mathematics

MDPI

Article

Regular CA-Groupoids and Cyclic Associative Neutrosophic Extended Triplet Groupoids (CA-NET-Groupoids) with Green Relations

Wangtao Yuan and Xiaohong Zhang *

Department of Mathematics, Shaanxi University of Science & Technology, Xi'an 710021, China; 1809007@sust.edu.cn
* Correspondence: zhangxiaohong@sust.edu.cn

Received: 18 December 2019; Accepted: 3 February 2020; Published: 6 February 2020

Abstract: Based on the theories of AG-groupoid, neutrosophic extended triplet (NET) and semigroup, the characteristics of regular cyclic associative groupoids (CA-groupoids) and cyclic associative neutrosophic extended triplet groupoids (CA-NET-groupoids) are further studied, and some important results are obtained. In particular, the following conclusions are strictly proved: (1) an algebraic system is a regular CA-groupoid if and only if it is a CA-NET-groupoid; (2) if $(S, *)$ is a regular CA-groupoid, then every element of S lies in a subgroup of S, and every $\mathcal{H}$-class in S is a group; and (3) an algebraic system is an inverse CA-groupoid if and only if it is a regular CA-groupoid and its idempotent elements are commutative. Moreover, the Green relations of CA-groupoids are investigated, and some examples are presented for studying the structure of regular CA-groupoids.

Keywords: semigroup; CA-groupoid; regular CA-groupoid; neutrosophic extended triplet (NET); Green relation

1. Introduction

The theory of group is an essential branch of algebra. The research of group has become an important trend in the theory of semigroup. Various algebraic structures are related to groups, such as regular semigroups, generalized groups, and neutrosophic extended triplet groups (see [1–6]). With the development of semigroup, the study of generalized regular semigroup has become an important topic. This paper focuses on the regularity of non-associative algebraic structures satisfying the cyclic associative law: $x(yz) = z(xy)$.

As early as 1954, Sholander [7] used the term of cyclic associative law to express the following operation law: $(ab)c = (bc)a$. Obviously, its dual form is as follows: $a(bc) = c(ab)$. At the same time, in 1954, Hosszu also used the term of cyclic associative law in the study of functional equation (see the introduction and explanation by Maksa [8]). In 1995, Kleinfeld [9] studied the rings with cyclic associative law $x(yz) = y(zx)$. Moreover, Zhan and Tan [10] introduced the notion of left weakly Novikov algebra. In many fields (such as non-associative rings and non-associative algebras [11–14]), image processing [15], and networks [16]), non-associativity has essential research significance. Since cyclic associative law is widely used in algebraic systems, we have been focusing on the basic algebraic structure of cyclic associative groupoids (CA-groupoids) and other relevant algebraic structures (see [17,18]).

Smarandache first proposed the new concept of neutrosophic set in [19]. The theory of neutrosophic set has been applied in many fields, such as applying neutrosophic soft sets in decision making, and proposing a new model of similarity in medical diagnosis and verifying its validity of l through a numerical example with practical background [20]. Later, Smarandache and colleagues extended the neutrosophic logic to the neutrosophic extended triplet group (NETG) [6]. In this paper, we analyze the structure of cyclic associative neutrosophic extended triplet groupoids (CA-NET-Groupoids).

Green's relations, first studied by Green [21] in 1951, have played a fundamental role in the development of regular semigroup theory. This has in turn completely illustrated the effectiveness of Green's method in studying semigroups, especially regular semigroups. Research on the Green relations of regular semigroups is at the core, and it involves almost all aspects of semigroup algebra theory. In 2011, Mary [22] studied the generalized inverse of semigroups by means of Green's relations. In 2017, Kufleitner and Manfred [23] considered the complexity of Green's relations when the semigroup is given by transformations on a finite set. This paper focuses on the Green's relations of CA-groupoids, in particular regular CA-groupoids. Recently, we analyzed these new results and studied them from the perspective of CA-groupoid theory. Miraculously, we obtained some unexpected results that, if S is a regular CA-groupoid, then every element of S lies in a subgroup of S, and every $\mathcal{H}$-class in S is a group.

The rest of this paper is organized as follows. In Section 2, we give the related concepts and results of the CA-groupoid. In Section 3, we give some basic concepts and examples of regular elements, strongly regular elements, inverse elements, and local associative and quasi-regular elements. In Section 4, we prove the equivalence of regular CA-groupoids and CA-NET-groupoids, and give corresponding examples. In Section 5, we discuss the Green's relations of CA-groupoids and the Green's relations of regular CA-groupoids. In Section 6, we propose a new concept of inverse CA-groupoids and prove that regular CA-groupoids, strongly regular CA-groupoids, CA-NET-groupoids, inverse CA-groupoids and commutative regular semigroups are equivalent. Finally, the summary and plans for future work are presented in Section 7.

2. Preliminaries

In this section, we give the related research and results of the CA-groupoid. Some related notions are introduced.

A groupoid is a pair $(S, \times)$ where S is a non-empty set together with a binary operation $\times$. Traditionally, the $\times$ operator is omitted without confusion.

Definition 1. *([4,5]) A groupoid $(S, \times)$ is called a neutrosophic extended triplet-groupoid NET-groupoid) if, for any $a \in S$, there exist a neutral of "a" (denoted by neut(a)), and an opposite of "a" (denoted by anti(a)), such that neut(a)$\in$S, anti(a)$\in$S, and:*

$$a \times neut(a) = neut(a) \times a = a; a \times anti(a) = anti(a) \times a = neut(a)$$

The triplet (a, neut(a), anti(a)) is called a neutrosophic extended triplet.

Let $(S, \times)$ be a groupoid. Some concepts are defined as follows:

(1) An element $a \in S$ is called idempotent if $a^2 = a$.
(2) S is called semigroup if, for any $a, b, c \in S$, $a \times (b \times c) = (a \times b) \times c$. A semigroup $(S, \times)$ is commutative if, for all $a, b \in S$, $a \times b = b \times a$.

Here, recall some basic concepts in the semigroup theory. A non-empty subset A of a semigroup $(S, \times)$ is called a left ideal if $SA \subseteq A$, a right ideal if $AS \subseteq A$, and an ideal if it is both a left and a right ideal. If a is an element of a semigroup $(S, \times)$, the smallest left ideal containing a is $Sa \cup \{a\}$, which we may conveniently write as S^1a.

An element a of a semigroup S is called regular if there exists x in S such that $a \times x \times a = a$. The semigroup S is called regular if all its elements are regular.

Among idempotents in an arbitrary semigroup, there is a natural (partial) order relation defined by the rule that $e \leq f$ if and only if $e \times f = f \times e = e$. It is easy to verify that the given relation has the properties (reflexive), (antisymmetric) that define an order relation. Certainly, it is clear that $e \leq e$, and that $e \le f$ and $f \le e$ together implies that $e = f$. To show transitivity, notice that, if $e \le f$ and $f \le g$, so that $e \times f = f \times e = e$ and $f \times g = g \times f = f$, then $e \times g = e \times f \times g = e \times f = e$ and $g \times e = g \times f \times e = f \times e = e$, and thus $e \le g$.

Let S be a regular semigroup and let $E(S)$ denote the set of idempotents of S. For each $e \in E(S)$, let G_e be a subgroup of S with identity e. If $T(S) = \cup\,(G_e\colon e \in E(S))$ is a subsemigroup and $e, f, g \in E(S)$, $e \ge f$, and $e \ge g$ imply $f \times g = g \times f$, we term S a strongly regular semigroup [24].

An equivalent relation $\mathcal{L}$ on S is defined by the rule that $a\mathcal{L}b$ if and only if $S^1a = S^1b$; an equivalent relation $\mathcal{R}$ on S is defined by the rule that $a\mathcal{R}b$ if and only if $aS^1 = bS^1$; denote $\mathcal{H} = \mathcal{L}\cap\mathcal{R}, \mathcal{D} = \mathcal{L}\cup\mathcal{R}$, that is, $a\mathcal{H}b$ if and only if $S^1a = S^1b$ and $aS^1 = bS^1$; $a\mathcal{D}b$ if and only if $S^1a = S^1b$ or $aS^1 = bS^1$. An equivalent relation $\mathcal{J}$ on S is defined by the rule that $a\mathcal{J}b$ if and only if $S^1aS^1 = S^1bS^1$, where:

$$S^1aS^1 = SaS \cup aS \cup Sa \cup \{a\}.$$

That is, $a\mathcal{J}b$ if and only if there exists $x, y, u, v \in S^1$ for which $x \times a \times y = b$, $u \times b \times v = a$. The $\mathcal{L}$-class ($\mathcal{R}$-class, $\mathcal{H}$-class,$\mathcal{D}$-class,$\mathcal{J}$-class) containing the element a is written $\mathcal{L}a$ ($\mathcal{R}a$,$\mathcal{H}a$,$\mathcal{D}a$,$\mathcal{J}a$).

