mathematics

MDPI

Article

Impulsive Stability of Stochastic Functional Differential Systems Driven by G-Brownian Motion

Lijun Pan [1,*,†], Jinde Cao [2,*,†] and Yong Ren [3,†]

1 School of Mathematics and Statistics, Lingnan Normal University, Zhanjiang 524048, China
2 School of Mathematics, Southeast University, Nanjing 210096, China
3 School of Mathematics and Statistics, Anhui Normal University, Wuhu 241000, China; renyong@126.com
* Correspondence: plj1977@126.com(L.P.); jdcao@seu.edu.cn(J.C.)
† These authors contributed equally to this work.

Received: 3 January 2020; Accepted: 3 February 2020; Published: 10 February 2020

check for updates

Abstract: This paper is concerned with the p-th moment exponential stability and quasi sure exponential stability of impulsive stochastic functional differential systems driven by G-Brownian motion (IGSFDSs). By using G-Lyapunov method, several stability theorems of IGSFDSs are obtained. These new results are employed to impulsive stochastic delayed differential systems driven by G-motion (IGSDDEs). In addition, delay-dependent method is developed to investigate the stability of IGSDDSs by constructing the G-Lyapunov–Krasovkii functional. Finally, an example is given to demonstrate the effectiveness of the obtained results.

Keywords: stability; stochastic systems; delay; impulse; G-Brownian motion

1. Introduction

In the past few decades, stochastic differential system has been widely applied to engineering science, electricity and economics [1,2]. G-expectation is a rising research owing to inclusive application in risk measures, volatility uncertainty, superpricing in finance, and so on. The initial research of G-expectation can be traced back to Peng [3], where G is an infinitesimal generator of a heat equation. Thereafter, its related stochastic calculus, strong laws of large numbers, and central limit theorem under sublinear expectation have been established [4–8]. Especially, based on sample path properties of G-Brownian motion, many studies are available discussing the stochastic differential equation driven by G-Brownian motion (GSDEs) [9–13]. For example, Gao [9] studied the existence for the solutions of GSDEs under Lipschitz coefficient. In [10], a Lyapunov differential operator under G-expectation is provided to deal with G-martingale problems. In [11], by using discrete time feedback control, the authors discussed stabilization of stochastic systems driven by G-Brownian motion.

As is well known, impulsive jumps are natural phenomena owing to abrupt changes at some instants; such jumps are characterized by an impulsive differential system, which has been applied practically in depicting some phenomena that emerge in the areas such as biology, science, and engineering. Therefore, the dynamics of impulsive differential systems is also becoming a research issue [14–18]. Recently, numerous impulsive input stability achievements of stochastic systems have been acquired [19–23]. For example, in [19], Liu studied the stability of solutions for stochastic impulsive differential systems via Lyapunov function method and comparison principles. The authors of [22] investigated the pth moment and almost sure exponential stability for stochastic systems with impulse and delay via Lyapunov technique. [23] discussed stochastic input-to-state stability for impulsive stochastic nonlinear systems by using fixed dwell-time condition and Lyapunov-based approach. However, the research of the stability of impulsive stochastic systems driven by G-Brown motion is a challenging topic since it is necessary to overcome the G-Brown disturbance and reduce the influence of the impulsive effect. [24,25] established the stability for impulsive stochastic differential equations driven by G-Brownian motion with the help of G-Lyapunov function technique. In the evolution of dynamical systems, it is impossible for systems to contact at the same time owing to time-delays. To get over the adverse impact of time-delays, delay-dependent scheme is a vigorous tool to verify the stability of dynamical systems (see [26–28]).

To our best knowledge, there is no literature reported on the pth moment exponential stability and quasi sure exponential stability of the zero solution for impulsive stochastic functional differential systems driven by G-Brownian motion (IGSFDSs) or impulsive stochastic delayed differential systems driven by G-Brownian motion. Therefore, the influence of between delay and impulse for stochastic systems driven by G-Brownian motion provide a motivation of the current study. The aim of this paper is to investigate G-Lyapunov method for IGSFDSs. The contributions in this paper are concluded as follows: (i) Some theorems on pth moment exponential stability and quasi sure exponentially stability of the zero solution of IGSFDSs are established by G-Lyapunov method and impulsive analysis. (ii) These new results are employed to impulsive stochastic delayed systems driven by G-Brownian motion. (iii) If the upper bound of delay may not surpass the length of impulsive gap, delay-dependent technique is utilized to get over the influences of impulses and time delay by G-Lyapunov–Krasovkii functional. The remaining part of this paper is arranged as follows. In Section 2, some definitions and lemmas on G-expectation are proposed and the model descriptions of IGSFDSs are presented. In Section 3, some pth moment exponential stability and quasi sure exponential stability criteria are obtained via stochastic analysis and impulse technique. Section 4 extends the above theorems to impulsive stochastic delayed differential systems driven by G-Brownian motion. In addition, delay-dependent method is employed to establish the stability theorem. In Section 5, an example is provided to show our results. Section 6 gives some conclusions.

2. Preliminaries and Model Description

Let R^n be n-dimensional Euclidean space. $|\cdot|$ denotes the norm for a vector and $\langle\cdot,\cdot\rangle$ represents the scalar product. S^n denotes the space $n\times n$ symmetric matrices. Ω represents the space of R^n-valued continuous functions on $[0,+\infty)$. For every $\omega\in\Omega$, $B_t(\omega)=\omega_t$ denotes the canonical process. Let $\{\mathcal{F}_t\}$ be the filtration generated by canonical process $(B_t)_{t\geq 0}$ as $\mathcal{F}_t=\sigma(B_s, 0\leq s\leq t)$. For each $T>0$, denote

$$L_{ip}(\Omega_T)\triangleq\{\psi(B_{t_1},\cdots,B_{t_n}): n\geq 1, t_1,\cdots,t_n\in[0,T], \psi\in C_{b,L_{ip}}(R^{n\times n})\},$$

and

$$L_{ip}(\Omega)\triangleq\bigcup_T L_{ip}(\Omega_T),$$

where $C_{b,L_{ip}}(R^{n\times n})$ is the space of all bounded Lipschitz functions defined on $R^{n\times n}$. Let $(\Omega, L_{ip}(\Omega), \widetilde{E})$ be the G-expectation space, where $G: S^n\to R$ is defined by

$$G(D)=\tfrac{1}{2}\widetilde{E}[\langle D\xi,\xi\rangle], D\in S^n,$$

$\xi\sim\mathcal{N}(0,\Upsilon)$, Υ denotes the bounded convex and closed subset of $R^{n\times n}$, and the function $\widetilde{E}: L_{ip}(\Omega)\to R$ is a sublinear expectation with zero mean uncertainty.

