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Article

Conjugacy of Dynamical Systems on Self-Similar Groups

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Abstract: We show that the limits for dynamical systems of self-similar groups are eventually conjugate if, and only if, there is an isomorphism between their Deaconu groupoid preserving cocycles. For limit solenoids of self-similar groups, we show that the conjugacy of limit solenoids is equivalent to existence of isomorphism between the Deaconu groupoids of limit solenoid preserving cocycles.

Keywords: self-similar group; eventual conjugacy; limit dynamical system; limit solenoid; groupoid

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1. Introduction

In recent years, Matsumoto [1] defined the eventual conjugacy of one-sided subshifts as a special case of continuous orbit equivalence [2]. Following Matsumoto, Carlsen and Rout [3] generalized eventual conjugacy to graphs. In this paper, we study the eventual conjugacy of dynamical systems associated to self-similar groups.

Introduced by Nekrashevych [4,5], self-similar groups have been an important example for combinatorial group theory, topological dynamics, and C^*-algebras. A self-similar group has two naturally associated dynamical systems, called the limit dynamical system and the limit solenoid. Naively speaking, the limit dynamical system is the quotient of the one-sided infinite path space by group action with the shift map, and the limit solenoid is the quotient of the two-sided infinite path space with the shift map. So the limit dynamical system and limit solenoid are generalizations of one-sided and two-sided subshifts of finite type, respectively, by group action with the shift map.

Then, it is rational to expect that limit dynamical systems and limit solenoids would have similar properties to one-sided and two-sided subshifts of finite type. Generalizing the results of Carlsen and Rout [3], we show that the eventual conjugacy of the limit dynamical systems of self-similar groups is equivalent to the existence of groupoid isomorphism preserving cocycles (Theorem 2). We also show that the limit solenoids are conjugate if, and only if, there is an isomorphism between the groupoids of the limit solenoid preserving cocycles (Theorem 4).

2. Self-Similar Groups

We review the properties of self-similar groups. All of the material in this section is taken from [4,5].

Suppose that X is a finite set. We denote by X^n the set of words of length n in X with $X^0 = \{\emptyset\}$, and let $X^* = \cup_{n=0}^{\infty} X^n$. We denote by X^ω the set of right-infinite paths of the form $x_0 x_1 \cdots$ where $x_i \in X$. The product topology of the discrete set X is given on X^ω. A cylinder set $Z(u)$ for each $u \in X^*$ is

$$Z(u) = \{x \in X^\omega \colon x = x_0 x_1 \cdots \text{ such that } x_0 \cdots x_{|u|-1} = u\}.$$

The collection of all such cylinder sets forms a basis for the product topology on X^ω. It is trivial that every cylinder set is a compact open set, and that X^ω is a compact metrizable space.

A *self-similar group* (G, X) consists of a finite set X and a *faithful action* of a finitely generated countable group G on X^* such that, for all $g \in G$ and $x \in X$, there exist unique $y \in X$ and $h \in G$, such that

$$g(xu) = yh(u) \text{ for every } u \in X^*.$$

The unique element h is called the *restriction* of g at x and is denoted by $g|_x$. The restriction extends to X^* via the inductive formula

$$g|_{xy} = (g|_x)|_y$$

so that for every $u, v \in X^*$ we have

$$g(uv) = g(u) g|_u(v).$$

The G-action extends to an action of G on X^ω given by

$$g(x_0 x_1 \cdots) = g(x_0) g|_{x_0}(x_1 x_2 \cdots).$$

2.1. Conditions on Self-Similar Groups

A self-similar group (G, X) is called *contracting* if there is a finite subset N of G satisfying the following: For every $g \in G$, there is $n \geq 0$ such that $g|_v \in N$ for every $v \in X^*$ of length $|v| \geq n$. If the group is contracting, the smallest set N satisfying this condition is called the *nucleus* of the group. We say that (G, X) is *regular* if, for every $g \in G$ and every $\xi \in X^\omega$, either $g(\xi) \neq \xi$ or there is a neighborhood of ξ, such that every point in the neighborhood is fixed by g. We say that (G, X) is *recurrent* if, for any two words a, b are of equal length and every $h \in G$, there is a $g \in G$, such that $g(a) = b$ and $g|_a = h$.

2.2. Limit Solenoids

Suppose that (G, X) is a self-similar group. We consider the space $X^{\mathbb{Z}}$ of bi-infinite paths $\cdots x_{-1}.x_0 x_1 x_2 \cdots$ over X and the shift map $\sigma \colon X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ given by $\sigma(x)_n = x_{n+1}$. The direct product topology of the discrete set X is given on $X^{\mathbb{Z}}$. We say that two paths $\cdots x_{-1}.x_0 x_1 x_2 \cdots$ and $\cdots y_{-1}.y_0 y_1 y_2 \cdots$ in $X^{\mathbb{Z}}$ are *asymptotically equivalent* if there is a finite set $I \subset G$ and a sequence $g_n \in I$ such that

$$g_n(x_n x_{n+1} \cdots) = y_n y_{n+1} \cdots$$

for every $n \in \mathbb{Z}$. The quotient of $X^{\mathbb{Z}}$ by the asymptotic equivalence relation is called the *limit solenoid* of (G, X) and is denoted S_G.

