x) \\ &= \frac{\partial}{\partial\alpha}\left[\exp\left\{-\frac{1}{2}\|\alpha\theta_0\|_{C_0'}^2\right\} \int_{C_0[0,T]} F_S^{h_1}(x)\exp\{(\alpha\theta_0, x)^\sim\}dm(x)\right]\Bigg|_{\alpha=0} \\ &= \frac{\partial}{\partial\alpha}\left[\exp\left\{-\frac{\alpha^2}{2}\|z_{x_0}h_2\|_2^2\right\} \int_{C_0[0,T]} F_S^{h_1}(x)\exp\{\alpha\langle z_{x_0}h_2, x\rangle\}dm(x)\right]\Bigg|_{\alpha=0} \\ &= \int_{C_0[0,T]} \langle z_{x_0}h_2, x\rangle \Phi_v(SZ_{h_1}(x,\cdot))dm(x). \end{aligned}$$

Since $(\theta_0, x)^{\sim} = \langle z_{x_0} h_2, x\rangle = (x_0, Z_{h_2}(x,\cdot))^{\sim}$, we complete the proof of Theorem 7 as desired. □

In the last theorem in this paper, we use Equation (29) to give an integration formula involving the generalized first variation and the generalized integral transform. This formula tells us that we can calculate the Wiener integral of generalized first variation for generalized integral transform directly without calculations of them.

Theorem 8. *Let $h_1, h_2, h_3 \in \mathcal{H}$ and let $S_1, S_2 \in \mathcal{L}$. In addition, let u, x_0, θ_0 be as in Theorem 7. Then,*

$$\int_{C_0[0,T]} \delta^{h_2,h_3}_{S_2,S_2} \mathcal{T}^{h_1}_{S_1,R}(\Phi_v)(y|u)dm(y) = M(S_1, RS_2 : h_1, h_2 : v)(h_3 z_{x_0}, h_2 z_{S_2^* R^* v})_2. \tag{30}$$

Proof. Applying Equation (29) to the functional $\mathcal{T}^{h_1}_{S_1,R}(\Phi_v)$ instead of Φ_v, we have

$$\begin{aligned}
&\int_{C_0[0,T]} \delta^{h_2,h_3}_{S_2,S_2} \mathcal{T}^{h_1}_{S_1,R}(\Phi_v)(y|u)dm(y) \\
&= \int_{C_0[0,T]} (x_0, Z_{h_3}(y,\cdot))^{\sim} \mathcal{T}^{h_1}_{S_1,R}(\Phi_v)(S_2 Z_{h_2}(y,\cdot))dm(y).
\end{aligned}$$

Now, using Equations (8) and (13), it becomes that

$$\begin{aligned}
&\int_{C_0[0,T]} \delta^{h_2,h_3}_{S_2,S_2} \mathcal{T}^{h_1}_{S_1,R}(\Phi_v)(y|u)dm(y) \\
&= M(S_1 : h_1 : v) \int_{C_0[0,T]} (x_0, Z_{h_3}(y,\cdot))^{\sim} \exp\{(R^* v, S_2 Z_{h_2}(y,\cdot))^{\sim}\}dm(y) \\
&= M(S_1 : h_1 : v) \int_{C_0[0,T]} \langle h_3 z_{x_0}, y\rangle \exp\{\langle h_2 z_{S_2^* R^* v}, y\rangle\}dm(y).
\end{aligned}$$