Remark 1. *For any given function $G: S^n\to R$, there exists a bounded, convex, and closed subset Υ of the space of all $n\times n$ symmetric matrices such that*

$$G(D)=\tfrac{1}{2}\sup_{K\in\Upsilon} tr[DK].$$

Definition 1 ([4])**.** *The n dimensional process $(B_t)_{t\geq 0}$ is said to be a G-Brownian motion under $(\Omega, L_{ip}(\Omega), \widetilde{E})$, if $B_0=0$ and:*

(i) For $s, t \geq 0$, $\psi \in L_{ip}(\Omega)$, $B_t \sim B_{t+s} - B_s \sim \mathcal{N}(0, G)$.
(ii) For $n = 1, 2, \cdots$, $0 = t_0 < t_1 < \cdots < t_n < \infty$, the increment $B_{t_n} - B_{t_{n-1}}$ is independent to $B_{t_1}, B_{t_2}, \cdots, B_{t_{n-1}}$.

For $p \geq 1$, $L_G^p(\Omega)$ denotes the completion of $L_{ip}(\Omega)$ with the norm $(\widetilde{E}|\cdot|^p)^{\frac{1}{p}}$.

Definition 2. *There exists a weakly compact set $\mathcal{J}$ defined on $(\Omega, B(\Omega))$ such that for $\chi \in L_G^1(\Omega)$*

$$\widetilde{E}[\chi] = \sup_{J \in \mathcal{J}} E_J[\chi].$$

Given a set $\mathcal{J}$, we define Choquet capacity as follows

$$\mathcal{U}(C) \triangleq \sup_{J \in \mathcal{J}} J(C), J \subset B(\Omega).$$

Definition 3. *A set $C \subset B(\Omega)$ is said to be polar if $\mathcal{U}(C) = 0$. A property is said to hold quasi-surely if it holds outside a polar set.*

For $p \geq 1$, $T \in [0, +\infty)$, a partition of $[0, T]$ is a finite order subset $\{\mathcal{D}_T^N : N \geq 1\}$ such that

$$\mathcal{D}_T^N : 0 = t_0 < t_1 < \cdots < t_N = T.$$

The space $\mathcal{M}_G^{p,0}([0, T])$ of simple processes can be defined by

$$\mathcal{M}_G^{p,0}([0, T]) \triangleq \{\delta_t(\omega) \triangleq \sum_{j=1}^{N-1} \varsigma_{t_j}(\omega) I_{[t_j, t_{j+1})} : \varsigma_{t_j}(\omega) \in L_G^p(\Omega)\}.$$

For $p \geq 1$, $\mathcal{M}_G^p([0, T])$ denotes the completion of $\mathcal{M}_G^{p,0}([0, T])$ with the norm

$$\|\delta\|_{\mathcal{M}_G^p([0,T])} = \frac{1}{T}(\int_0^T \widetilde{E}[\delta_s^p]ds)^{\frac{1}{p}} = \frac{1}{T}(\sum_{j=1}^{N-1} \widetilde{E}[|\varsigma_{t_j}(\omega)|^p](t_{j+1} - t_j))^{\frac{1}{p}}.$$

Definition 4. *(Itô Integral) For $\delta_t(\omega) \in \mathcal{M}_G^{p,0}([0, T])$, define the Itô Integral as follows*

$$\int_0^T \delta_t dB_t \triangleq \sum_{j=1}^{N-1} \varsigma_{t_j}(\omega)(B_{t_{j+1}} - B_{t_j}).$$

Definition 5. *(Quadratic Variation Process) For a partition of $[0, t](t > 0)$, $0 = t_0 < t_1 < \cdots < t_{N-1} = t$, the quadratic variation process of G-Brownian motion B is defined by*

$$\langle B \rangle_t \triangleq \lim_{N \to \infty} \sum_{j=1}^{N-1} (B_{t_{j+1}} - B_{t_j})^2 = B_t^2 - 2\int_0^t B_s dB_s.$$

Moreover, we define the mutual variation of $\overline{B}$ and $\widehat{B}$ as follows

$$\langle \overline{B}, \widehat{B} \rangle_t \triangleq \frac{1}{4}(\langle \overline{B} + \widehat{B} \rangle_t - \langle \overline{B} - \widehat{B} \rangle_t).$$

Definition 6. *(Integral with respect to $\langle B \rangle_t$) For $\delta_t \in \mathcal{M}_G^{1,0}([0, T])$, we define Itô Integral as follows*

$$\int_0^T \delta_t d\langle B \rangle_t \triangleq \sum_{j=1}^{N-1} \varsigma_{t_j}(\omega)(\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}).$$

Lemma 1 ([4]). *Assume that* $\chi \in L_G^1(\Omega)$, $p > 0$ *and* $\widetilde{E}[|\chi|^p] < +\infty$. *Then, for any* $\epsilon > 0$,

$$\mathcal{U}(|\chi| > \epsilon) \leq \frac{\widetilde{E}[|\chi|^p]}{\epsilon^p}.$$

Lemma 2 ([4]). *Assume that* $p \geq 1$, $c, d \in R^n$, $\delta_t \in \mathcal{M}_G^p([0,T])$ *and* $0 \leq u \leq v \leq T$. *Then,*

$$\widetilde{E}(\sup_{u \leq y \leq v} |\int_u^y \delta_t d\langle \overline{B}, \widehat{B}\rangle_t|^p) \leq C_p^{(1)}|v-u|^{p-1}\int_u^v \widetilde{E}[|\delta_t|^p]dt,$$

where $C_p^{(1)}$ *is a positive constant independent of* δ_t.

Lemma 3 ([4]). *Assume that* $p \geq 1$, $\delta_t \in \mathcal{M}_G^p([0,T])$ *and* $0 \leq u \leq v \leq T$. *Then,*

$$\widetilde{E}(\sup_{u \leq \chi \leq v} |\int_u^\chi \delta_t dB_t|^p) \leq C_p^{(2)}\widetilde{E}(|\int_u^v |\delta_t|^2 dt|^{\frac{p}{2}}) \leq C_p^{(2)}|v-u|^{\frac{p}{2}-1}\int_u^v \widetilde{E}[|\delta_t|^p]dt,$$

where $C_p^{(2)}$ *is a positive constant independent of* δ_t.