The topology on S_G is given as follows: The product topology of the discrete set X is given on the bi-infinite path space $X^{\mathbb{Z}}$. Thus, for each finite path $u = u_{-m} \cdots u_m \in X^*$,

$$Z(u) = \{x \in X^{\mathbb{Z}} \colon x = \cdots x_{-m-1} u x_{m+1} \cdots\}$$

is a compact open base of $X^{\mathbb{Z}}$. Let $q\colon X^{\mathbb{Z}} \to S_G$ be the quotient map by the asymptotic equivalence relation. Then S_G is given the quotient topology by q so that $q(Z(u))$ is a compact open base of S_G [4] (Proposition 2.5).

The shift map on $X^{\mathbb{Z}}$ is transferred to an induced homeomorphism on S_G, which we will denote by σ when there is no confusion. We also use the term limit solenoid of (G, X) for the dynamical system (S_G, σ).

2.3. Limit Dynamical Systems

Let $\pi\colon X^{\mathbb{Z}} \to X^{\omega}$ be the canonical projection map. We restrict the asymptotic equivalence relation on $X^{\mathbb{Z}}$ to X^{ω} so that $x_0x_1x_2\cdots$ and $y_0y_1y_2\cdots$ in X^{ω} are *asymptotically equivalent* if there is a $g \in G$, such that

$$g(x_0x_1\cdots) = y_0y_1\cdots,$$

with the property that the collection $\{g|_{x_0\cdots x_{n-1}}\colon n \in \mathbb{N}\}$ is a finite set. The quotient of X^{ω} by the asymptotic equivalence is called the *limit space* of (G, X) and we denote by T_G. The quotient topology by asymptotic equivalence relation is given on T_G. Then the canonical projection $\pi\colon X^{\mathbb{Z}} \to X^{\omega}$ and the shift map $\sigma\colon X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ induce a natural projection map $S_G \to T_G$ and a shift map $T_G \to T_G$

$$q(\cdots x_{-1}.x_0x_1\cdots) \mapsto q(x_0x_1\cdots) \text{ and } q(x_0x_1\cdots) \mapsto q(x_1x_2\cdots), \text{ respectively.}$$

We denote these induced projection, quotient, and shift maps as π, q, and σ, respectively, when there is no confusion. The restricted dynamical system (T_G, σ) is called the *limit dynamical system* of (G, X). Then it is easy to check that the projection maps on $X^{\mathbb{Z}}$ and S_G, quotient maps on $X^{\mathbb{Z}}$ and X^{ω}, and shift maps on $X^{\mathbb{Z}}$, X^{ω}, S_G, and T_G are commuting with each other.

Theorem 1. *[4] (Proposition 2.6) The limit solenoid of a self-similar group is the inverse limit of the limit dynamical system.*

Remark 1.
1. *In [4,5], Nekrashevych used the shift map defined by $\sigma(x)_n = x_{n-1}$ so that the limit space is given as the quotient of left-hand-sided full shift.*
2. *The limit solenoid S_G and limit space T_G are compact metrizable spaces. If (X, G) satisfies the recurrent condition, then S_G and T_G are connected [4] (Proposition 2.4).*
3. *If (X, G) satisfies the contracting, recurrent, and regular conditions, then the limit solenoid (S_G, σ) is a mixing Smale space [4] (Proposition 6.10).*
4. *If (X, G) satisfies the contracting and regular conditions, then the shift map $\sigma\colon T_G \to T_G$ is a covering map [4] (Proposition 6.1).*

2.4. Deaconu Groupoids

Let Y be a compact Hausdorff space and $f\colon Y \to Y$ be continuous onto a map. The *Deaconu groupoid* of (Y, f) is

$$D_{Y,f} = \{(y_1, m-n, y_2)\colon y_1, y_2 \in Y, m, n \in \mathbb{N}, f^m(y_1) = f^n(y_2)\}.$$

A pair $\{(y_1, m-n, y_2), (y_3, k-l, y_4)\} \in D^{(2)}_{Y,f}$ is composable if $y_2 = y_3$, and the multiplication and inverse are given by

$$(y_1, m-n, y_2)(y_2, k-l, y_4) = (y_1, m-n+k-l, y_4) \text{ and}$$
$$(y_1, m-n, y_2)^{-1} = (y_2, n-m, y_1).$$

With these operations, $D_{Y,f}$ is a groupoid. For a $(y_1, m-n, y_2) \in D_{Y,f}$, the domain and range are given by

$$d(y_1, m-n, y_2) = (y_1, 0, y_1) \text{ and } r(y_1, m-n, y_2) = (y_2, 0, y_2).$$

The unit space of $D_{Y,f}$ denoted by $D_{Y,f}^{(0)}$ is identified with Y via the diagonal map, and the isotropy group bundle is given by $I = \{(y_1, m, y_1) \in D_{Y,f}\}$. For open sets U, V of Y and $k, l \geq 0$, let

$$Z(U, m, n, V) = \{(y_1, m-n, y_2) \colon y_1 \in U, y_2 \in V, f^m(y_1) = f^n(y_2)\}.$$

Then the collection of these sets is the basis for a second countable locally compact Hausdorff topology on $D_{Y,f}$, and the counting measure is a Haar system of $D_{Y,f}$ if f is a local homeomorphism [6].