Definition 2. *([7–10,25]) Let $(S, \times)$ be a groupoid. If, for all $a, b, c \in S$,*

$$a \times (b \times c) = c \times (a \times b),$$

then $(S, \times)$ is called a cyclic associative groupoid (shortly, CA-groupoid).

Proposition 1. *([25]) Let $(S, \times)$ be a CA-groupoid. Then, for any $a, b, c, d, x, y \in S$,*

(1) $(a \times b) \times (c \times d) = (d \times a) \times (c \times b)$; and
(2) $(a \times b) \times ((c \times d) \times (x \times y)) = (d \times a) \times ((c \times b) \times (x \times y))$.

Definition 3. *([25]) A NET-groupoid $(S, \times)$ is called cyclic associative (shortly, CA-NET-groupoid) if it is cyclic associative as a groupoid. S is called a commutative CA-NET-groupoid if, for all $a, b \in N$, $a \times b = b \times a$.*

Theorem 1. *([25]) Let $(S, \times)$ be a CA-NET-groupoid. Then, for any $a, p, q \in N$ and $anti(a) \in \{anti(a)\}$,*

(1) $q \times neut(a) \in \{anti(a)\}$, for all $q \in \{anti(a)\}$;
(2) $p \times neut(a) = q \times neut(a)$, for all $p, q \in \{anti(a)\}$; and
(3) $neut(p) \times neut(a) = neut(a) \times neut(p) = neut(a)$, for all $p \in \{anti(a)\}$.

Remark 1. *Since there may be more than one anti-element of an element a, the symbol $\{anti(a)\}$ is used to represent the set of all anti elements of a. Therefore, the meaning of $q \in \{anti(a)\}$ is that q is an anti-element of a.*

Theorem 2. *([25]) Let $(S, \times)$ be a CA-NET-groupoid. Denote the set of all different neutral element in S by $E(S)$. For any $e \in E(S)$, denote $S(e) = \{a \in S|\ neut(a) = e\}$. Then, for any $e \in E(S)$, $S(e)$ is a subgroup of S.*

3. Regular and Inverse Elements in Cyclic Associative Groupoids (CA-Groupoids)

Definition 4. *An element a of a CA-groupoid $(S, \times)$ is called regular if there exists $x \in S$ such that*

$$a = a \times (x \times a)$$

$(S, \times)$ is called a regular CA-groupoid if all its elements are regular.

Definition 5. *An element a of a CA-groupoid $(S, \times)$ is called strongly regular if there exists $x \in S$ such that*

$$a = a \times (x \times a) \text{ and } a = (a \times x) \times a$$

$(S, \times)$ is called strongly regular CA-groupoid if all its elements are strongly regular.

Example 1. *Denote $S = \{a, b, c\}$ and define operations $\times$ on S, as shown in Table 1. We can verify that a is strongly regular, since $a = a \times (a \times a) = (a \times a) \times a$; b is regular, since $b = b \times (b \times b)$. However, b is not strongly regular, since $b \neq (b \times b) \times b$, and there does not exist $x \in S$ such that $b = b \times (x \times b) = (b \times x) \times b$.*

Table 1. The operation $\times$ on S.

$\times$	a	b	c
a	a	a	c
b	b	a	c
c	c	c	c

Example 2. *Let $S = \{1, 2, 3, 4\}$. The operation $\times$ on S is defined as Table 2. We can verify that $(S, \times)$ is a commutative semigroup, then for any $a, b, c \in S$, we have $a \times (b \times c) = (a \times b) \times c = c \times (a \times b)$. Thus, $(S, \times)$ is a commutative CA-groupoid. Moreover, $(S, \times)$ is an AG-groupoid because $(S, \times)$ is a commutative CA-groupoid. In addition, $(S, \times)$ is a regular semigroup, because $1 = 1 \times 1 \times 1$, $2 = 2 \times 2 \times 2$, $3 = 3 \times 1 \times 3$, $4 = 4 \times 2 \times 4$. $(S, \times)$ is also a regular CA-groupoid, since $1 = 1 \times (1 \times 1)$, $2 = 2 \times (2 \times 2)$, $3 = 3 \times (1 \times 3)$, $4 = 4 \times (2 \times 4)$. $(S, \times)$ is also a regular AG-groupoid, since $1 = (1 \times 1) \times 1$, $2 = (2 \times 2) \times 2$, $3 = (3 \times 1) \times 3$, $4 = (4 \times 4) \times 4$.*

Table 2. The operation $\times$ on S.

$\times$	**1**	**2**	**3**	**4**
1	1	2	3	4
2	2	1	4	3
3	3	4	3	4
4	4	3	4	3

Example 3. *Let $S = \{1, 2, 3, 4, 5\}$. The operation $\times$ on S is defined as Table 3. We can verify that $(S, \times)$ is a strongly regular semigroup. However, $(S, \times)$ is not a CA-groupoid because $3 \times (4 \times 5) \neq 5 \times (3 \times 4)$.*

Table 3. The operation $\times$ on S.

$\times$	**1**	**2**	**3**	**4**	**5**
1	1	1	1	1	1
2	1	2	1	1	5
3	1	1	3	4	1
4	1	4	1	1	3
5	1	1	5	2	1

Example 4. *Let $S = \{1, 2, 3, 4\}$. The operation $\times$ on S is defined as Table 4. We can verify that $(S, \times)$ is a strongly regular CA-groupoid, since $1 = 1 \times (1 \times 1) = (1 \times 1) \times 1$, $2 = 2 \times (4 \times 2) = (2 \times 4) \times 2$, $3 = 3 \times (3 \times 3) = (3 \times 3) \times 3$, $4 = 4 \times (2 \times 4) = (4 \times 2) \times 4$. $(S, \times)$ is also a strongly regular semigroup.*

Table 4. The operation $\times$ on S.

$\times$	**1**	**2**	**3**	**4**
1	1	1	1	1
2	1	4	2	3
3	1	2	3	4
4	1	3	4	2

An idea of great important in CA-groupoid theory is that of an inverse of an element.

Definition 6. *For any element a in a CA-groupoid S, we say that* a^{-1} *is an inverse of a if satisfied*

$$a = a \times (a^{-1} \times a), a^{-1} \times (a \times a^{-1}) = a^{-1} \tag{1}$$

Notice that an element with an inverse is necessarily regular. Less obviously, each regular element has an inverse; for if $a \times (x \times a) = a$ *we need only define* $a^{-1} = x \times (a \times x)$ *and verify that Equation (1) are satisfied.*

Theorem 3. *Let* $(S, \times)$ *be a regular CA-groupoid; then, each of its elements has an inverse and the inverse is unique.*

Proof. Let x_1, x_2 be inverses of a in S. Then, we have $a = a \times (x_1 \times a)$, $x_1 = x_1 \times (a \times x_1)$ and $a = a \times (x_2 \times a)$, $x_2 = x_2 \times (a \times x_2)$,

$$x_1 = x_1 \times (a \times x_1) = x_1 \times (x_1 \times a) = x_1 \times (x_1 \times (a \times (x_2 \times a))) = x_1 \times (x_1 \times (a \times (a \times x_2)))$$

$$= x_1 \times ((a \times x_2) \times (x_1 \times a))$$

$$= x_1 \times ((a \times a) \times (x_1 \times x_2)) \text{ (Applying Proposition 1)}$$

$$= (x_1 \times x_2) \times (x_1 \times (a \times a)) = (x_1 \times x_2) \times (a \times (x_1 \times a)) = (x_1 \times x_2) \times a.$$

Similarly, we can get that $x_2 = (x_2 \times x_1) \times a$.
Then, we have

$$(x_1 \times a) \times x_2 = (x_1 \times a) \times ((x_2 \times x_1) \times a) = a \times ((x_1 \times a) \times (x_2 \times x_1)) = (x_2 \times x_1) \times (a \times (x_1 \times a))$$
$$= (x_2 \times x_1) \times a = x_2,$$

$$x_1 \times x_2 = x_1 \times ((x_2 \times x_1) \times a) = a \times (x_1 \times (x_2 \times x_1)) = (x_2 \times x_1) \times (a \times x_1)$$

$$= (x_1 \times x_2) \times (a \times x_1) \text{ (Applying Proposition 1)}$$

$$= x_1 \times ((x_1 \times x_2) \times a) = x_1 \times x_1.$$

Similarly, we can get that $x_2 \times x_1 = x_2 \times x_2$. Further, we have,

$$x_1 \times x_2 = x_1 \times ((x_1 \times a) \times x_2) = x_2 \times (x_1 \times (x_1 \times a)) = x_2 \times (a \times (x_1 \times x_1)) = (x_1 \times x_1) \times (x_2 \times a)$$
$$= (x_1 \times x_2) \times (x_2 \times a)$$

$$= (a \times x_1) \times (x_2 \times x_2) \text{ (Applying Proposition 1)}$$

$$= (a \times x_1) \times (x_2 \times x_1)$$

$$= (x_1 \times a) \times (x_2 \times x_2) \text{ (Applying Proposition 1 and } x_2 \times x_1 = x_2 \times x_2)$$

$$= x_2 \times ((x_1 \times a) \times x_2) = x_2 \times x_2.$$

Thus, $x_1 \times x_2 = x_2 \times x_1$, $x_1 = (x_1 \times x_2) \times a = (x_2 \times x_1) \times a = x_2$.