Let $PC([-\tau,0];R^n) \triangleq \{\sigma : [-\tau,0] \to R^n | \sigma(t^+), \sigma(t)$ exist and $\sigma(t^-) = \sigma(t)\}$ under the norm $\|\sigma\| = \sup_{-\tau \leq \theta \leq 0} |\sigma(\theta)|$. $PC_{\mathcal{F}_0}^b([-\tau,0];R^n)$ denotes the set of all bounded, $\mathcal{F}_0$-measurable, $PC([-\tau,0];R^n)$-valued random variables. For $p > 0$, $PC_{\mathcal{F}_t}^p([-\tau,0],R^n)$ represents the family of all $\mathcal{F}_t$ measurable $PC([-\tau,0],R^n)$ valued random variables σ such that $\int_{-\tau}^0 \widetilde{E}[|\sigma(s)|^p]ds < +\infty$.

For $i, j = 1, 2, \cdots, l$, by the Einstein convention, the repeated indices of i and j within one term means the summation from 1 to l, i.e.,

$$\int_0^t g_{ij}(s, x(s), x_s) d\langle B^i, B^j\rangle_s \triangleq \sum_{i,j=1}^{l} \int_0^t g_{ij}(s, x(s), x_s) d\langle B^i, B^j\rangle_s$$
$$\int_0^t \rho_j(s, x(s), x_s) dB_s^j \triangleq \sum_{j=1}^{l} \int_0^t \rho_j(s, x(s), x_s) dB_s^j$$

We consider stochastic functional differential system driven by G-Brownian motion under impulsive controller (IGSFDS):

$$\begin{cases} dx(t) = f(t, x(t), x_t)dt + g_{ij}(t, x(t), x_t)d\langle B^i, B^j\rangle_t + \rho_j(t, x(t), x_t)dB_t^j, t \geq 0, t \neq t_k, \\ \triangle x(t_k) = I_k(t_k, x(t_k)), \\ x(t) = \xi, t \in [-\tau, 0], \end{cases} \tag{1}$$

where $\xi \in PC_{\mathcal{F}_0}^b([-\tau,0],R^n)$, $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ and $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$, $x(t_k^+) = \lim_{h \to 0^+} x(t_k + h)$, $x(t_k) = \lim_{h \to 0^-} x(t_k + h)$, $t_k \geq 0$ are impulsive moments satisfying $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$, $\lim_{k \to +\infty} t_k = +\infty$, $\triangle x(t_k) = x(t_k^+) - x(t_k)$ denotes the jump in the state x at t_k, $f, h_{ij}, \rho_j : [0,+\infty) \times M_G^2([0,T];R^n) \times PC([-\tau,0];R^n) \to R^n$, $I_k : [0,+\infty) \times M_G^2([0,T];R^n) \to R^n$, $B_t^i, B_t^j (i,j = 1,2,\cdots,l)$ are G-Brownian, $\langle B^i, B^j\rangle_t$ is the mutual variation of B^i, B^j.

Throughout this paper, set $\triangle_k = t_{k+1} - t_k$, thus $\overline{\Delta} = \sup_{k \geq 0}\{\triangle_k\} < +\infty$. We also assume that f, g_{ij}, ρ_j, and I_k satisfy the classical Lipschitz condition of the global existence and uniqueness of solutions for $t \geq 0$. Given ξ, there exists a unique stochastic process $x(t,\xi)$ satisfying system (1). In addition, we assume that $f(t,0,0) \equiv 0, g_{ij}(t,0,0) \equiv 0, \rho_j(t,0,0) \equiv 0$ and $I_k(t,0) \equiv 0, k = 1,2,\cdots$, which ensures that $x(t) \equiv 0$ is a trivial solution.

Let $\mathcal{C}_1^2([-\tau,\infty) \times R^n; [0,+\infty))$ be the space of nonnegative functions $V(t,x(t))$ on $[-\tau,+\infty) \times R^n$ which are continuous on $(t_{k-1},t_k] \times R^n$, V_t, V_x, V_{xx} are continuous on $(t_{k-1},t_k] \times R^n$. For each $V \in$

$\mathcal{C}_1^2([-\tau,\infty)\times R^n;[0,+\infty))$, we define an operator $\mathcal{L}V:(t_{k-1},t_k]\times PC^p_{\mathcal{F}_t}([-\tau,0];R^n)\to R$ related with system (1) as the following form

$$\begin{aligned}\mathcal{L}V(t,\sigma) &= V_t(t,\sigma(0))+\langle V_x(t,\sigma(0),f(t,\sigma(0),\sigma)\rangle\\ &+G(\langle V_x(t,\sigma(0)),g(t,\sigma(0),\sigma)\rangle+\langle V_{xx}(t,\sigma(0))\rho(t,\sigma(0),\sigma),\rho(t,\sigma(0),\sigma)\rangle),\end{aligned}$$

where $\sigma=\{\sigma(\theta):-\tau\le\theta\le 0\}\in PC^p_{\mathcal{F}_t}([-\tau,0];R^n)$,

$$V_t(t,x)=\frac{\partial V(t,x)}{\partial t},V_x(t,x)=(\frac{\partial V(t,x)}{\partial x_1},\cdots,\frac{\partial V(t,x)}{\partial x_n}),V_{xx}(t,x)=(\frac{\partial^2 V(t,x)}{\partial x_i\partial x_j})_{n\times n}$$

and

$$\begin{aligned}&\langle V_x(t,\sigma(0)),g(t,\sigma(0),\sigma)\rangle+\langle V_{xx}(t,\sigma(0))\rho(t,\sigma(0),\sigma),\rho(t,\sigma(0),\sigma)\rangle\\ =&[\langle V_x(t,\sigma),g_{ij}(t,\sigma(0),\sigma)+g_{ji}(t,\sigma(0),\sigma)\rangle+\langle V_{xx}(t,\sigma)\rho_i(t,\sigma(0),\sigma),\rho_i(t,\sigma(0),\sigma)\rangle]^l_{i,j=1}.\end{aligned}$$

Remark 2. *Obviously, the operator $\mathcal{L}V$ is different from the operator associated with general stochastic system. Therefore, the nonnegative function $V(t,x(t))$ in $\mathcal{C}_1^2([-\tau,\infty)\times R^n;[0,+\infty))$ can be called as G-Lyapunov function.*

Definition 7. *The zero solution of system (1) are called to be pth moment exponentially stable if there exist $\lambda>0$ and $L>0$ such that for $\xi\in PC^b_{\mathcal{F}_0}([-\tau,0];R^n)$*

$$\widetilde{E}[|x(t,\xi)|^p]\le Le^{-\lambda t}\widetilde{E}[\|\xi\|^p],t\ge 0.$$

Remark 3. *When $p=2$, the zero solution of system (1) are said to be exponentially stable in mean square.*

Definition 8. *The zero solution of system (1) are called to be quasi sure exponentially stable if there exists $\lambda>0$ such that for $\xi\in PC^b_{\mathcal{F}_0}([-\tau,0];R^n)$*

$$\limsup_{t\to\infty}\frac{1}{t}\ln|x(t,\xi)|\le-\lambda,\quad q.s.$$

3. Exponential Stability Analysis

This section is devoted to pth moment exponential stability and quasi sure exponential stability of the zero solution of system (1).