Definition 1. *Let (G, X) be a self-similar group. We denote the Deaconu groupoids of (T_G, σ) and (S_G, σ) by D_G and E_G, respectively.*

We summarize the basic properties of D_G and E_G as follows.

Remark 2. *Let (G, X) be a self-similar group. Then the Deaconu groupoids, D_G and E_G, are topologically principal, locally compact, and Hausdorff groupoid. The locally compact and Hausdorff properties come from the definition of Deaconu groupoids. The topologically principal property is by [7] (Corollary 14.14). If (G, X) is a regular self-similar group, then D_G is étale by [4] (Proposition 6.1). If (G, X) satisfies the contracting and recurrent conditions, then D_G and E_G are amenable by [4] (Theorem 5.6).*

We refer the reader to [6,8] for the definition and properties of groupoids and groupoid algebras.

3. Eventual Conjugacy of Limit Dynamical Systems

We generalize Matsumoto's definition of eventual conjugacy of one-sided SFTs to limit dynamical systems of self-similar groups. See [1,3] for more details.

Definition 2. *Suppose that (G, X) and (H, Y) are self-similar groups and that (T_G, σ) and (T_H, σ) are their corresponding limit dynamical systems, respectively. The limit dynamical systems (T_G, σ) and (T_H, σ) are said to be eventually conjugate if there are a homeomorphism $h \colon T_G \to T_H$ and continuous maps $k_1 \colon T_G \to \mathbb{N} \cup \{0\}$ and $k_2 \colon T_H \to \mathbb{N} \cup \{0\}$, such that*

$$\sigma^{k_1(\xi)} \circ h \circ \sigma(\xi) = \sigma^{k_1(\xi)+1} \circ h(\xi) \text{ and } \sigma^{k_2(\eta)} \circ h^{-1} \circ \sigma(\eta) = \sigma^{k_2(\eta)+1} \circ h^{-1}(\eta)$$

for every $\xi \in T_G$ and $\eta \in T_H$.

Remark 3. *If self-similar groups (G, X) and (H, Y) both satisfy the recurrent condition, then T_G and T_H are connected spaces by Remark 1 and so the maps, k_1 and k_2 above, are constant maps.*

Lemma 1. *Suppose that (G, X) and (H, Y) are self-similar groups and that their corresponding limit dynamical systems, (T_G, σ) and (T_H, σ), are eventually conjugate. Then, for every natural number n, we have*

$$\begin{aligned}&\sigma^{k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{n-1}(\xi))} \circ h \circ \sigma^n(\xi) \\ &\qquad = \sigma^{k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{n-1}(\xi))+n} \circ h(\xi) \text{ and} \\ &\sigma^{k_2(\eta)+k_2(\sigma(\eta))+\cdots+k_2(\sigma^{n-1}(\eta))} \circ h^{-1} \circ \sigma^n(\eta) \\ &\qquad = \sigma^{k_2(\eta)+k_2(\sigma(\eta))+\cdots+k_2(\sigma^{n-1}(\eta))+n} \circ h^{-1}(\eta).\end{aligned}$$

Proof. We use induction. By eventual conjugacy, it is trivial that $\sigma^{k_1(\xi)} \circ h \circ \sigma(\xi) = \sigma^{k_1(\xi)+1} \circ h(\xi)$ holds. Assume that

$$\sigma^{k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{n-1}(\xi))} \circ h \circ \sigma^n(\xi) = \sigma^{k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{n-1}(\xi))+n} \circ h(\xi)$$

is true for every $\xi \in T_G$. Then we have

$$\begin{aligned}&\sigma^{k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{n-1}(\xi))+k_1(\sigma^n(\xi))} \circ h \circ \sigma^{n+1}(\xi) \\ &= \sigma^{k_1(\xi)} \circ \sigma^{k_1(\sigma(\xi))+k_1(\sigma^2(\xi))+\cdots+k_1(\sigma^n(\xi))} \circ h \circ \sigma^n(\sigma(\xi)) \\ &= \sigma^{k_1(\xi)} \circ \left\{\sigma^{k_1(\sigma(\xi))+k_1(\sigma(\sigma(\xi)))+\cdots+k_1(\sigma^{n-1}(\sigma(\xi)))} \circ h \circ \sigma^n(\sigma(\xi))\right\} \\ &= \sigma^{k_1(\xi)} \circ \left\{\sigma^{k_1(\sigma(\xi))+k_1(\sigma(\sigma(\xi)))+\cdots+k_1(\sigma^{n-1}(\sigma(\xi)))+n} \circ h \circ \sigma(\xi)\right\} \\ &= \sigma^{k_1(\sigma(\xi))+k_1(\sigma(\sigma(\xi)))+\cdots+k_1(\sigma^{n-1}(\sigma(\xi)))+n} \circ \left\{\sigma^{k_1(\xi)} \circ h \circ \sigma(\xi)\right\} \\ &= \sigma^{k_1(\sigma(\xi))+k_1(\sigma(\sigma(\xi)))+\cdots+k_1(\sigma^{n-1}(\sigma(\xi)))+n} \circ \left\{\sigma^{k_1(\xi)+1} \circ h(\xi)\right\} \\ &= \sigma^{k_1(\xi)+k_1(\sigma(\xi))+k_1(\sigma(\sigma(\xi)))+\cdots+k_1(\sigma^{n-1}(\sigma(\xi)))+n+1} \circ h(\xi).\end{aligned}$$