Therefore, in a regular CA-groupoid, each of its elements has an inverse and the inverse is unique. □

Example 5. *Let S = {1, 2, 3, 4, 5, 6}. The operation × on S is defined as Table 5. We can verify that (S, ×) is a CA-groupoid; element 3 is an inverse of 3 because 3 = 3 × (3 × 3), 3 = 3 × (3 × 3), obviously element 3 is a regular; and element 5 is an inverse of 5 since 5 = 5 × (5 × 5), 5 = 5 × (5 × 5), obviously element 5 is a regular. However, elements 1, 2, 4, and 6 have no inverses because there exists no x, y, p, q∈S such that 1 = 1 × (x × 1), x = x × (1 × x); 2 = 2 × (y × 2), y = y × (2 × y); 4 = 4 × (p × 4), p = p × (4 × p); and 6 = 6 × (q × 6), q = q × (6 × q). Obviously, for any a ∈ S, if a ∉ a × S, then a has not inverse.*

Table 5. The operation × on *S*.

×	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	3	3	5	2
2	4	3	3	3	5	2
3	3	3	3	3	5	2
4	3	3	3	3	5	2
5	5	5	5	5	3	5
6	4	3	3	3	5	3

Example 6. *Let S = {1, 2, 3, 4, 5, 6}. The operation × on S is defined as Table 6. We can verify that (S, ×) is a regular CA-groupoid, since 1 = 1 × (1 × 1), 2 = 2 × (2 × 2), 3 = 3 × (3 × 3), 4 = 4 × (4 × 4), 5 = 5 × (5 × 5), 6 = 6 × (6 × 6), and the inverse is unique.*

Table 6. The operation × on *S*.

×	**1**	**2**	**3**	**4**	**5**	**6**
1	1	2	5	5	5	6
2	2	1	5	5	5	6
3	5	5	3	4	5	6
4	5	5	4	3	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	5

Definition 7. *An element a of a CA-groupoid (S, ×) is called locally associative if satisfied*

$$a \times (a \times a) = (a \times a) \times a.$$

(S, ×) is called locally associative CA-groupoid if all its elements are locally associative.

Example 7. *Let S = {1, 2, 3, 4, 5}. The operation × on S is defined as Table 7. We can verify that (S, ×) is a locally associative CA-groupoid, since 1 × (1 × 1) = (1 × 1) × 1, 2 × (2 × 2) = (2 × 2) × 2, 3 × (3 × 3) = (3 × 3) × 3, 4 × (4 × 4) = (4 × 4) × 4, and 5 × (5 × 5) = (5 × 5) × 5. However, (S, ×) is not a semigroup because (3 × 4) × 3≠ 3 × (4 × 3).*

Table 7. The operation × on *S*.

×	**1**	**2**	**3**	**4**	**5**
1	1	1	1	1	2
2	1	1	2	1	2
3	1	1	4	2	4
4	1	1	2	1	2
5	1	1	4	2	4

Definition 8. *An element a of a CA-groupoid (S, ×) is called quasi-regular if there exists $x \in S$, $m \in N$ such that*

$$a^m \times (x \times a^m) = a^m.\ (a^m \text{ is defined by } a \times a^{m-1})$$

(S, ×) is called quasi-regular CA-groupoid if all its elements are quasi-regular.

Example 8. *Let S = {1, 2, 3, 4}. The operation × on S is defined as Table 8. We can verify that (S, ×) is a quasi-regular CA-groupoid, since $1 = 1^2 \times (3 \times 1^2)$, $2 = 2 \times (2 \times 2)$, $3 = 3 \times (3 \times 3)$, $4^2 = 4^2 \times (2 \times 4^2)$. However, (S, ×) is not a regular CA-groupoid because there exists no x, y∈S such that 1 = 1 × (x × 1), 4 = 4 × (y × 4). Moreover, (S, ×) is not a semigroup because (4 × 1) × 1 ≠ 4 × (1 × 1).*

Table 8. The operation × on *S*.

×	**1**	**2**	**3**	**4**
1	3	2	3	2
2	2	2	2	2
3	3	2	3	2
4	4	2	2	2

Definition 9. *Let (S, ×) be a groupoid. If for all a, b, c ∈ S,*

$$a \times (b \times c) = (a \times b) \times c,\ a \times (b \times c) = c \times (a \times b),$$

then (S, ×) is called cyclic associative semigroup (shortly, CA-semigroup).

Example 9. *Suppose S = {1, 2, 3, 4} and define a binary operation × on S as shown in Table 9. We can verify that (S, ×) is a CA-groupoid, but (S, ×) is not a CA-semigroup because (3 × 4) × 3 ≠ 3 × (4 × 3).*

Table 9. The operation × on *S*.

×	**1**	**2**	**3**	**4**
1	1	1	1	1
2	1	1	2	1
3	1	1	4	2
4	1	1	2	1

Obviously on the CA-groupoid *S*, there is: strongly regular element ⇒ regular element ⇒ inverse element ⇒ quasi-regular element.

According to Examples 1, 2, and 5–9, we can get the relationship between CA-groupoids and related algebraic systems, which we can be expressed as Figure 1.

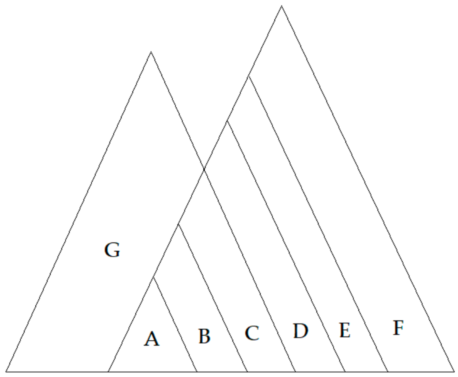

Figure 1. The relationships among some algebraic systems.

Remark 2. *In Figure 1, each letter only indicates the smallest area in which it is located. Here, A represents the set of all strongly regular CA-groupoids, and*

A∪B represents the set of all regular CA-groupoids;
A∪B∪C represents the set of all CA-semigroups;
A∪B∪C∪D represents the set of all quasi-regular CA-groupoids;
A∪B∪C∪D∪E represents the set of all locally associative CA-groupoids;
A∪B∪C∪D∪E∪F represents the set of all CA-groupoids; and
A∪B∪C∪G represents the set of all semigroups.

4. Regular Cyclic Associative Groupoids (CA-Groupoids) and Cyclic Associative Neutrosophic Extended Triplet Groupoids (CA-NET-Groupoids)

Theorem 4. *Let $(S, \times)$ be a CA-NET-groupoid. Then, its idempotents are commutative.*

Proof. Let a, b an idempotent in S; then, we have

$$(a \times b) \times (a \times b) = (b \times a) \times (a \times b) \text{ (Applying Proposition 1)}$$

$$= (b \times b) \times (a \times a) \text{ (Applying Proposition 1)} = b \times a.$$

Moreover

$$(a \times b) \times (a \times b) = (b \times (neut(b) \times a)) \times (a \times b)$$

$$= (b \times b) \times (a \times (neut(b) \times a)) \text{ (Applying Proposition 1)}$$

$$= (b \times b) \times (a \times (a \times neut(b))) = (b \times b) \times (neut(b) \times (a \times a))$$

$$= b \times (neut(b) \times a) = a \times (b \times neut(b)) = a \times b.$$

Therefore, $a \times b = b \times a$. In a CA-NET-groupoid, its idempotents are commutative. □