Theorem 1. *Assume that $V\in\mathcal{C}_1^2([-\tau,+\infty)\times R^n;[0,+\infty))$ and there exist constants $c_1>0$, $c_2>0$, $\eta_k>0,k=1,2,\cdots,\overline{\mu}\ge 0$, μ,δ such that*
(i) $c_1|x|^p\le V(t,x)\le c_2|x|^p$;
(ii) $\mathcal{L}V(t,\sigma)\le\mu V(t,\sigma(0))+\overline{\mu}\sup_{-\tau\le\theta\le 0}V(t+\theta,\sigma(\theta))$ for all $t\in(t_{k-1},t_k]$;
(iii) $V(t_k^+,x(t_k)+I_k(t_k,x(t_k)))\le\eta_k V(t_k,x(t_k))$;
(iv) $\ln\eta_k\le\delta\triangle_{k-1}$, $k=1,2,\cdots$; and
(v) $\mu+\epsilon\overline{\mu}+\delta<0$.

Then, the zero solution of system (1) is p-th moment exponentially stable with p-th moment exponent λ, where $\epsilon=\sup_{1\le k<+\infty}\{\epsilon_k\}$, $\epsilon_k=\max\{e^{\delta\triangle_{k-1}},e^{-\delta\triangle_{k-1}}\}$, λ is the unique solution of $\lambda+\mu+\epsilon e^{\lambda\tau}\overline{\mu}+\delta=0$.

Proof. $x(t)=x(t;\xi)$ denotes any solution of system (1) with the initial value $x_0=\xi$. By G-Itô formula, we can obtain

$$\begin{aligned} d[e^{-\mu t}V(t,x(t))] = &\ e^{-\mu t}[-\mu V(t,x(t)) + V_t(t,x(t)) + \langle V_x(t,x(t)), f(t,x(t),x_t)\rangle]dt \\ &+ e^{-\mu t}\langle V_x(t,x), \rho_j(t,x(t),x_t)\rangle B_t^j + e^{-\mu t}\langle V_x(t,x(t)), g_{ij}(t,x(t),x_t)\rangle d\langle B^i, B^j\rangle_t \\ &+ \frac{1}{2}e^{-\mu t}\langle V_{xx}(t,x(t))\rho_i(t,x(t),x_t), \rho_j(t,x(t),x_t)\rangle d\langle B^i, B^j\rangle_t. \end{aligned} \tag{2}$$

For $t \in [0, t_1]$, integrating (2) from 0 to t, we obtain

$$\begin{aligned} e^{-\mu t}V(t,x(t)) = &\ V(0,x(0)) + \int_0^t e^{-\mu s}[-\mu V(s,x(s)) + \mathcal{L}V(s,x(s))]ds \\ &+ \prod_t^0 + \int_0^t e^{-\mu s}\langle V_x(s,x), \rho_j(s,x(s),x_s)\rangle B_s^j, \end{aligned} \tag{3}$$

where

$$\begin{aligned} \prod_t^u = &\int_u^t e^{-\mu s}[\langle V_x(s,x(s)), g_{ij}(s,x(s),x_s)\rangle \\ &+ \frac{1}{2}\langle V_{xx}(s,x(s))\rho_i(s,x(s),x_s), \rho_j(s,x(s),x_s)\rangle]d\langle B^i, B^j\rangle_s \\ &- \int_u^t e^{-\mu s}G(\langle V_x(s,x(s)), g(s,x(s),x_s)\rangle + \langle V_{xx}(s,x(s))\rho(s,x(s),x_s), \rho(s,x(s),x_s)\rangle)ds \end{aligned} \tag{4}$$

is a G-martingale and $\widetilde{E}[\prod_t^u|\mathcal{F}_t] = 0$. Taking the expectations on both sides of (3) and by (ii), it yields that

$$\begin{aligned} e^{-\mu t}\widetilde{E}[V(t,x(t))] =& \widetilde{E}[V(0,x(0))] + \widetilde{E}\int_0^t e^{-\mu s}[-\mu V(s,x(s)) + \mathcal{L}V(s,x(s))]ds \\ &\le \widetilde{E}[V(0,x(0))] + \int_0^t \overline{\mu}e^{-\mu s}\sup_{-\tau\le\theta\le 0}\widetilde{E}[V(t+\theta, x(t+\theta))]ds, \end{aligned} \tag{5}$$

Set $V(t) = V(t,x(t))$ and $z(t) = e^{-\mu t}\widetilde{E}[V(t)]$. For $t = t_k, k = 1,2,\cdots$, by (iii), we have

$$z(t_k^+) = e^{-\mu t_k}\widetilde{E}[V(t_k^+)] \le \eta_k z(t_k). \tag{6}$$

For $t \in [0, t_1]$, by (5), we have

$$z(t) \le z(0) + \int_0^t \overline{\mu}e^{-\mu s}\sup_{-\tau\le\theta\le 0}\widetilde{E}[V(s+\theta)]ds, \tag{7}$$

and

$$z(t_1) \le z(0) + \int_0^{t_1} \overline{\mu}e^{-\mu s}\sup_{-\tau\le\theta\le 0}\widetilde{E}[V(s+\theta)]ds. \tag{8}$$