So

$$\sigma^{k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{n-1}(\xi))} \circ h \circ \sigma^n(\xi) = \sigma^{k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{n-1}(\xi))+n} \circ h(\xi)$$

holds for every natural number n. By the same argument, we have the second equality for h^{-1}. □

Recall that the Deaconu groupoid of (T_G, σ) is

$$D_G = \{(\xi, m-n, \eta) \colon \xi, \eta \in T_G, m, n \in \mathbb{N}, \sigma^m(\xi) = \sigma^n(\eta)\}.$$

Lemma 2. *For every $(\xi, m-n, \eta) \in D_G$, there are unique minimal nonnegative integers m_0 and n_0, such that $m - n = m_0 - n_0$ and $\sigma^k(\xi) = \sigma^l(\eta)$ holds for all $k \geq m_0, l \geq n_0$ with $k - l = m - n$.*

Proof. If $k \geq m$ and $l \geq n$ satisfy $k - l = m - n$, then $k - m = l - n$ implies

$$\sigma^k(\xi) = \sigma^{k-m} \circ \sigma^m(\xi) = \sigma^{l-n} \circ \sigma^n(\eta) = \sigma^l(\eta).$$

Let

$$\begin{aligned}m_0 &= \min\{k \in \mathbb{N} \cup \{0\} \colon \exists l \in \mathbb{N} \cup \{0\} \text{ such that } \sigma^k(\xi) = \sigma^l(\eta)\} \\ n_0 &= \min\{l \in \mathbb{N} \cup \{0\} \colon \exists k \in \mathbb{N} \cup \{0\} \text{ such that } \sigma^k(\xi) = \sigma^l(\eta)\}.\end{aligned}$$

Then the conclusion is trivial. □

The next property is a special case of [9] (Theorem 8.10). For the Deaconu groupoid D_G, we define a groupoid 1-cocycle $c_G\colon D_G \to \mathbb{Z}$ by $(\xi, m-n, \eta) \mapsto m-n$.

Theorem 2. *[3] (Theorem 4.1) Let (T_G, σ) and (T_H, σ) be the limit dynamical systems of regular self-similar groups (G, X) and (H, Y), respectively. Then the following assertions are equivalent:*

1. *(T_G, σ) and (T_H, σ) are eventually conjugate.*
2. *There is an isomorphism $\psi\colon D_G \to D_H$, such that*

$$c_H(\psi(\xi, m-n, \eta)) = c_G(\xi, m-n, \eta)$$

for every $(\xi, m-n, \eta) \in D_G$.

Proof. (1) $\Longrightarrow$ (2). Suppose that (T_G, σ) and (T_H, σ) are eventually conjugate and that $h\colon T_G \to T_H$ is the corresponding homeomorphism. For a $(\xi, m-n, \eta) \in D_G$, $\sigma^m(\xi) = \sigma^n(\eta)$ and Lemma 1 imply

$$\begin{aligned}
&\sigma^{k_1(\eta)+k_1(\sigma(\eta))+\cdots+k_1(\sigma^{n-1}(\eta))} \circ \sigma^{k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{m-1}(\xi))} \circ h \circ \sigma^m(\xi) \\
&= \sigma^{k_1(\eta)+k_1(\sigma(\eta))+\cdots+k_1(\sigma^{n-1}(\eta))} \circ \sigma^{k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{m-1}(\xi))+m} \circ h(\xi) \\
&= \sigma^{k_1(\eta)+k_1(\sigma(\eta))+\cdots+k_1(\sigma^{n-1}(\eta))+k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{m-1}(\xi))+m} \circ h(\xi) \\
&= \sigma^{k_1(\eta)+k_1(\sigma(\eta))+\cdots+k_1(\sigma^{n-1}(\eta))} \circ \sigma^{k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{m-1}(\xi))} \circ h \circ \sigma^n(\eta) \\
&= \sigma^{k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{m-1}(\xi))} \circ \sigma^{k_1(\eta)+k_1(\sigma(\eta))+\cdots+k_1(\sigma^{n-1}(\eta))} \circ h \circ \sigma^n(\eta) \\
&= \sigma^{k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{m-1}(\xi))} \circ \sigma^{k_1(\eta)+k_1(\sigma(\eta))+\cdots+k_1(\sigma^{n-1}(\eta))+n} \circ h(\eta) \\
&= \sigma^{k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{m-1}(\xi))+k_1(\eta)+k_1(\sigma(\eta))+\cdots+k_1(\sigma^{n-1}(\eta))+n} \circ h(\eta)
\end{aligned}$$