Corollary 1. *Every CA-NET-groupoid is commutative.*

Proof. Let $(S, \times)$ be a CA-NET-groupoid. By Theorem 4, for any $x \in S$, $neut(x)$ is idempotent. Then, for any $a, b \in S$, we have

$$neut(a) \times neut(b) = neut(b) \times neut(a),$$

Furthermore,

$$neut(a) \times b = neut(a) \times (neut(b) \times b) = b \times (neut(a) \times neut(b))$$

$$= neut(b) \times (b \times neut(a)) = (neut(b) \times neut(b)) \times (b \times neut(a))$$

$$= (neut(a) \times neut(b)) \times (b \times neut(b)) \text{ (Applying Proposition 1)}$$

$$= (neut(a) \times neut(b)) \times (neut(b) \times b)$$

$$= (b \times neut(a)) \times (neut(b) \times neut(b)) \text{ (Applying Proposition 1)}$$

$$= (b \times neut(a)) \times neut(b)$$

Further, for any $a, b \in S$, we have

$$a \times b = (neut(a) \times a) \times (neut(b) \times b)$$

$$= (b \times neut(a)) \times (neut(b) \times a) \text{ (Applying Proposition 1)}$$

$$= a \times ((b \times neut(a)) \times neut(b))$$

$$= a \times (neut(a) \times b) \text{ (by } neut(a) \times b)$$

$$= (b \times neut(a)) \times neut(b))$$

$$= b \times (a \times neut(a))$$

$$= b \times a$$

Therefore, every CA-NET-groupoid is commutative. □

Example 10. *Let S = {1, 2, 3, 4, 5}. The operation × on S is defined as Table 10. We can verify that (S, ×) is a CA-NET-groupoid, and*

$$neut(1) = 1, anti(1) = \{1, 5\}; neut(2) = 2, anti(2) = \{1, 2, 3, 4, 5\};$$

$$neut(3) = 3, anti(3) = \{3, 5\}; neut(4) = 4, anti(4) = \{1, 3, 4, 5\}; neut(5) = 5, anti(3) = 5.$$

Obviously, (S, ×) is a commutative.

Table 10. The operation × on S.

×	1	2	3	4	5
1	1	2	4	4	1
2	2	2	2	2	2
3	4	2	3	4	3
4	4	2	4	4	4
5	1	2	3	4	5

Theorem 5. *Let (S, ×) be a groupoid. Then, S is a CA-NET-groupoid if and only if it is a regular CA-groupoid.*

Proof. Assume that S is a CA-NET-groupoid. For any a in S, by Definitions 1 and 3, we have

$$a \times (anti(a) \times a) = a \times neut(a) = a.$$

From this and Definition 4, we know that element a is a regular element and S is a regular CA-groupoid.

Therefore, we prove that S is a regular CA-groupoid.

Now, we assume that S is a regular CA-groupoid. For any a in a regular CA-groupoid S, we have

$$a \times (x \times a) = a.$$

Furthermore,

$$(x \times a) \times a = (x \times a) \times (a \times (x \times a)) = (x \times a) \times ((x \times a) \times a) = a \times ((x \times a) \times (x \times a))$$

$$= a \times (a \times ((x \times a) \times x))$$

$$= a \times (x \times (a \times (x \times a)))$$

$$= a \times (x \times a) = a.$$

Therefore, there exists $(x \times a) \in S$, such that $(x \times a) \times a = a \times (x \times a) = a$.
Moreover, we have

$$(x \times a) = x \times (a \times (x \times a)) = (x \times a) \times (x \times a) = a \times ((x \times a) \times x).$$

Furthermore,

$$((x \times a) \times x) \times a = ((x \times a) \times x) \times (a \times (x \times a))$$

$$= (x \times a) \times (((x \times a) \times x) \times a) = a \times ((x \times a) \times ((x \times a) \times x))$$

$$= a \times (x \times ((x \times a) \times (x \times a)))$$

$$= a \times (x \times (x \times a)) \ (by\ (x \times a) \times (x \times a) = (x \times a))$$

$$= (x \times a) \times (a \times x)$$

$$= x \times ((x \times a) \times a) \ (by\ (x \times a) \times a = a)$$

$$= x \times a$$

Therefore, there exists $((x \times a) \times x) \in S$, such that $a \times ((x \times a) \times x) = ((x \times a) \times x) \times a = x \times a$. Then, S is a CA-NET-groupoid. □

Example 11. *Let $S = \{1, 2, 3, 4\}$. The operation $\times$ on S is defined as Table 11. We can verify that $(S, \times)$ is a CA-NET- groupoid, and $neut(1) = 1$, $anti(1) = \{1, 2, 3, 4\}$; $neut(2) = 3$, $anti(2) = 4$; $neut(3) = 3$, $anti(3) = 3$; $neut(4) = 3$, $anti(4) = 2$.*

Table 11. The operation $\times$ on S.

$\times$	**1**	**2**	**3**	**4**
1	1	1	1	1
2	1	4	2	3
3	1	2	3	4
4	1	3	4	2

Moreover, $(S, \times)$ is a regular CA-groupoid, since $1 = 1 \times (1 \times 1)$, $2 = 2 \times (4 \times 2)$, $3 = 3 \times (3 \times 3)$, $4 = 4 \times (2 \times 4)$.

Definition 10. *Let $(S, \times)$ be a groupoid.*

(1) If for any $a \in S$, there exist two elements b and c in S satisfying the condition $a \times b = a$ and $c \times a = b$, then S is called a CA-(r, l)-NET-groupoid.

(2) If for any $a \in S$, there exist two elements b and c in S satisfying the condition $a \times b = a$ and $a \times c = b$, then S is called a CA-(r, r)-NET-groupoid.

(3) If for any $a \in S$, there exist two elements b and c in S satisfying the condition $b \times a = a$ and $a \times c = b$, then S is called a CA-(l, r)-NET-groupoid.

(4) If for any $a \in S$, there exist two elements b and c in S satisfying the condition $b \times a = a$ and $c \times a = b$, then S is called a CA-(l, l)-NET-groupoid.

Theorem 6. *Let (S, ×) be a groupoid. Then, S is a CA-(r, l)-NET-groupoid if and only if it is a regular CA-groupoid.*

Proof. Assume that S is a *CA-(r, l)-NET*-groupoid. For any a in S, by Definitions 1 and 10(1), we have

$$a \times neut(a) = a, anti(a) \times a = neut(a)$$

$$a \times (anti(a) \times a) = a \times neut(a) = a$$

From this and Definition 4, we know that element a is a regular element and S is a regular CA-groupoid. Therefore, we prove that S is a regular CA-groupoid.

Now, we assume that S is a regular CA-groupoid. For any a in a regular CA-groupoid S, we have

$$a \times (x \times a) = a.$$

Thus, there exists $(x \times a) \in S$, such that $a \times (x \times a) = a$.

Moreover, we have:

$$x \times a = (x \times a).$$

Therefore, there exists $x \in S$, such that $x \times a = (x \times a)$. Then, S is a CA-(r, l)-NET-groupoid. □

Theorem 7. *Let (S, ×) be a groupoid. Then, S is a CA-(r, r)-NET-groupoid if and only if it is a regular CA-groupoid.*

Proof. Assume that S is a CA-(r, r)-NET-groupoid. For any a in S, by Definitions 1 and 10(2), we have

$$a \times neut(a) = a, a \times anti(a) = neut(a),$$

$$a \times (anti(a) \times a) = a \times (a \times anti(a)) = a \times neut(a) = a$$

From this and Definition 4, we know that element a is a regular element and S is a regular CA-groupoid. Therefore, we prove that S is a regular CA-groupoid.

Now, we assume that S is a regular CA-groupoid, for any a in a regular CA-groupoid S, we have

$$a \times (x \times a) = a \times (a \times x) = a$$

Thus, there exists $(a \times x) \in S$, such that $a \times (a \times x) = a$.

Moreover, we have

$$a \times x = (a \times x)$$

Therefore, there exists $x \in S$, such that $a \times x = (a \times x)$. Then, S is a CA-(r, r)-NET-groupoid. □

Theorem 8. *Let (S, ×) be a groupoid. Then, S is a CA-(l, r)-NET-groupoid if and only if it is a regular CA-groupoid.*

Proof. Assume that S is a CA-(l, r)-NET-groupoid. For any a in S, by Definitions 1 and 10(3), we have

$$neut(a) \times a = a, a \times anti(a) = neut(a)$$

$$neut(a) \times a = (a \times anti(a)) \times a = (a \times anti(a)) \times (neut(a) \times a)$$

$$= (a \times a) \times (neut(a) \times anti(a)) \text{ (Applying Proposition 1)}$$

$$= (anti(a) \times a) \times (neut(a) \times a) \text{ (Applying Proposition 1)}$$

$$= (anti(a) \times a) \times a.$$

Thus, $a \times anti(a) = anti(a) \times a = neut(a)$.

Moreover, we have

$$a \times neut(a) = (neut(a) \times a) \times (anti(a) \times a)$$

$$= (a \times neut(a)) \times (anti(a) \times a) \text{ (Applying Proposition 1)}$$

$$= (a \times neut(a)) \times neut(a).$$

Thus, a $\times$ $neut(a) = neut(a) \times a = a$.