For $t \in (t_1, t_2]$, by using the same method, together with (6) and (8), we obtain

$$\begin{aligned} z(t) &\le z(t_1^+) + \int_{t_1}^t \overline{\mu}e^{-\mu s}\sup_{-\tau\le\theta\le 0}\widetilde{E}[V(s+\theta)]ds \\ &\le \eta_1\{z(0) + \int_0^{t_1} \overline{\mu}e^{-\mu s}\sup_{-\tau\le\theta\le 0}\widetilde{E}[V(s+\theta)]ds\} + \int_{t_1}^t \overline{\mu}e^{-\mu s}\sup_{-\tau\le\theta\le 0}\widetilde{E}[V(s+\theta)]ds \\ &= \eta_1 z(0) + \eta_1\int_0^{t_1} \overline{\mu}e^{-\mu s}\sup_{-\tau\le\theta\le 0}\widetilde{E}[V(s+\theta)]ds + \int_{t_1}^t \overline{\mu}e^{-\mu s}\sup_{-\tau\le\theta\le 0}\widetilde{E}[V(s+\theta)]ds. \end{aligned} \tag{9}$$

By induction, it yields that for $t \in (t_{k-1}, t_k]$

$$z(t) \leq z(0) \prod_{0 \leq t_m < t} \eta_m + \overline{\mu} \int_0^t \prod_{s \leq t_m < t} \eta_m e^{-\mu s} \sup_{-\tau \leq \theta \leq 0} \widetilde{E}[V(s+\theta)]ds, \tag{10}$$

which yields that

$$\widetilde{E}[V(t)] \leq \widetilde{E}[V(0)]e^{\mu t} \prod_{0 \leq t_m < t} \eta_m + \overline{\mu} \int_0^t e^{\mu(t-s)} \prod_{s \leq t_m < t} \eta_i \sup_{-\tau \leq \theta \leq 0} \widetilde{E}[V(s+\theta)]ds. \tag{11}$$

Let $t_{m1}, _{m2}, \cdots, t_{mp}$ be the impulsive points in $[s, t)$ and t_{m1-1} be the first impulsive point before t_{m1}. If $\delta \geq 0$, by (iv), we have

$$\begin{aligned}
\prod_{s \leq t_m < t} \eta_m =& \eta_{m1}\eta_{m2} \cdots \eta_{mp} \leq e^{\delta \Delta_{m1-1}} e^{\delta \Delta_{m1}} \ldots e^{\delta \Delta_{mp-1}} \\
&= e^{\delta(t_{mp} - t_{m1-1})} = e^{\delta(t-s)} e^{\delta(t_{mp}-t)} e^{\delta(s-t_{m1-1})} \\
&\leq e^{\delta(t-s)} e^{\delta(s-t_{m1-1})} \leq \beta e^{\delta(t-s)}.
\end{aligned} \tag{12}$$

If $\delta < 0$, it follows from the similar techniques that

$$\widetilde{E}[V(t)] \leq \beta \widetilde{E}[V(0)] e^{(\mu+\delta)t} + \beta \overline{\mu} \int_0^t e^{(\mu+\delta)(t-s)} \sup_{-\tau \leq \theta \leq 0} \widetilde{E}[V(s+\theta)]ds. \tag{13}$$

Let $\phi(\lambda) = \lambda + \mu + \beta\overline{\mu}e^{\lambda\tau} + \delta$. By (v), we see that $\phi(0) < 0, \phi(+\infty) = +\infty$ and $\phi'(\lambda) = 1 + \beta\overline{\mu}\tau e^{\lambda\tau} > 0$. It follows that $\phi(\lambda) = 0$ has a unique positive solution λ. We conclude that, for $t \geq -\tau$,

$$\widetilde{E}[V(t)] \leq \beta e^{-\lambda t} \sup_{-\tau \leq \varsigma \leq 0} \widetilde{E}[V(\varsigma)]. \tag{14}$$

When $t \in [-\tau, 0]$

$$\widetilde{E}[V(t)] \leq \epsilon \sup_{-\tau \leq \kappa \leq 0} \widetilde{E}[V(\kappa)] \leq \epsilon e^{-\lambda t} \sup_{-\tau \leq \kappa \leq 0} \widetilde{E}[V(\kappa)]. \tag{15}$$

Next, we only show that (14) holds for $t > 0$. If it is not true, there exists $\tilde{t} > 0$ such that

$$\widetilde{E}[V(\tilde{t})] > \epsilon e^{-\lambda \tilde{t}} \sup_{-\tau \leq \kappa \leq 0} \widetilde{E}[V(\kappa)], \quad \widetilde{E}[V(t)] \leq \epsilon e^{-\lambda t} \sup_{-\tau \leq \kappa \leq 0} \widetilde{E}[V(\kappa)], -\tau \leq t < \tilde{t}. \tag{16}$$

Noting $\phi(\lambda) = 0$ and (13), (16) yields

$$\begin{aligned}
\widetilde{E}[V(\tilde{t})] &\leq \epsilon \widetilde{E}[V(0)] e^{(\mu+\delta)\tilde{t}} + \epsilon \overline{\mu} \int_0^{\tilde{t}} e^{(\mu+\delta)(\tilde{t}-s)} \sup_{-\tau \leq \theta \leq 0} \widetilde{E}[V(s+\theta)]ds \\
&\leq \epsilon \sup_{-\tau \leq \kappa \leq 0} \widetilde{E}[V(\kappa)] e^{(\mu+\delta)\tilde{t}} + \epsilon^2 \overline{\mu} e^{\lambda\tau} \sup_{-\tau \leq \kappa \leq 0} \widetilde{E}[V(\kappa)] \int_0^{\tilde{t}} e^{(\mu+\delta)(\tilde{t}-s)} e^{-\lambda s} ds \\
&= \epsilon e^{-\lambda \tilde{t}} \sup_{-\tau \leq \kappa \leq 0} \widetilde{E}[V(\kappa)].
\end{aligned} \tag{17}$$

It leads to a contradiction, which means that (14) holds. By (i) and (14), we have

$$c_1 \widetilde{E}[|x(t)|^p] \leq E[V(t)] \leq \epsilon e^{-\lambda t} \sup_{-\tau \leq \kappa \leq 0} \widetilde{E}[V(\kappa)] \leq \epsilon c_2 e^{-\lambda t} \widetilde{E}[\|\xi\|^p]. \tag{18}$$

Then,

$$\widetilde{E}[|x(t)|^p] \leq \frac{\epsilon c_2}{c_1} e^{-\lambda t} \widetilde{E}[\|\xi\|^p]. \tag{19}$$

Therefore, the zero solution of (1) is p-th moment exponentially stable in the sense of Definition 6. The proof of Theorem 1 is complete. □