so that

$$\begin{aligned}
&\sigma^{k_1(\eta)+k_1(\sigma(\eta))+\cdots+k_1(\sigma^{n-1}(\eta))+k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{m-1}(\xi))+m} \circ h(\xi) \\
&\quad = \sigma^{k_1(\xi)+k_1(\sigma(\xi))+\cdots+k_1(\sigma^{m-1}(\xi))+k_1(\eta)+k_1(\sigma(\eta))+\cdots+k_1(\sigma^{n-1}(\eta))+n} \circ h(\eta).
\end{aligned}$$

Since

$$\begin{aligned}
&k_1(\eta) + \cdots + k_1(\sigma^{n-1}(\eta)) + k_1(\xi) + \cdots + k_1(\sigma^{m-1}(\xi)) + m \\
&\qquad - (k_1(\xi) + \cdots + k_1(\sigma^{m-1}(\xi)) + k_1(\eta) + \cdots + k_1(\sigma^{n-1}(\eta)) + n)) \\
&\qquad = m - n,
\end{aligned}$$

we have

$$(h(\xi), m-n, h(\eta)) \in D_H.$$

We define $\psi\colon D_G \to D_H$ by

$$(\xi, m-n, \eta) \mapsto (h(\xi), m-n, h(\eta)).$$

It is not difficult to check that ψ is a continuous groupoid isomorphism, such that

$$c_H(\psi(\xi, m-n, \eta)) = c_G(\xi, m-n, \eta)$$

for every $(\xi, m-n, \eta) \in D_G$.

(2) $\Longrightarrow$ (1). Suppose that $\psi\colon D_G \to D_H$ is a continuous isomorphism, satisfying $c_H(\psi(\xi, m-n, \eta)) = c_G(\xi, m-n, \eta)$ for every $(\xi, m-n, \eta) \in D_G$. As $D_G^{(0)} = T_G$ and $D_H^{(0)} = T_H$, the restriction $h = \psi|_{D_G^{(0)}}\colon T_G \to T_H$ is obviously a homeomorphism. Since ψ is a groupoid isomorphism,

$$\begin{aligned} d(\psi(\xi,m,\eta)) &= \psi(d(\xi,m,\eta)) = \psi(\xi,0,\xi) = h(\xi) \text{ and} \\ r(\psi(\xi,m,\eta)) &= \psi(r(\xi,m,\eta)) = \psi(\eta,0,\eta) = h(\eta) \end{aligned}$$

imply

$$\psi(\xi,m,\eta) = (h(\xi), m, h(\eta)).$$

For every $\xi \in T_G$, consider $(\xi, 1, \sigma(\xi)) \in D_G$ and

$$\psi(\xi,1,\sigma(\xi)) = (h(\xi),1,h(\sigma(\xi))) \in D_H.$$

Then, Lemma 2 implies that there is a unique nonnegative integer m_0, such that

$$\sigma^{m_0+1} \circ h(\xi) = \sigma^{m_0} \circ h \circ \sigma(\xi).$$

For $\xi \in T_G$, we define $k_1(\xi) = m_0$. Then we have

$$\sigma^{k_1(\xi)+1} \circ h(\xi) = \sigma^{k_1(\xi)} \circ h \circ \sigma(\xi).$$

We need to show that $k_1\colon T_G \to \mathbb{N} \cup \{0\}$ is a continuous map, i.e., for every $\xi \in T_G$, there is a neighborhood W of ξ, such that k_1 is a constant on W. Consider $(\xi, 1, \sigma(\xi))$ in D_G. Then Lemma 2 and $\sigma^{m_0+1} \circ h(\xi) = \sigma^{m_0} \circ \sigma \circ h(\xi) = \sigma^{m_0} \circ h \circ \sigma(\xi)$ imply

$$h(\xi) = q(x_0 \cdots x_{m_0}\alpha) \text{ and } h \circ \sigma(\xi) = q(y_1 \cdots y_{m_0}\beta)$$

where

1. $q\colon Y^\omega \to T_H$ is the quotient map by asymptotic equivalence relation,
2. $x_0 \cdots x_{m_0}\alpha$ and $y_1 \cdots y_{m_0}\beta$ are elements of Y^ω,
3. $\sigma \circ h(\xi) = \sigma \circ q(x_0 \cdots x_{m_0}\alpha) = q \circ \sigma(x_0 \cdots x_{m_0}\alpha) = q(x_1 \cdots x_{m_0}\alpha)$
4. $\alpha = \sigma^{m_0+1}(x_0 \cdots x_{m_0}\alpha)$ and $\beta = \sigma^{m_0}(y_1 \cdots y_{m_0}\beta)$ are asymptotically equivalent, and
5. for any $1 \le i \le m_0$,

$$\sigma^{m_0+1-i}(x_0 \cdots x_{m_0}\alpha) = x_{m_0+1-i} \cdots x_{m_0}\alpha \text{ and } \sigma^{m_0-i}(y_1 \cdots y_{m_0}\beta) = y_{m_0+1-i} \cdots y_{m_0}\beta$$

are not asymptotically equivalent.