Then,

$$(anti(a) \times a) \times a = neut(a) \times a = a \times neut(a) = a \times (anti(a) \times a) = a$$

From this and Definition 4, we know that element a is a regular element and S is a regular CA-groupoid. Therefore, we prove that S is a regular CA-groupoid.

Now, we assume that S is a regular CA-groupoid. For any a in a regular CA-groupoid S, let $a = (a \times x) \times a$. We have

$$x \times a = x \times ((a \times x) \times a) = a \times (x \times (a \times x)) = ((a \times x) \times (a \times x)),$$

$$(a \times x) \times a = (a \times x) \times ((a \times x) \times a) = a \times ((a \times x) \times (a \times x)) = a \times (x \times a) = a, a \times x = (a \times x)$$

Therefore, S is a CA-(l, r)-NET-groupoid. □

Theorem 9. *Let $(S, \times)$ be a groupoid. Then, S is a CA-(l, l)-NET-groupoid if and only if it is a regular CA-groupoid.*

Proof. Assume that S is a CA-(l, l)-NET-groupoid. For any a in S, by Definitions 1 and 10(4), we have

$$neut(a) \times a = a, anti(a) \times a = neut(a),$$

$$a \times neut(a) = (neut(a) \times a) \times (anti(a) \times a)$$

$$= (a \times neut(a)) \times (anti(a) \times a) \text{ (Applying Proposition 1)}$$

$$= (a \times neut(a)) \times neut(a)$$

Thus, $a \times neut(a) = neut(a) \times a = a$.

Then,

$$(anti(a) \times a) \times a = neut(a) \times a = a \times neut(a) = a \times (anti(a) \times a) = a.$$

From this and Definition 4, we know that element a is a regular element and S is a regular CA-groupoid. Therefore, we prove that S is a regular CA-groupoid.

Now, we assume that S is a regular CA-groupoid. For any a in a regular CA-groupoid S, let $a = (x \times a) \times a$, we have

$$x \times a = x \times ((a \times x) \times a) = a \times (x \times (a \times x)) = ((a \times x) \times (a \times x)),$$

$$(x \times a) \times a = (x \times a) \times ((x \times a) \times a) = a \times ((x \times a) \times (x \times a)) = (x \times a) \times (a \times (x \times a))$$

$$= (x \times a) \times (a \times (a \times x)) = (a \times x) \times ((x \times a) \times a) = (a \times x) \times a$$

$$= (a \times x) \times ((a \times x) \times a) \; (by\; (a \times x) \times a = a)$$

$$= a \times ((a \times x) \times (a \times x))$$

$$= a \times (x \times a) = a.$$

Moreover, we have $x \times a = (x \times a)$. Therefore, S is a CA-(l, l)-NET-groupoid. □

Example 12. *Denote S = {1, 2, 3, 4} and define operations × on S as shown in* Table 12. *We can verify that (S, ×) is a CA-(r, l)-NET-groupoid, and,*

$$neut_{(r,\ l)}(1) = 1,\ anti_{(r,\ l)}(1) = \{1, 2, 3, 4\};\ neut_{(r,\ l)}(2) = 4,\ anti_{(r,\ l)}(2) = 2;$$

$$neut_{(r,\ l)}(3) = 3,\ anti_{(r,\ l)}(3) = 3;\ neut_{(r,\ l)}(4) = 4,\ anti_{(r,\ l)}(4) = 4$$

Table 12. The operation × on *S*.

×	**1**	**2**	**3**	**4**
1	1	1	1	1
2	1	4	1	2
3	1	1	3	1
4	1	2	1	4

It is easy to verify that S is also a CA-(r, r)-NET-groupoid, CA-(l, r)-NET-groupoid, CA-(l, l)-NET-groupoid.

Moreover, (S, ×) is a regular CA-groupoid, since 1 = 1 × (2 × 1), 2 = 2 × (2 × 2), 3 = 3 × (3 × 3), and 4 = 4 × (4 × 4).

5. Green Relations in Cyclic Associative Groupoids (CA-Groupoids)

If *a* is an element of a CA-groupoid *S*, the smallest left ideal of *S* containing *a* is $Sa \cup \{a\}$.

Definition 11. *Let (S, ×) be a CA-groupoid, for any a, b ∈ S, define the following binary relationships:*

$$a\mathcal{L}b \Leftrightarrow Sa \cup \{a\} = Sb \cup \{b\};$$
$$a\mathcal{R}b \Leftrightarrow aS \cup \{a\} = bS \cup \{b\};$$
$$a\mathcal{J}b \Leftrightarrow (Sa \cup \{a\})S \cup (Sa \cup \{a\}) = (Sb \cup \{b\})S \cup (Sb \cup \{b\});$$
$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

We call $\mathcal{L}$,$\mathcal{R}$,$\mathcal{J}$, and $\mathcal{H}$ the Green's relations on the CA-groupoid.

Definition 12. *Let (S, ×) be a CA-groupoid. A relation R on the set S is called left compatible (with the operation on S) if*

$$(\forall a,\ s,\ t \in S)\ (s,\ t)\ \in\ R\ \Rightarrow (a \times s\ , a \times t)\ \in R,$$

and right compatible if

$$(\forall a,\ s,\ t \in S)\ (s,\ t)\ \in\ R\ \Rightarrow (s \times a\ , t \times a)\ \in R.$$

It is called compatible if

$$(\forall s,\ t, s'\ t'\ \in S)\ [(s,\ t)\ \in\ R \text{ and } (s',\ t'\ \in R)] \Rightarrow (s \times s'\ , t \times t')\ \in R.$$

A left [right] compatible equivalence is called a left [right] congruence. A compatible equivalence relation is called a congruence.

Proposition 2. *Let a, b be elements of a CA-groupoid S. If a = b, then $a\mathcal{L}a$, $a\mathcal{R}a$. If a≠b, then $a\mathcal{L}b$ if and only if there exists x, y in S such that x × a = b, y × b = a. In addition, $a\mathcal{R}\,b$ if and only if there exists u, v in S such that a × u = b, b × v = a.*

Another immediate property of this is as follows:

Proposition 3. *$\mathcal{L}$ is a left congruence and $\mathcal{R}$ is a right congruence.*

Corollary 2. *In a CA-groupoid S, $\mathcal{L}$ and $\mathcal{R}$ are not commutative. That is, as a binary relationship, $\mathcal{L}\circ\mathcal{R}\neq\mathcal{R}\circ\mathcal{L}$.*

Example 13. *Let S = {1, 2, 3, 4, 5, 6}. The operation × on S is defined as Table 13. Then, (S, ×) is a CA-groupoid.*

Table 13. The operation × on *S*.

×	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	3	3	5	2
2	4	3	3	3	5	2
3	3	3	3	3	5	4
4	3	3	3	3	5	4
5	5	5	5	5	3	5
6	3	3	3	3	5	3

$\mathcal{L}$ = {< 3, 5>, <5, 3>}, $\mathcal{R}$ = {<3, 4>, <4, 3>}. $\mathcal{L}\circ\mathcal{R}$ = {<5, 4>} ≠ $\mathcal{R}\circ\mathcal{L}$ = {<4, 5>}. Then, $\mathcal{L}$ and $\mathcal{R}$ are not commutative.

In a regular CA-groupoid *S* we have a particularly useful way of looking at the equivalences $\mathcal{L}$ and $\mathcal{R}$. First, notice that if *S* is regular then $a = a \times (x \times a) \in aS$, and similarly $a \in Sa$, $a \in SaS$. Hence, in describing the Green equivalences for a regular CA-groupoid we can drop all reference to $Sa\cup\{a\}$, and assert simply that

$$a\mathcal{L}b \Leftrightarrow Sa = Sb;$$
$$a\text{b} \Leftrightarrow aS = bS;$$
$$a\mathcal{J}b \Leftrightarrow SaS = SbS;$$
$$\mathcal{H} = \mathcal{L}\cap\mathcal{R}.$$

Definition 13. *Let (S, ×) be a regular CA-groupoid, define the following binary relationship:*

$$\mathcal{D} = \mathcal{L}\cup\mathcal{R}$$

We call $\mathcal{D}$ the Green's relations on the regular CA-groupoid.