Remark 4. *As a G-martingale, $\prod_t^u$ satisfies that $\widetilde{E}[\prod_t^u |\mathcal{F}_t] = 0, t \geq u$, which plays a vital role in the proof of Theorem 1. For details about $\prod_t^u$, we can see Peng [4].*

Theorem 2. *Let $p \geq 2$ and $\widehat{\triangle} = \inf_{k\geq 0}\{\triangle_k\} > 0$. Assume that the conditions of Theorem 1 hold. If there exist constants $L_1 > 0$, $L_2 > 0$, $M_k > 0, k = 1, 2, \cdots$, such that for all $t \geq 0$, $i, j = 1, 2, \cdots, l$, $k = 1, 2, \cdots$, $\sigma = \{\sigma(\theta) : -\tau \leq \kappa \leq 0\} \in PC^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$*

$$\widetilde{E}[|f(t,\sigma(0),\sigma)|^p + |g_{ij}(t,\sigma(0),\sigma)|^p + |\rho_j(t,\sigma(0),\sigma)|^p] \leq L_1\widetilde{E}[|\sigma(0)|^p] + L_2 \sup_{-\tau\leq\kappa\leq 0} \widetilde{E}[|\sigma(\kappa)|^p], \tag{20}$$

and

$$\widetilde{E}[|I_k(t_k, x(t_k))|] \leq M_k\widetilde{E}[|x(t_k)|], \tag{21}$$

then the zero solution of system (1) is quasi sure exponentially stable.

Proof. Since the conditions of Theorem 1 hold, it follows that

$$\widetilde{E}[|x(t)|^p] \leq \frac{\epsilon c_2}{c_1} e^{-\lambda t}\widetilde{E}[\|\xi\|^p]. \tag{22}$$

For $t \geq \tau, 0 \leq \varsigma \leq \tau$, we have

$$\begin{aligned} x(t+\varsigma) =& x(t) + \int_t^{t+\varsigma} f(s,x(s),x_s)ds + \int_t^{t+\varsigma} g_{ij}(s,x(s),x_s)d\langle B^i, B^j\rangle_s \\ &+ \int_t^{t+\varsigma} \rho_j(s,x(s),x_s)dB^j_s + \sum_{t\leq t_k\leq t+\varsigma} I_k(t_k, x(t_k)). \end{aligned} \tag{23}$$

Then,

$$\begin{aligned} \widetilde{E}[\|x_{t+\tau}\|^p] =& \widetilde{E} \sup_{0\leq\varsigma\leq\tau} [|x(t+\varsigma)|^p] \\ &\leq 5^{p-1}\{|x(t)|^p + \widetilde{E}[\int_t^{t+\tau} |f(s,x(s),x_s)|ds]^p + \widetilde{E}[\int_t^{t+\varsigma} |g_{ij}(s,x(s),x_s)|d\langle B^i, B^j\rangle_s]^p \\ &+ \widetilde{E}[\int_t^{t+\varsigma} |\rho_j(s,x(s),x_s)|dB^j_s]^p + \widetilde{E}[\sum_{t\leq t_k\leq t+\tau} |I_k(t_k, x(t_k))|]^p\}. \end{aligned} \tag{24}$$

By Hölder inequality, combining (20) and (22), one can obtain

$$\begin{aligned} &\widetilde{E}[\int_t^{t+\tau} |f(s,x(s),x_s)|ds]^p \\ &\leq \tau^{p-1} \int_t^{t+\tau} \widetilde{E}[|f(s,x(s),x_s)|^p]ds \\ &\leq L_1\tau^{p-1} \int_t^{t+\tau} \widetilde{E}|x(s)|^p ds + L_2\tau^{p-1} \int_t^{t+\tau} \sup_{-\tau\leq\kappa\leq 0} \widetilde{E}|x(s+\kappa)|^p ds \\ &\leq \frac{\epsilon L_1 c_2\tau^{p-1}}{c_1}\widetilde{E}[\|\xi\|^p] \int_t^{t+\tau} e^{-\lambda s}ds + \frac{\epsilon L_2 c_2 \tau^{p-1}}{c_1}\widetilde{E}[\|\xi\|^p] \int_t^{t+\tau} e^{-\lambda(s-\tau)}ds \\ &\leq \frac{\epsilon c_2\tau^{p-1}}{c_1\lambda}(L_1 e^{\lambda\tau} + L_2)e^{-\lambda(t-\tau)}\widetilde{E}[\|\xi\|^p]. \end{aligned} \tag{25}$$

By (20), (22), and Lemma 2, we obtain

$$\begin{aligned}
&\widetilde{E}[\sup_{0\leq\varsigma\leq\tau}\int_t^{t+\varsigma}|g_{ij}(s,x(s),x_s)|^p d\langle B^i,B^j\rangle_s] \\
&\leq C_p^{(1)}\tau^{p-1}\int_t^{t+\varsigma}\widetilde{E}[|g_{ij}(s,x(s),x_s)|^p]ds \\
&\leq L_1 C_p^{(1)}\tau^{p-1}\int_t^{t+\tau}\widetilde{E}[|x(s)|^p]ds + L_2 C_p^{(1)}\tau^{p-1}\int_t^{t+\tau}\sup_{-\tau\leq\kappa\leq 0}\widetilde{E}[|x(s+\kappa)|^p]ds \\
&\leq \frac{\epsilon L_1 c_2 C_p^{(1)}\tau^{p-1}}{c_1}\widetilde{E}[\|\xi\|^p]\int_t^{t+\tau}e^{-\lambda s}ds + \frac{\epsilon L_2 c_2 C_p^{(1)}\tau^{p-1}}{c_1}\widetilde{E}[\|\xi\|^p]\int_t^{t+\tau}e^{-\lambda(s-\tau)}ds \\
&\leq \frac{\epsilon c_2 C_p^{(1)}\tau^{p-1}}{c_1\lambda}(L_1 e^{\lambda\tau}+L_2)e^{-\lambda(t-\tau)}\widetilde{E}[\|\xi\|^p].
\end{aligned} \tag{26}$$