Let $U = Z(x_0 \cdots x_{m_0})$ and $V = Z(y_1 \cdots y_{m_0})$ in Y^ω. Then $q(U)$ and $q(V)$ are compact open sets in T_H, by definition of the topology on T_H. So

$$Z(q(U), m_0+1, m_0, q(V))$$

is a compact open set in D_H. We consider

$$W_1 = Z(q(U), m_0+1, m_0, q(V)) \setminus \bigcup_{1 \le i \le m_0} Z(q(U), m_0+1-i, m_0-i, q(V)).$$

As $Z(q(U), m_0 + 1 - i, m_0 - i, q(V))$ is a compact set, W_1 is an open set in D_H containing $(h(\xi), 1, h(\sigma(\xi)))$. Then $\psi^{-1}(W_1)$ is an open set in D_G, and so is

$$W_2 = \psi^{-1}(W_1) \cap Z(T_G, 1, 0, T_G),$$

which contains $(\xi, 1, \sigma(\xi))$. It is easy to observe that every element in W_2 is of the form $(\eta, 1, \sigma(\eta))$ because of $Z(T_G, 1, 0, T_G)$. For the domain map, d of D_G. we let

$$W = d(W_2).$$

Then W is a neighborhood of $\xi \in T_G$ because D_G is an étale groupoid so that the domain map d is a local homeomorphism.

Now, we show that k_1 is a constant on W. For any $\eta \in W$ and $(\eta, 1, \sigma(\eta)) \in W_2$, we have

$$\psi(\eta, 1, \sigma(\eta)) = (h(\eta), 1, h \circ \sigma(\eta)) \in W_1$$

and that, from the construction of W_1, m_0 is the smallest nonnegative integer satisfying

$$\sigma^{k+1} \circ h(\eta) = \sigma^k \circ h \circ \sigma(\eta).$$

Hence we have $k_1(\eta) = m_0$ for every $\eta \in W$, and this shows that k_1 is a continuous map.

For ψ^{-1}, we have a continuous map k_2 from the above method so that

$$\sigma^{k_2(\eta)+1} \circ h^{-1}(\eta) = \sigma^{k_2(\eta)} \circ h^{-1} \circ \sigma(\eta)$$

holds. Therefore (T_G, σ) and (T_H, σ) are eventually conjugate. □

Remark 4. *For the proof of continuity of $k_1 \colon T_G \to \mathcal{N} \cup \{0\}$ in Theorem 2, a reviewer suggested the following elegant and short argument: Let κ be the depth-kore operator introduced in [10] (Chpater 3). Then we have*

$$k_1 = \kappa \circ \psi \circ \alpha$$

where $\alpha \colon T_G \to D_G$ is defined by $\xi \mapsto (\xi, 1, \sigma(\xi))$. It is easy to verify that α is continuous, and that the set $\{\alpha^{-1}(W_2)\}$ forms a basis for the topology of T_G. The depth-kore operator κ is locally constant on this basis, thus, it is obviously continuous. Hence, k_1 is a composition of continuous functions. One remarkable property of this argument is that we do not require the étale property of the Deaconu groupoids, i.e., it works for every graph.

Recurrent Self-Similar Groups

If (G, X) and (H, Y) are recurrent self-similar groups, then their limit spaces T_G and T_H, respectively, are connected spaces by Remark 1. Thus, when the limit dynamical systems of recurrent self-similar groups are eventually conjugate, the connection maps k_1 and k_2 are constant maps by Remark 3. For recurrent self similar groups, we can strengthen Theorem 2. First we refine Lemmas 1 and 2 as follows.

Lemma 3. *[11] (Lemma 3.3) Suppose that (G, X) and (H, Y) are recurrent and regular self-similar groups and that their corresponding limit dynamical systems (T_G, σ) and (T_H, σ) are eventually conjugate. Then, for every natural number n, we have*

$$\sigma^{nk_1} \circ h \circ \sigma^n(\xi) = \sigma^{n(k_1+1)} \circ h(\xi) \text{ and}$$
$$\sigma^{nk_2} \circ h^{-1} \circ \sigma^n(\eta) = \sigma^{n(k_2+1)} \circ h^{-1}(\eta).$$