Theorem 10. *In a regular CA-groupoid* S*, the relations$\mathcal{L}$ and $\mathcal{R}$ are commutative. That is, as a binary relationship, $\mathcal{L}\circ\mathcal{R} = \mathcal{R}\circ\mathcal{L}$.*

Proof. Let (*S*, ×) be a regular CA-groupoid, let $a, b \in S$, and suppose that $(a, b) \in \mathcal{L}\circ\mathcal{R}$. Then, there exists *c* in *S* such that $a\mathcal{L}c$ and $c\mathcal{R}b$. That is, there exist *x*, *y*, *u*, *v* in *S* such that

$$x \times a = c, c \times u = b,$$

$$y \times c = a, b \times v = c.$$

If we now write *d* for the element $(y \times c) \times u$ of *S*, applying Theorem 5, *S* is a CA-NET-groupoid. As such, we have

$$a \times u = (y \times c) \times u = d,$$

$$a = y \times c = y \times (b \times v) = v \times (y \times b) = b \times (v \times y) = (c \times u) \times (v \times y)$$

$$= (y \times c) \times (v \times u) \text{ (Applying Proposition 1)}$$

$$= a \times (v \times u) = u \times (a \times v) = v \times (u \times a)$$

$$= v \times (u \times (neut(a) \times a)) = v \times (a \times (u \times neut(a)))$$

$$= v \times (neut(a) \times (a \times u)) = v \times (neut(a) \times d) = d \times (v \times neut(a))$$

hence $a\mathcal{R}d$. In addition,

$$b = c \times u = (x \times a) \times u = (x \times a) \times (u \times neut(u)) = neut(u) \times ((x \times a) \times u) = neut(u) \times b$$

$$d = (y \times c) \times u = (y \times c) \times (neut(u) \times u) = u \times ((y \times c) \times neut(u)) = neut(u) \times (u \times (y \times c))$$

$$= neut(u) \times (c \times (u \times y)) = neut(u) \times (y \times (c \times u)) = neut(u) \times (y \times b) = neut(u) \times (y \times (neut(u) \times b))$$

$$= neut(u) \times (b \times (y \times neut(u))) = (y \times neut(u)) \times (neut(u) \times b) = (y \times neut(u)) \times b,$$

$$d = a \times u = (y \times c) \times u = (y \times c) \times (u \times neut(u)) = neut(u) \times ((y \times c) \times u) = neut(u) \times d,$$

$$b = c \times u = (x \times a) \times u = (x \times a) \times (neut(u) \times u) = u \times ((x \times a) \times neut(u)) = neut(u) \times (u \times (x \times a))$$

$$= neut(u) \times (u \times c) = neut(u) \times (u \times (x \times a)) = neut(u) \times (u \times (x \times (y \times c)))$$

$$= neut(u) \times ((y \times c) \times (u \times x)) = neut(u) \times (a \times (u \times x)) = neut(u) \times (x \times (a \times u)) = neut(u) \times (x \times d)$$

$$= neut(u) \times (x \times (neut(u) \times d)) = neut(u) \times (d \times (x \times neut(u))) = (x \times neut(u)) \times (neut(u) \times d)$$

$$= (x \times neut(u)) \times d,$$

thus $d\mathcal{L}b$. We deduce that $(a, b) \in \mathcal{R} \circ \mathcal{L}$. We have shown that $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{L}$; the reverse inclusion follows in a similar way. □

Theorem 11. *In a regular CA-groupoid* S, $\mathcal{L}$ *is equivalent to* $\mathcal{R}$. *That is, as a binary relationship,* $\mathcal{L} = \mathcal{R}$.

Proof. By Theorem 10, we have $d\mathcal{L}\, b$. Then,

$$b = c \times u = (x \times a) \times u = (x \times a) \times (neut(u) \times u) = u \times ((x \times a) \times neut(u))$$

$$= neut(u) \times (u \times (x \times a)) = neut(u) \times (u \times c) = neut(u) \times (u \times (x \times a)) = neut(u) \times (u \times (x \times (y \times c)))$$

$$= neut(u) \times ((y \times c) \times (u \times x)) = neut(u) \times (a \times (u \times x)) = neut(u) \times (x \times (a \times u))$$

$$= neut(u) \times (x \times d) = d \times (neut(u) \times x).$$

$$d = (y \times c) \times u = (y \times c) \times (neut(u) \times u) = u \times ((y \times c) \times neut(u)) = neut(u) \times (u \times (y \times c))$$

$$= neut(u) \times (c \times (u \times y)) = neut(u) \times (y \times (c \times u)) = neut(u) \times (y \times b) = neut(u)(y \times (c \times u))$$

$$= neut(u) \times (y \times b) = b \times (neut(u) \times y).$$

Thus, $d\mathcal{R}b$.

Therefore, in a regular CA-groupoid S, $\mathcal{L}$ *is equivalent to* $\mathcal{R}$. □

Example 14. *Let* $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. *The operation* $\times$ *on* S *is defined as Table 14. Then,* $(S, \times)$ *is a regular CA-groupoid.* $\mathcal{L} = \{<1, 2>, <2, 1>, <3, 4>, <4, 3>, <5, 6>, <6, 5>, <7, 8>, <8, 7>\}$, $\mathcal{R} = \{<1, 2>, <2, 1>, <3, 4>, <4, 3>, <5, 6>, <6, 5>, <7, 8>, <8, 7>\}$, $\mathcal{L} \circ \mathcal{R} = \{<1, 1>, <2, 2>, <3, 3>, <4, 4>, <5, 5>, <6, 6>, <7, 7>, <8, 8>\} = \mathcal{R} \circ \mathcal{L} = \{<1, 1>, <2, 2>, <3, 3>, <4, 4>, <5, 5>, <6, 6>, <7, 7>, <8, 8>\}$. *Thus,* $\mathcal{L}$ *and* $\mathcal{R}$ *are commutative, and* $\mathcal{L} = \mathcal{R}$.

Table 14. The operation $\times$ on S.

$\times$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	1	2	5	5	5	6	7	8
2	2	1	5	5	5	6	7	8
3	5	5	3	4	5	6	7	8
4	5	5	4	3	5	6	7	8
5	5	5	5	5	5	6	7	8
6	6	6	6	6	6	5	7	8
7	7	7	7	7	7	7	7	8
8	8	8	8	8	8	8	8	7

Obviously on the regular CA-groupoid S, there is

$$\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J}.$$

Lemma 1. *In a regular CA-groupoid S, each $\mathcal{L}$-class contains at least one idempotent.*

Proof. For any $a \in S$, there exist $x \in S$, such that $a = a \times (x \times a)$; then,

$$(x \times a) \times (x \times a) = a \times ((x \times a) \times x) = x \times (a \times (x \times a)) = x \times a$$

Therefore, $(x \times a)$ is idempotent and $a\mathcal{L}(x \times a)$. □

Lemma 2. *Every idempotent e in a regular CA-groupoid S is a left identity for $\mathcal{L}e$.*

Proof. If $a \in \mathcal{L}e$, then $a = x \times e$. For some x in S and

$$e \times a = e \times (x \times e) = e \times (e \times x) = x \times e^2 = x \times e = a.$$

□

Proposition 4. *Let a be an element of a regular $\mathcal{L}$-class L in a regular CA-groupoid S. If $\mathcal{L}a$ contains idempotents e, then $\mathcal{L}e$ contains an inverse a^{-1} of a such that $a \times a^{-1} = a^{-1} \times a = e$.*

Proof. Since $a\mathcal{L}e$ it follows by Lemma 2 that $e \times a = a$. Again, from $a\mathcal{R}e$, it follows that there exists x in S such that $a \times x = e$. If we denote $x \times e$ by a^{-1}, we easily see that

$$a \times (a^{-1} \times a) = a \times ((x \times e) \times a) = a \times (a \times (x \times e)) = a \times (e \times (a \times x))$$

$$= a \times (e \times e) = e \times (a \times e) = e \times (e \times a) = e \times a = a,$$

$$a^{-1} \times (a \times a^{-1}) = (x \times e) \times (a \times (x \times e)) = (x \times e) \times (e \times (a \times x)) = (x \times e) \times (e \times e) = e \times ((x \times e) \times e)$$

$$= e \times (e \times (x \times e)) = e \times (e \times (e \times x)) = e \times (x \times e) = e \times (e \times x) = x \times (e \times e) = x \times e = a^{-1}.$$

Thus, a^{-1} is an inverse of a. Moreover,

$$a \times a^{-1} = a \times (x \times e) = e \times (a \times x) = e \times e = e$$

Further,

$$a \times a = (x \times e) \times a = (x \times e) \times (e \times a) = a \times ((x \times e) \times e)$$

$$= e \times (a \times (x \times e)) = e \times (e \times (a \times x)) = e \times (e \times e) = e.$$

It now follows easily that

$$a \times a^{-1} = a^{-1} \times a = e.$$

□

Theorem 12. *Let* $(S, \times)$ *be a CA-groupoid. Then, the following statements are equivalent:*

(1) *S is regular;*
(2) *Every element of S lies in a subgroup of S; and*
(3) *Every $\mathcal{H}$-class in S is a group.*

Proof. (1)$\Rightarrow$ (2). Assume that S is a regular CA-groupoid. By Theorem 5, we know that S is a CA-NET-groupoid. By Theorem 2, we know that, in a CA-NET-groupoid S, every element of S lies in a subgroup of S. Thus, if S is a regular CA-groupoid, then every element of S lies in a subgroup of S.