Similarly, from (20), (22), and Lemma 3, it yields that

$$\begin{aligned}
&\widetilde{E}[\sup_{0\leq\varsigma\leq\tau}\int_t^{t+\varsigma}|\rho_j(s,x(s),x_s)|^p dB_s^j] \\
&\leq C_p^{(2)}\tau^{\frac{p}{2}-1}\int_t^{t+\varsigma}\widetilde{E}[|\rho_j(s,x(s),x_s)|^p]ds \\
&\leq L_1 C_p^{(2)}\tau^{\frac{p}{2}-1}\int_t^{t+\tau}\widetilde{E}[|x(s)|^p]ds + L_2 C_p^{(2)}\tau^{\frac{p}{2}-1}\int_t^{t+\tau}\sup_{-\tau\leq\kappa\leq 0}\widetilde{E}[|x(s+\kappa)|^p]ds \\
&\leq \frac{\epsilon L_1 c_2 C_p^{(2)}\tau^{\frac{p}{2}-1}}{c_1}\widetilde{E}[\|\xi\|^p]\int_t^{t+\tau}e^{-\lambda s}ds + \frac{\epsilon L_2 c_2 C_p^{(2)}\tau^{\frac{p}{2}-1}}{c_1}\widetilde{E}[\|\xi\|^p]\int_t^{t+\tau}e^{-\lambda(s-\tau)}ds \\
&\leq \frac{\epsilon c_2 C_p^{(2)}\tau^{\frac{p}{2}-1}}{c_1\lambda}(L_1 e^{\lambda\tau}+L_2)e^{-\lambda(t-\tau)}\widetilde{E}[\|\xi\|^p].
\end{aligned} \tag{27}$$

Noting (21), we have the following estimation

$$\widetilde{E}[\sum_{t\leq t_k\leq t+\tau}|I_k(t_k,x(t_k))|]^p \leq [\frac{\tau}{\triangle}]^p\frac{\epsilon c_2}{c_1}\widetilde{E}\sup_{1\leq k<+\infty}[M_k|x(t_k)|^p] \leq [\frac{\tau}{\triangle}]^p\frac{\epsilon c_2}{c_1}\sup_{1\leq k<+\infty}\{M_k^p\}e^{-\lambda t}\widetilde{E}\|\xi\|^p. \tag{28}$$

Substituting (25)–(28) into (24), we have

$$\widetilde{E}\|x_{t+\tau}\|^p \leq \chi e^{-\lambda t}, t\geq\tau, \tag{29}$$

where $\chi > 0$. Hence, for $\varepsilon\in(0,\lambda)$, $n=1,2,\cdots$, by Lemma 1, we have

$$\mathcal{U}\{\omega : \|x_{(n+1)\tau}\|^p > e^{-(\lambda-\varepsilon)n\tau}\} \leq e^{-(\lambda-\varepsilon)n\tau}\widetilde{E}\|x_{(n+1)\tau}\|^p \leq \chi e^{-\varepsilon n\tau}. \tag{30}$$

According to the Borel–Cantelli lemma, there exists $n_0(\omega)$ such that for almost all $\omega\in\Omega$, $n\geq n_0(\omega)$

$$\|x_{(n+1)\tau}\|^p \leq e^{-(\lambda-\varepsilon)n\tau}, \quad q.s., \tag{31}$$

which implies that for $n\tau\leq t\leq(n+1)\tau$, $n\geq n_0(\omega)$

$$\frac{\ln|x(t)|}{t} \leq -\frac{\lambda-\varepsilon}{p}. \tag{32}$$

Thus,

$$\limsup_{t\to\infty}\frac{\ln|x(t)|}{t} \leq -\frac{\lambda-\varepsilon}{p}, \quad q.s. \tag{33}$$

The remaining proof is to let $\varepsilon \to 0$. This completes the proof. $\square$

Remark 5. *Theorems 1 and 2 are developed by stability of impulsive stochastic functional differential systems with standard Brownian motion [1]. However, due to different random processes, the operator $\mathcal{L}V$ is also different from the operator associated with general stochastic system. For the specific operation of $\mathcal{L}V$, we can see the following two sections.*

4. Some Generalized Results

This section applies the above new consequences to IGSDSs. In addition, delay-dependent method is developed to the stable of the zero solution by constructing G-Lyapunov–Krasovskii functional.

Consider the following impulsive stochastic delayed differential systems driven by G-Brownian motion (IGSDDSs):

$$\begin{cases} dx(t) = f(t, x(t), x(t-\tau(t)))dt + g_{ij}(t, x(t), x(t-\tau(t)))d\langle B^i, B^j\rangle_t \\ \qquad +\rho_j(t, x(t), x(t-\tau(t)))dB_t^j, t \geq 0, t \neq t_k, \\ \triangle x(t_k) = I_k(t_k, x(t_k)), \\ x(t) = \xi, t \in [-\tau, 0], \end{cases} \tag{34}$$

where $0 \leq \tau(t) \leq \tau$, $\tau^{'}(t) \leq \vartheta < 1$, τ is a positive constant, ϑ is a constant, and $f, g_{ij}, \rho_j : [0, +\infty) \times R^n \times R^n \to R^n$ satisfy $f(t, 0, 0) \equiv 0$, $g_{ij}(t, 0, 0) \equiv 0$, $\rho_j(t, 0, 0) \equiv 0$.

Theorem 3. *Assume that $V \in \mathcal{C}_1^2([-\tau, +\infty) \times R^n; [0, +\infty))$ and there exist constants $c_1 > 0$, $c_2 > 0$, $\eta_k > 0, k = 1, 2, \cdots, \overline{\mu} \geq 0$, μ, δ such that:*
(i) $c_1|x|^2 \leq V(t, x) \leq c_2|x|^2$;
(ii) $\mathcal{L}V(t, x(t)) \leq \mu V(t, x(t)) + \overline{\mu}V(t-\tau(t), x(t-\tau(t)))$; $t \in (t_{k-1}, t_k]$;
(iii) $V(t_k^+, x(t_k) + I_k(t_k, x(t_k))) \leq \eta_k V(t_k, x(t_k))$;
(iv) $\ln \eta_k \leq \delta\triangle_{k-1}, k = 1, 2, \cdots$; and
(v) $\mu + \epsilon\overline{\mu} + \delta < 0$.

Then, the zero solution of system (34) is p-th moment exponentially stable with pth moment exponent λ, where $\epsilon = \sup\limits_{1\leq k<+\infty}\{\epsilon_k\}$, $\epsilon_k = \max\{e^{\delta\triangle_{k-1}}, e^{-\delta\triangle_{k-1}}\}$, λ is the unique solution of $\lambda + \mu + \beta e^{\lambda\tau}\overline{\mu} + \delta = 0$.