The following integration formula

$$\int_{C_0[0,T]} \langle w, x\rangle \exp\{\langle p, x\rangle\}dm(x) = (w,p)_2 \exp\left\{\frac{1}{2}\|p\|_2^2\right\}, \qquad w, p \in L_2[0,T]$$

and Equation (12) yield that

$$\begin{aligned}
&\int_{C_0[0,T]} \delta^{h_2,h_3}_{S_2,S_2} \mathcal{T}^{h_1}_{S_1,R}(\Phi_v)(y|u)dm(y) \\
&= M(S_1 : h_1 : v) \int_{C_0[0,T]} \langle h_3 z_{x_0}, y\rangle \exp\{\langle h_2 z_{S_2^* R^* v}, y\rangle\}dm(y) \\
&= M(S_1 : h_1 : v)(h_3 z_{x_0}, h_2 z_{S_2^* R^* v})_2 \exp\left\{\frac{1}{2}\|h_2 z_{S_2^* R^* v}\|_2^2\right\} \\
&= M(S_1 : h_1 : v) M(RS_2 : h_2 : v)(h_3 z_{x_0}, h_2 z_{S_2^* R^* v})_2.
\end{aligned}$$

Finally, by using Equation (16) in Lemma 1, we establish Equation (30) as desired. □

6. Application

We finish this paper by giving some examples to illustrate the usefulness of our results and formulas.

We first give a simple example used in the stack exchange and the signal process. For $x \in C_0[0,T]$, let $K_s(x)(t) = \int_0^t x(s)ds$. Then, the adjoint is given by the formula $K_s^*(x)(t) = \int_t^T x(s)ds$.

Example 1. *Let $S = K_s$ and let $v(t) = -t + \frac{T}{2}$ and $h(t) = t^2$ on $[0, T]$. Then, $h \in \mathcal{H}$. In addition, we have*

$$S^*v(t) = \int_t^T v(s)ds = \frac{1}{2}t^2 - \frac{t}{2}T = \int_0^t (s - \frac{1}{2}T)ds.$$

This means that $z_{S^*v}(t) = t - \frac{1}{2}T$ on $[0, T]$ and hence $\|hz_{S^*v}\|_2^2 = \frac{1}{12}T^4$. Thus, we obtain that

$$\mathcal{T}^h_{S,R}(\Phi_v)(y) = \exp\left\{\frac{1}{24}T^4\right\}\Phi_{R^*v}(y).$$

We give two examples in the quantum mechanics. To do this, we consider useful operators used in quantum mechanics. We consider two cases. However, various cases can be applied in appropriate methods as examples.

Case 1 : Multiplication operator.

In the next examples, we consider the multiplication operator T_m, which plays a role in physics (quantum theories) (see [21]). Before do this, we introduce some observations to proceed obtaining examples. Let $R \in \mathcal{L}$ such that

$$R(xy) = xR(y) \tag{31}$$

for all $x, y \in C_0[0, T]$. In addition, for $t \in [0, T]$ on $C_0[0, T]$, we define a multiplication operator T_m by

$$(T_m(x))(t) \equiv T_m(x(t)) = tx(t). \tag{32}$$

Then, we have $T_m(xy) = tx(t)y(t)$ and $xT_m(y) = x(t)ty(t)$. Hence, Equation (31) holds. In addition, one can easily check that $T_m^*v(t) = tv(t)$ for all $v \in C_0'$. Note that the expected value or corresponding mean value is

$$E(x) \equiv \int_0^T t|x(t)|^2 dt = \int_0^T T_m(|x|^2)(t)dt,$$

where x is the state function of a particle in quantum mechanics and $\int_0^T |x(t)|^2 dt$ is the probability that the particle will be found in $[0, T]$.

In the first and second examples, we give some formula with respect to the multiplication operator T_m.

Example 2. *Let $S = T_m$ and let $v(t) = \frac{1}{2}t^2$ and $h(t) = t^2$ on $[0, T]$. Then, $h \in \mathcal{H}$. In addition, we have*

$$v(t) = \frac{1}{2}t^2 = \int_0^t s ds$$

and

$$S^*v(t) = \frac{1}{2}t^3 = \int_0^t \frac{3}{2}s^2 ds.$$

This means that $z_v(t) = t$ and $z_{S^*v}(t) = \frac{3}{2}t^2$ on $[0, T]$ and hence $\|hz_{S^*v}\|_2^2 = \frac{3}{10}T^5$. Thus, we obtain that

$$\mathcal{T}^h_{S,R}(\Phi_v)(y) = \exp\left\{\frac{3}{10}T^5\right\}\Phi_{R^*v}(y).$$