(2)$\Rightarrow$(3). Assume that every element of S lies in a subgroup of S. Let $a \in S$; then, $a \in G$ for some subgroup G of S. Denote the identity element of G by e, and the inverse of a within G by a^{-1}. Then, from

$$e \times a = a \times e = a \text{ and } a \times a^{-1} = a^{-1} \times a = e$$

it follows that $a\mathcal{H}e$, and hence H_a= H_e, every $\mathcal{H}$-class in S is a group.

(3)$\Rightarrow$(1). Assume that every $\mathcal{H}$-class in S is a group. For each a in S, $a \in H_a$, because H_a is a group, then element a has a unique inverse a^{-1} within the group H_a. Let x= a^{-1}; then, it is clear that

$$a \times (x \times a) = a.$$

Therefore, S is a regular CA-groupoid. □

Example 15. *Let* $S = \{a, b, c, d, e\}$*. Define operation* $\times$ *on S as Table 15. Then,* $(S, \times)$ *is a CA-groupoid.*

Table 15. The operation $\times$ on S.

$\times$	**a**	**b**	**c**	**d**	**e**
a	a	b	c	d	c
b	b	a	d	c	d
c	c	d	c	d	c
d	d	c	d	c	d
e	c	d	c	d	e

$(S, \times)$ is a regular CA-groupoid, since $a = a \times (a \times a)$, $b = b \times (b \times b)$, $c = c \times (c \times c)$, $d = d \times (d \times d)$, and $e = e \times (e \times e)$. Every element of CA-groupoid S lies in a subgroup of S, because $\{a, b\}$, $\{c, d\}$, $\{e\}$ is a subgroup of S. Moreover, $a, b \in \{a, b\}$, $c, d \in \{c, d\}$, and $e \in \{e\}$. Every$\mathcal{H}$-class in S is a group. Then, H_1, H_2, H_3 of $\mathcal{H}$-class in S, $H_1 = \{a, b\}$, $H_2 = \{c, d\}$, $H_3 = \{e\}$. Moreover, $a \times b = b$, $b \times b = a$; $c \times d = d$, $d \times d = c$ and $e \times e = e$, H_1, H_2, H_3 is a group.

6. Relationships between Some Cyclic Associative Groupoids (CA-Groupoids)

Definition 14. *A CA-groupoid* $(S, \times)$ *is called inverse CA-groupoid if there exists a unary operation* a^{-1} *on S with the properties*

$$(a^{-1})^{-1} = a, a \times (a^{-1} \times a) = a,$$

and for any $x, y \in S$,

$$(x \times x^{-1}) \times (y \times y^{-1}) = (y \times y^{-1}) \times (x \times x^{-1})$$

Theorem 13. *Let (S, ×) be a CA-groupoid. Then, S is an inverse CA-groupoid if and only if it is a regular CA-groupoid and its idempotent is commutative.*

Proof. Let S be an inverse CA-groupoid, which follows if we show that every idempotent in S can be expressed in the form xx^{-1}. Let e be an idempotent in S. Then, the inverse CA-groupoid property ensures that there is an element e^{-1} in S such that $e \times (e^{-1} \times e) = e$, $(e^{-1})^{-1} = e$. Hence,

$$e^{-1} = e^{-1} \times ((e^{-1})^{-1} \times e^{-1}) = e^{-1} \times (e \times e^{-1}) = e^{-1} \times ((e \times e) \times e^{-1}) = e^{-1} \times (e^{-1} \times (e \times e)) = e^{-1} \times (e \times (e^{-1} \times e)) = e^{-1} \times e = e^{-1} \times (e \times e) = e \times (e^{-1} \times e) = e.$$

and thus $e = e^2 = e \times e = e \times e^{-1}$.

According to the definition of an inverse CA-groupoid, idempotents commute. If x, y are idempotent, then $x \times y = (x \times x^{-1}) \times (y \times y^{-1}) = (y \times y^{-1}) \times (x \times x^{-1}) = y \times x$.

Therefore, S is a regular CA-groupoid and its idempotents are commutative.

Now, we assume that S is a regular CA -groupoid and its idempotents are commutative. Then, according to regularity, for any $x \in S$, there exists $neut(x) \in S$, $anti(x) \in S$. By Theorem 1, let $anti(x) \times neut(x) = x^{-1}$; then, we have

$$(x^{-1})^{-1} = x,\ x \times (x^{-1} \times x) = x,$$

$$(x \times x^{-1}) \times (y \times y^{-1}) = neut(x) \times neut(y) = neut(y) \times neut(x) = (y \times y^{-1}) \times (x \times x^{-1})$$

Therefore, S is an inverse CA-groupoid. □

Corollary 3. *Let (S, ×) be a regular CA-groupoid. Then, S is a commutative CA-groupoid.*

Proof. Let $(S, \times)$ be a regular CA-groupoid. By Theorem 5, S is a CA-NET-groupoid. By Corollary 1, S is a commutative CA-groupoid. □

Theorem 14. *Let (S, ×) be a CA-groupoid. Then, the following statements are equivalent:*

(1) *S is a regular CA-groupoid;*
(2) *S is a strongly regular CA-groupoid;*
(3) *S is a CA-NET-groupoid;*
(4) *S is an inverse CA-groupoid; and*
(5) *S is a commutative regular semigroup.*

Proof. (1)⇒(2). Assume that S is a regular CA-groupoid. By Corollary 3, we know that S is a commutative CA-groupoid. Then, for any $a \in S$, there exists $x \in S$, such that $a = a \times (x \times a)$ and $a = (a \times x) \times a$. According to the definition of strongly regular CA-groupoid (Definition 5), S is a strongly regular CA-groupoid.

(2)⇒(3). Assume that S is a strongly regular CA-groupoid. By Definitions 4 and 5, S is a regular CA-groupoid. By Theorem 5, S is a CA-NET-groupoid.

(3)⇒(4). Let $(S, \times)$ be a CA-NET-groupoid. According to Theorem 4, the idempotent of S is commutative. By Theorem 5, S is a regular CA-groupoid. By Theorem 13, S is an inverse CA-groupoid.

(4)⇒(5). Let $(S, \times)$ be an inverse CA-groupoid. By Theorem 13, S is a regular CA-groupoid and its idempotent is commutative. Then, we only need proof a regular CA-groupoid is a commutative regular semigroup. By Corollary 3, S is a commutative CA-groupoid. For any $a, b, c \in S$, we have

$$a \times (b \times c) = c \times (a \times b) = (a \times b) \times c$$

and there exists $x \in S$, such that $a = a \times (x \times a) = a \times (a \times x) = (a \times x) \times a = a \times x \times a$.

Therefore, S is a commutative regular semigroup.

(5)⇒(1). Assume that $(S, \times)$ is a commutative regular semigroup. For any $a, b, c \in S$, we have

$$a \times (b \times c) = (a \times b) \times c = c \times (a \times b)$$

and there exists $x \in S$, such that $a = a \times x \times a = a \times (x \times a)$.

Therefore, S is a regular CA-groupoid. □

Example 16. *Let $S = \{1, 2, 3, 4\}$. The operation $\times$ on S is defined as Table 16. Then, $(S, \times)$ is a regular CA-groupoid, since $1 = 1 \times (1 \times 1)$, $2 = 2 \times (4 \times 2)$, $3 = 3 \times (3 \times 3)$, and $4 = 4 \times (4 \times 4)$. $(S, \times)$ is also a strongly regular CA-groupoid because $1 = 1 \times (1 \times 1)$, $1 = (1 \times 1) \times 1$; $2 = 2 \times (4 \times 2)$, $2 = (2 \times 4) \times 2$; $3 = 3 \times (3 \times 3)$, $3 = (3 \times 3) \times 3$; $4 = 4 \times (4 \times 4)$, and $4 = (4 \times 4) \times 4$. We can verify that $(S, \times)$ is a CA-NET-groupoid, and neut(1) = 1, anti(1) = 1; neut(2) = 2, anti(2) = \{1, 2, 3, 4\}; neut(3) = 3, anti(3) = 3; and neut(4) = 4, anti(4) = \{1, 3, 4\}. $(S, \times)$ is an inverse CA-groupoid, since $1 \times 2 = 2 \times 1$, $1 \times 3 = 3 \times 1$, $1 \times 4 = 4 \times 1$, $2 \times 3 = 3 \times 2$, $2 \times 4 = 4 \times 2$, and $3 \times 4 = 4 \times 3$. $(S, \times)$ is also a commutative regular semigroup because $1 = 1 \times 1 \times 1$, $2 = 2 \times 2 \times 2$, $3 = 3 \times 3 \times 3$, and $4 = 4 \times 4 \times 4$.*

Table 16. The operation × on *S*.