Proof. We use induction. For every $\xi \in T_G$, assume that $\sigma^{nk_1} \circ h \circ \sigma^n(\xi) = \sigma^{n(k_1+1)} \circ h(\xi)$ holds for some $n \in \mathbb{N}$. Then we have $\sigma^{nk_1} \circ h \circ \sigma^n(\sigma(\xi)) = \sigma^{n(k_1+1)} \circ h \circ \sigma(\xi)$ as $\sigma^n(\xi) \in T_G$, so that

$$\begin{aligned}
\sigma^{(n+1)k_1} \circ h \circ \sigma^{n+1}(\xi) &= \sigma^{k_1} \circ \sigma^{nk_1} \circ h \circ \sigma^{n+1}(\xi) \\
&= \sigma^{k_1} \circ \sigma^{nk_1} \circ h \circ \sigma^{n}(\sigma(\xi)) \\
&= \sigma^{k_1} \circ \sigma^{n(k_1+1)} \circ h \circ \sigma(\xi) \\
&= \sigma^{n(k_1+1)} \circ \sigma^{k_1} \circ h \circ \sigma(\xi) \\
&= \sigma^{n(k_1+1)} \circ \sigma^{(k_1+1)} \circ h(\xi) \\
&= \sigma^{(n+1)(k_1+1)} \circ h(\xi).
\end{aligned}$$

By the same argument, we have the second equality for h^{-1}. □

We recall that a self-similar group (G, X) satisfies the recurrent condition if, and only if, for any two words, a, b of equal length, and every $h \in G$, there is a $g \in G$ such that $g(a) = b$ and $g|_a = h$ [12] (p. 235). Then the proof of the following Lemma is basically the same as that of [11] (Proposition 3.5).

Lemma 4. *Suppose that (G, X) is a recurrent and regular self-similar group with the Deaconu groupoid D_G. Then, for every $(\xi, m-n, \eta) \in D_G$, $\sigma^k(\xi) = \sigma^l(\eta)$ holds for all nonnegative integers k and l such that $k - l = m - n$.*

Proof. We consider any $(\xi, m-n, \eta) \in D_G$. If $k > m$, then $\sigma^m(\xi) = \sigma^n(\eta)$ and $k - m = l - n$ imply

$$\sigma^k(\xi) = \sigma^{k-m} \circ \sigma^m(\xi) = \sigma^{k-m} \circ \sigma^n(\eta) = \sigma^{l-n} \circ \sigma^n(\eta) = \sigma^l(\eta).$$

If $0 \le k < m$, we choose any $x = x_0x_1 \cdots \in q^{-1}(\xi)$ and $y = y_0y_1 \cdots \in q^{-1}(\eta)$ where $q \colon X^\omega \to T_G$ is the quotient map. As the shift maps on X^ω and T_G, respectively, and the quotient maps are commuting to each other, we have

$$\sigma^m(x) = x_m x_{m+1} \cdots \in q^{-1}(\sigma^m(\xi)) \text{ and } \sigma^n(y) = y_n y_{n+1} \cdots \in q^{-1}(\sigma^n(\eta)).$$

Then

$$q(\sigma^m(x)) = \sigma^m(q(x)) = \sigma^m(\xi) = \sigma^n(\eta) = \sigma^n(q(y)) = q(\sigma^n(y))$$

implies that $\sigma^m(x) = x_m x_{m+1} \cdots$ and $\sigma^n(y) = y_n y_{n+1} \cdots$ are asymptotically equivalent. So there is an $h \in G$, such that

$$h(x_m x_{m+1} \cdots) = y_n y_{n+1} \cdots .$$

Because of $m - k = n - l$, we have $|x_k \cdots x_{m-1}| = |y_l \cdots y_{n-1}|$. Thus the recurrent condition implies that there is a $g \in G$ such that

$$g(x_k \cdots x_{m-1}) = y_l \cdots y_{n-1} \text{ and } g|_{x_k \cdots x_{m-1}} = h.$$

Then,

$$\begin{aligned} g(\sigma^k(x)) &= g(x_k \cdots x_{m-1} x_m x_{m+1} \cdots) \\ &= g(x_k \cdots x_{m-1}) g|_{x_k \cdots x_{m-1}} (x_m x_{m+1} \cdots) \\ &= y_l \cdots y_{n-1} h(x_m x_{m+1} \cdots) \\ &= y_l \cdots y_{n-1} y_n y_{n+1} \cdots \\ &= \sigma^l(y) \end{aligned}$$

implies that $\sigma^k(x)$ and $\sigma^l(y)$ are asymptotically equivalent to each other. Hence, we have

$$q(\sigma^k(x)) = \sigma^k \circ q(x) = \sigma^k(\xi) = \sigma^l(\eta) = \sigma^l \circ q(y) = q(\sigma^l(y)).$$

Therefore $\sigma^k(\xi) = \sigma^l(\eta)$ holds for all $k, l \in \mathbb{N} \cup \{0\}$ such that $k - l = m - n$. □

Theorem 3. *Let (T_G, σ) and (T_H, σ) be the limit dynamical systems of recurrent and regular self-similar groups (G, X) and (H, Y), respectively. Then, the following assertions are equivalent:*

1. *(T_G, σ) and (T_H, σ) are eventually conjugate.*
2. *There is an isomorphism $\psi \colon D_G \to D_H$, such that*

$$c_G(\xi, m-n, \eta) = c_H(\psi(\xi, m-n, \eta))$$

for every $(\xi, m-n, \eta) \in D_G$.