Example 3. *Let $S = T_m$ and let $v(t) = e^t - 1$ and $h(t) = t$ on $[0, T]$. Then, $h \in \mathcal{H}$. In addition, we have*

$$v(t) = e^t - 1 = \int_0^t e^s ds$$

and

$$S^*v(t) = te^t - t = \int_0^t (se^s + e^s - 1)ds.$$

This means that $z_v(t) = e^t$ *and* $z_{S^*v}(t) = te^t + e^t - 1$ *on* $[0, T]$ *and hence*

$$\|hz_{S^*v}\|_2^2 = \frac{1}{4}e^{2T}(2T^4 + 2T^2 - 2T + 1) - 2e^T(T^2 + 2T - 4) + \frac{1}{3}T^3 - \frac{33}{4}.$$

Thus, we obtain that

$$\mathcal{T}^h_{S,R}(\Phi_v)(y) = \exp\bigg\{\frac{1}{8}e^{2T}(2T^4 + 2T^2 - 2T + 1) - e^T(T^2 + 2T - 4) + \frac{1}{6}T^3 - \frac{33}{8}\bigg\}\Phi_{R^*v}(y).$$

Case 2 : Quantum mechanics operators.

In the next examples, we consider some linear operators which are used to explain the solution of the diffusion equation and the Schrôdinger equation (see [24]).

Let $S : C_0'[0, T] \to C_0'[0, T]$ be the linear operator defined by

$$Sw(t) = \int_0^t w(s)ds. \tag{33}$$

Then, the adjoint operator S^* of S is given by the formula

$$S^*w(t) = w(T)t - \int_0^t w(s)ds = \int_0^t [w(T) - w(s)]ds$$

and the linear operator $A = S^*S$ is given by the formula

$$Aw(t) = \int_0^T \min\{s, t\}w(s)ds.$$

In addition, A is self-adjoint on $C_0'[0, T]$ and so

$$(w_1, Aw_2)_{C_0'} = (Sw_1, Sw_2)_{C_0'} = \int_0^T w_1(s)w_2(s)ds$$

for all $w_1, w_2 \in C_0'[0, T]$. Hence, A is a positive definite operator, i.e., $(w, Aw)_{C_0'} \geq 0$ for all $w \in C_0'[0, T]$. This means that the orthonormal eigenfunctions $\{e_m\}$ of A are given by

$$e_m(t) = \frac{\sqrt{2T}}{(m - \frac{1}{2})\pi} \sin\left(\frac{(m - \frac{1}{2})\pi}{T}t\right) \equiv \int_0^t \alpha_m(s)ds$$

with corresponding eigenvalues $\{\beta_m\}$ given by

$$\beta_m = \left(\frac{T}{(m - \frac{1}{2})\pi}\right)^2.$$

Furthermore, it can be shown that $\{e_m\}$ is a basis of $C_0'[0, T]$ and so $\{\alpha_m\}$ is a basis of Ł2, and that A is a trace class operator and so S is a Hilbert–Schmidt operator on $C_0'[0, T]$. In fact, the trace of A is given by $TrA = \frac{1}{2}T^2 = \int_0^T tdt$. By using the concept of m-lifting on abstract Wiener space, the operators S and A can be extended on $C_0[0, T]$ (see [19,25]).

We now give formulas with respect to the operators S and A, respectively.