×	*1*	**2**	*3*	*4*
1	*1*	*2*	*4*	*4*
2	*2*	*2*	*2*	*2*
3	*4*	*2*	*3*	*4*
4	*4*	*2*	*4*	*4*

Corollary 4. *Let $(S, \times)$ be a strongly regular CA-groupoid. Then, S is a strongly regular semigroup.*

Proof. Let $(S, \times)$ be a strongly regular CA-groupoid. By Theorem 14 (2), (5), S is a strongly regular semigroup. □

7. Conclusions

Starting from various backgrounds (for examples, non-associative rings with $x(yz) = y(zx)$, cyclic associative Abel-Grassman groupoids, regular semigroup, and regular AG-groupoid), this paper introduces the concept of regular cyclic associative groupoid (CA-groupoid) for the first time. Furthermore, we study the relationship between regular CA-groupoids and other relevant algebraic structures. The research shows that the regular CA-groupoids, as a kind of non-associative algebraic structures, has typical representativeness and rich connotation, and is closely related to many kinds of algebraic structures. This paper concludes some important results, which are listed as follows:

(1) If an algebraic system is a regular CA-groupoid, then, each of its elements has an inverse and the inverse is unique (see Theorem 3 and Example 6).
(2) If an algebraic system is a CA-NET-groupoid, then, its idempotents are commutative (see Theorem 4).
(3) Every CA-NET-groupoid is commutative (see Corollary 1 and Example 10).
(4) An algebraic system is a regular CA-groupoid if and only if it is a CA-NET-groupoid (see Theorem 5 and Example 11).
(5) If an algebraic system is a CA-groupoid, then, $\mathcal{L}$ and $\mathcal{R}$ are not commutative. That is, as a binary relationship, $\mathcal{L}\circ\mathcal{R}\neq\mathcal{R}\circ\mathcal{L}$ (see Corollary 2 and Example 13).
(6) If an algebraic system is a regular CA-groupoid, then, the relations $\mathcal{L}$ and $\mathcal{R}$ commute. That is, as a binary relationship, $\mathcal{L}\circ\mathcal{R}=\mathcal{R}\circ\mathcal{L}$ (see Theorem 10 and Example 14).

(7) If an algebraic system is a regular CA-groupoid, then every element of S lies in a subgroup of S, and every $\mathcal{H}$-class in S is a group (see Theorem 12 and Example 15).
(8) An algebraic system is an inverse CA-groupoid if and only if it is a regular CA-groupoid and its idempotent is commutative (see Theorem 13 and Example 16).
(9) If an algebraic system is a regular CA-groupoid, then, it is a commutative CA-groupoid (see Corollary 3 and Example 16).
(10) An algebraic system is a regular CA-groupoid if and only if it is a commutative regular semigroup (see Theorem 14 and Example 16).

These results are important for exploring the structure characterizations of regular CA-groupoids and CA-NET-groupoids.

For future research, we will discuss the integration of the related topics, such as the ideals in CA-groupoids and the relationships among some algebraic structures (see [26–28]).

Author Contributions: W.Y. and X.Z. initiated the research and wrote the paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Natural Science Foundation of China (Grant No. 61976130).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Petrich, M. The structure of completely regular semigroups. *Trans. Am. Math. Soc.* **1974**, *189*, 211–236. [CrossRef]
2. Petrich, M.; Reilly, N.R. *Completely Regular Semigroups*; John Wiley & Sons: New York, NY, USA, 1999.
3. Akinmoyewa, J.T. A study of some properties of generalized groups. *Octogon Math. Mag.* **2009**, *17*, 599–626.
4. Smarandache, F. *Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras and Applications*; Pons Publishing House: Brussels, Belgium, 2017.
5. Smarandache, F.; Ali, M. Neutrosophic triplet group. *Neural Comput. Appl.* **2018**, *29*, 595–601. [CrossRef]
6. Zhang, X.H.; Wang, X.J.; Smarandache, F.; Jaiyeola, T.G.; Lian, T.Y. Singular neutrosophic extended triplet groups and generalized groups. *Cogn. Syst. Res.* **2019**, *57*, 32–40. [CrossRef]
7. Sholander, M. Medians, lattices, and trees. *Proc. Am. Math. Soc.* **1954**, *5*, 808–812. [CrossRef]
8. Maksa, G. CM solutions of some functional equations of associative type. *Ann. Univ. Sci. Bp. Sect. Comput.* **2004**, *24*, 125–132.
9. Kleinfeld, M. Rings with x(yz)=y(zx). *Commun. Algebra* **1995**, *23*, 5085–5093. [CrossRef]
10. Zhan, J.M.; Tan, Z.S. Left weakly Novikov algebra. *J. Math.* **2005**, *25*, 135–138.
11. Schafer, R.D. *An Introduction to Nonassociative Algebras*; Academic Press: Cambridge, MA, USA, 1966.
12. Sabinin, L.; Sbitneva, L.; Shestakov, I. *Non-Associative Algebra and Its Applications*; CRC Press: Boca Raton, FL, USA, 2006.
13. Chajda, I.; Halaš, R.; Länger, H. Operations and structures derived from non-associative MV-algebras. *Soft Comput.* **2019**, *23*, 3935–3944. [CrossRef]
14. Shah, T.; Razzaque, A.; Rehman, I.; Gondal, M.A.; Faraz, M.I.; Shum, K.P. Literature survey on non-associative rings and developments. *Eur. J. Pure Appl. Math.* **2019**, *12*, 370–408. [CrossRef]
15. Lazendic, S.; Pizurica, A.; De Bie, H. Hypercomplex algebras for dictionary learning. In Proceedings of the 7th Conference on Applied Geometric Algebras in Computer Science and Engineering–AGACSE 2018, Campinas, Brazil, 23–27 July 2018; pp. 57–64.
16. Hirsch, R.; Jackson, M.; Kowalski, T. Algebraic foundations for qualitative calculi and networks. *Theor. Comput. Sci.* **2019**, *768*, 99–116. [CrossRef]
17. Zhang, X.H.; Hu, Q.Q.; Smarandache, F.; An, X.G. On neutrosophic triplet groups: Basic properties, NT-subgroups, and some notes. *Symmetry* **2018**, *10*, 289. [CrossRef]
18. Ma, Y.C.; Zhang, X.H.; Yang, X.F.; Zhou, X. Generalized neutrosophic extended triplet group. *Symmetry* **2019**, *11*, 327. [CrossRef]
19. Smarandache, F. Neutrosophic set—A generialization of the intuituionistics fuzzy sets. *Int. J. Pure Appl. Math.* **2005**, *3*, 287–297.

20. Tehrim, S.T.; Riaz, M. A novel extension of TOPSIS to MCGDM with bipolar neutrosophic soft topology. *J. Intell. Fuzzy Syst.* **2019**, *37*, 5531–5549. [CrossRef]
21. Green, J.A. The structure of semigroups. *Ann. Math.* **1951**, *54*, 163–172. [CrossRef]
22. Mary, X. On generalized inverses and Green's relations. *Linear Algebra Appl.* **2011**, *434*, 1836–1844. [CrossRef]
23. Fleischer, L.; Kufleitner, M. Green's Relations in Finite Transformation Semigroups. *Comput. Sci. Theory Appl.* **2017**, *10304*, 112–125.
24. Warne, R.J. Orthodox congruences on strongly regular semigroups. *Comment. Math. Univ. St. Paul.* **1980**, *2*, 203–208.
25. Zhang, X.H.; Ma, Z.R.; Yuan, W.T. Cyclic associative groupoids (CA-groupoids) and cyclic associative neutrosophic extended triplet groupoids (CA-NET-groupoids). *Neutrosophic Sets Syst.* **2019**, *29*, 19–29.
26. Zhang, X.H.; Borzooei, R.A.; Jun, Y.B. Q-filters of quantum B-algebras and basic implication algebras. *Symmetry* **2018**, *10*, 573. [CrossRef]
27. Zhang, X.H.; Wu, X.Y.; Mao, X.Y.; Smarandache, F.; Park, C. On neutrosophic extended triplet groups (loops) and Abel-Grassmann's groupoids (AG-groupoids). *J. Intell. Fuzzy Syst.* **2019**, *37*, 5743–5753. [CrossRef]
28. Zhang, X.H.; Smarandache, F.; Ma, Y.C. Symmetry in hyperstructure: neutrosophic extended triplet semihypergroups and regular hypergroups. *Symmetry* **2019**, *11*, 1217. [CrossRef]