3. *(T_G, σ) and (T_H, σ) are conjugate.*

Proof. (1) $\Longrightarrow$ (2) follows from Theorem 2.

(2) $\Longrightarrow$ (3). Suppose that $\psi \colon D_G \to D_H$ is a continuous cocycle preserving isomorphism. When we define $h = \psi|_{D_G^{(0)}} \colon T_G \to T_H$, as in the proof of Theorem 2, h is a homeomorphism and

$$\psi(\xi, m, \eta) = (h(\xi), m, h(\eta)).$$

For every $\xi \in T_G$, we consider $(\xi, 1, \sigma(\xi)) \in D_G$. Due to

$$\psi(\xi, 1, \sigma(\xi)) = (h(\xi), 1, h(\sigma(\xi))) \in D_H,$$

Lemma 4 implies $k_1 = 0$ and $\sigma \circ h(\xi) = h \circ \sigma(\xi)$. Therefore (T_G, σ) and (T_H, σ) are conjugate.

(3) $\Longrightarrow$ (1) is trivial. □

We omit the definitions for equivalence of self-similar groups, and refer the reader to [4] for details. The next property follows directly from Theorem 3 and [4] (Theorem 6.4).

Corollary 1. *Suppose that (G, X) and (H, Y) are contracting, recurrent, and regular self-similar groups. Then the following are equivalent.*

1. *The self-similar groups (G, X) and (H, Y) are equivalent in the sense of Nekrashevych.*
2. *(T_G, σ) and (T_H, σ) are eventually conjugate.*
3. *There is an isomorphism $\psi \colon D_G \to D_H$, such that*

$$c_G(\xi, m-n, \eta) = c_H(\psi(\xi, m-n, \eta))$$

for every $(\xi, m-n, \eta) \in D_G$.

4. (T_G, σ) *and* (T_H, σ) *are conjugate.*

4. Conjugacy of Limit Solenoids

While $\sigma\colon T_G \to T_G$ is an epimorphism, $\sigma\colon S_G \to S_G$ is a homeomorphism. Thus, it is natural that the limit solenoid (S_G, σ) of a self-similar group (G, X) has stronger properties than the limit dynamical system (T_G, σ).

Theorem 4. *Suppose that* (G, X) *and* (H, Y) *are self-similar groups with limit solenoids* (S_G, σ) *and* (S_H, σ)*, respectively, and groupoids* E_G *and* E_H*, respectively. Then the following are equivalent.*

1. (S_G, σ) *and* (S_H, σ) *are conjugate.*
2. *There is an isomorphism* $\psi\colon E_G \to E_H$*, such that*

$$c_G(\xi, m-n, \eta) = c_H(\psi(\xi, m-n, \eta))$$

for every $(\xi, m-n, \eta) \in E_G$.

Proof. Suppose that (S_G, σ) and (S_H, σ) are conjugate. Then there is a homeomorphism $h\colon S_G \to S_H$ such that $\sigma \circ h = h \circ \sigma$. We define $\psi\colon E_G \to E_H$ by

$$(\xi, m-n, \eta) \mapsto (h(\xi), m-n, h(\eta)).$$

Due to $\sigma^m(\xi) = \sigma^n(\eta)$ and $\sigma \circ h = h \circ \sigma$, we have

$$h \circ \sigma^m(\xi) = h \circ \sigma^n(\eta) = \sigma^m \circ h(\xi) = \sigma^n \circ h(\eta),$$

so that $(h(\xi), m-n, h(\eta)) \in E_H$. Then, it is routine to check that ψ is a groupoid isomorphism satisfying $c_G(\xi, m-n, \eta) = c_H(\psi(\xi, m-n, \eta))$.

Conversely, assume that $\psi\colon E_G \to E_H$ is an isomorphism with

$$c_G(\xi, m-n, \eta) = c_H(\psi(\xi, m-n, \eta)).$$

Then, $E_G^{(0)} = S_G$ and $E_H^{(0)} = S_H$ imply that $h = \psi|_{E_G^{(0)}}\colon S_G \to S_H$ is a homeomorphism.

Now, we show $\sigma \circ h = h \circ \sigma$. For every $(\xi, m-n, \eta) \in E_G$, we have

$$\psi(\xi, m-n, \eta) = (h(\xi), m-n, h(\eta))$$

due to $c_G(\xi, m-n, \eta) = c_H(\psi(\xi, m-n, \eta))$. Then, for every $(\xi, 1, \sigma(\xi)) \in E_G$, $\psi(\xi, 1, \sigma(\xi)) = (h(\xi), 1, h(\sigma(\xi)) \in E_H$ and the fact that the shift map on S_G is a homeomorphism imply

$$\sigma \circ h(\xi) = h \circ \sigma(\xi).$$

Therefore, (S_G, σ) and (S_H, σ) are conjugate. □

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