Example 4. *Let S be given by Equation (33) and let $v(t) = \frac{1}{2}t^2$ and $h(t) = t$ on $[0, T]$. Then, $h \in \mathcal{H}$. In addition, we have*

$$v(t) = \frac{1}{2}t^2 = \int_0^t sds$$
$$Sv(t) = \int_0^t \frac{1}{2}s^2 ds = \frac{1}{6}t^3$$

and

$$S^*v(t) = tv(T) - Sv(t) = \frac{1}{2}tT^2 - \frac{1}{6}t^3 = \int_0^t \left[\frac{1}{2}T - \frac{1}{2}s^2\right] ds.$$

This means that $z_v(t) = t$ and $z_{S^*v}(t) = \frac{1}{2}T - \frac{1}{2}t^2$ on $[0, T]$ and hence $\|hz_{S^*v}\|_2^2 = \frac{1}{40}T^7$. Thus, we obtain that

$$\mathcal{T}^h_{S,R}(\Phi_v)(y) = \exp\left\{\frac{1}{80}T^7\right\}\Phi_{R^*v}(y).$$

Example 5. *Let $S = A$ and let $v(t) = \frac{1}{2}t^2$ and $h(t) = t$ on $[0, T]$. Then, $h \in \mathcal{H}$. In addition, we have*

$$v(t) = \frac{1}{2}t^2 = \int_0^t sds,$$

$$\begin{aligned} Av(t) = SS^*v(t) &= \int_0^t S^*v(s)ds = \int_0^t \int_0^s [v(T) - v(u)du]ds \\ &= \int_0^t \left[sv(T) - \int_0^s \frac{1}{2}u^2 du\right] ds = \int_0^t \left[\frac{1}{2}sT^2 - \frac{1}{6}s^3\right] ds \\ &= \frac{1}{4}T^2t^2 - \frac{1}{24}t^4 \end{aligned}$$

and

$$A^*v(t) = Av(t) = \int_0^t \left[\frac{1}{2}sT^2 - \frac{1}{6}s^3\right] ds.$$

This means that $z_v(t) = t$ and $z_{S^*v}(t) = \frac{1}{2}tT^2 - \frac{1}{6}t^3$ on $[0, T]$ and hence $\|hz_{A^*v}\|_2^2 = \frac{58}{2835}T^9$. Thus, we obtain that

$$\mathcal{T}^h_{A,R}(\Phi_v)(y) = \exp\left\{\frac{29}{2835}T^9\right\}\Phi_{R^*v}(y).$$

We now give an example with respect to Theorem 8.

Example 6. *Let $s_1 = T_m$ and $S_2 = S$, as used in the examples above. Let $R = I$ and let $h_1(t) = h_2(t) = t, h_3(t) = t^2$ on $[0, T]$. Furthermore, let $v(t) = \frac{1}{2}t^2$ on $[0, T]$ and let $x_0(t) = t = \int_0^t 1ds \in C_0'$. Then, we have $z_v(t) = t, z_{S_1^*v}(t) = \frac{3}{2}t^2, z_{S_2^*v}(t) = \frac{1}{2}T - \frac{1}{2}t^2$ and $z_{x_0}(t) = 1$ on $[0, T]$. Furthermore, we have*

$$M(S_1, RS_2 : h_1, h_2 : v) = \exp\left\{\frac{5}{28}T^7 + \frac{1}{24}T^4 - \frac{1}{20}T^5\right\}$$

and

$$(h_3 z_{x_0}, h_2 z_{S_2^* R^* v})_2 = \frac{1}{8}T^4 - \frac{1}{12}T^6.$$

Hence, by using Equation (30) in Theorem 8, we can conclude that

$$\begin{aligned} &\int_{C_0[0,T]} \delta^{h_2,h_3}_{S_2,S_2} \mathcal{T}^{h_1}_{S_1,R}(\Phi_v)(y|u)dm(y) \\ &= \exp\left\{\frac{5}{28}T^7 + \frac{1}{24}T^4 - \frac{1}{20}T^5\right\}\left(\frac{1}{8}T^4 - \frac{1}{12}T^6\right). \end{aligned} \tag{34}$$

7. Conclusions

In Sections 3 and 4, we establish some fundamental formulas for the generalized integral transform, the generalized convolution product and the generalized first variation involving the generalized Cameron–Storvick theorem. As shown in Examples 2, 4 and 6, various applications are established by choosing the kernel functions and operators. The results and formulas are more generalized forms than those in previous papers. From these, we can conclude that various examples can also be explained very easily.

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