Theorem 4. *Let $p \geq 2$ and $\widehat{\triangle} = \inf\limits_{k\geq 0}\{\triangle_k\} > 0$. Assume that the conditions of Theorem 3 hold. If there exist constants $L_1 > 0$, $L_2 > 0$, $M_k > 0, k = 1, 2, \cdots$, such that, for all $t \geq 0$, $i, j = 1, 2, \cdots, l$, $k = 1, 2, \cdots$, $x, y \in R^n$*

$$\widetilde{E}[|f(t, x, y)|^p + |g_{ij}(t, x, y)|^p + |\rho_j(t, x, y)|^p] \leq L_1\widetilde{E}[|x|^p] + L_2\widetilde{E}[|y|^p], \tag{35}$$

and

$$\widetilde{E}[|I_k(t, x)|] \leq M_k\widetilde{E}[|x|], \tag{36}$$

then the zero solution of system (34) is quasi sure exponentially stable.

Theorem 5. *Assume that $V \in \mathcal{C}_1^2([-\tau, +\infty) \times R^n; [0, +\infty))$ and there exist constants $c_1 > 0$, $c_2 > 0$, $\eta_k > 0, k = 1, 2, \cdots, \overline{\mu} \geq 0$, $\lambda > 0$, μ such that*
(i) $c_1|x|^2 \leq V(t, x) \leq c_2|x|^2$;
(ii) $\mathcal{L}V(t, x(t)) \leq \mu V(t, x(t)) + \overline{\mu}V(t-\tau(t), x(t-\tau(t)))$, $t \in (t_{k-1}, t_k]$;
(iii) $V(t_k^+, x(t_k) + I_k(t_k, x(t_k))) \leq \eta_k V(t_k, x(t_k))$;
(iv) $\triangle_{k-1} \geq \tau, k = 1, 2, \cdots$; and
(v) $\ln(\eta_k + \frac{\overline{\mu}}{1-\vartheta}\triangle_{k-1}) + \nu\triangle_{k-1} \leq -\lambda, k = 1, 2, \cdots$,

then the zero solution of system (34) is pth moment exponentially stable with pth moment exponent $\frac{\lambda}{\triangle}$*, where* $\nu = \mu + \frac{\overline{\mu}}{1-\vartheta}$.

Proof. Let

$$W(t) = V(t, x(t)) + \frac{\overline{\mu}}{1-\vartheta}\int_{t-\tau(t)}^{t} V(t, x(s))ds, \tag{37}$$

Similar to the proof of Theorem 1, By using G-Itô formula and (ii), for $t \in (t_k, t_{k+1}]$, we have

$$D^{+}\widetilde{E}[W(t)] = \widetilde{E}[\mathcal{L}W(t)] \leq (\mu + \frac{\overline{\mu}}{1-\vartheta})\widetilde{E}V(t, x(t)) \leq \nu \widetilde{E}W(t), \tag{38}$$

where $\nu = \mu + \frac{\overline{\mu}}{1-\vartheta}$. It follows that for $t \in (t_k, t_{k+1}]$

$$\widetilde{E}[W(t)] \leq \widetilde{E}[W(t_k^{+})]e^{\nu(t-t_k)}. \tag{39}$$

When $t = t_k$, by (iii) and the above inequality, we can obtain

$$\widetilde{E}[V(t_k^{+}, x(t_k^{+}))] \leq \eta_k \widetilde{E}[V(t_k, x(t_k)))] \leq \eta_k \widetilde{E}[W(t_k)] \leq \eta_k e^{\nu\triangle_{k-1}}\widetilde{E}[W(t_{k-1}^{+})]. \tag{40}$$

Furthermore, in view of (iv), there exists a $\bar{t}_k \in (t_{k-1}, t_k]$ such that

$$\frac{\overline{\mu}}{1-\vartheta}\int_{t_k-\tau(t_k)}^{t_k} V(s, x(s))ds \leq \frac{\overline{\mu}}{1-\vartheta}\int_{t_{k-1}}^{t_k} V(s, x(s))ds \leq \frac{\overline{\mu}}{1-\vartheta}\triangle_{k-1}V(\bar{t}_k, x(\bar{t}_k)). \tag{41}$$

It follows from(39) and (41) that

$$\widetilde{E}[\frac{\overline{\mu}}{1-\vartheta}\int_{t_k-\tau(t_k)}^{t_k} V(s, x(s))ds] \leq \frac{\overline{\mu}}{1-\vartheta}\triangle_{k-1}\widetilde{E}[W(\bar{t}_k)] \leq \frac{\overline{\mu}}{1-\vartheta}\triangle_{k-1}e^{\nu\triangle_{k-1}}\widetilde{E}[W(t_{k-1}^{+})]. \tag{42}$$

Submitting (40) and (42) into (37), by (v), we have

$$\widetilde{E}[W(t_k^{+})] \leq (\eta_k + \frac{\overline{\mu}}{1-\vartheta}\triangle_{k-1})e^{\nu\triangle_{k-1}}\widetilde{E}[W(t_{k-1}^{+})] \leq e^{-\lambda}\widetilde{E}[W(t_{k-1}^{+})], \tag{43}$$

which yields that

$$\widetilde{E}[W(t_k^{+})] \leq e^{-\lambda k}\widetilde{E}[W(0)]. \tag{44}$$

For $t \in (t_k, t_{k+1}]$, by (39)and (41), we see that

$$\begin{aligned}
\widetilde{E}[W(t)] &\leq e^{\nu(t-t_k)}\widetilde{E}[W(t_k^{+})] \leq e^{\nu\triangle_k}e^{-\lambda k}\widetilde{E}[W(0)] \\
&\leq e^{\nu\triangle_k}e^{\frac{-\lambda t_k}{\triangle}}\widetilde{E}[W(0)] \leq e^{\nu\triangle_k}e^{\frac{-\lambda(t_k-t_{k+1})}{\triangle}}e^{\frac{-\lambda t_{k+1}}{\triangle}}(1+\frac{\overline{\mu}}{1-\vartheta})\widetilde{E}[\sup_{-\tau\leq\theta\leq 0} V(\theta, x(\theta))] \\
&\leq e^{\nu\overline{\triangle}+\lambda}e^{\frac{-\lambda t}{\triangle}}(1+\frac{\overline{\mu}}{1-\vartheta})\widetilde{E}[\sup_{-\tau\leq\theta\leq 0} V(\theta, x(\theta))].
\end{aligned} \tag{45}$$

Thus, by (i), we have

$$\widetilde{E}[|(t)|^{p}] \leq \frac{c_2}{c_1}(1+\frac{\overline{\mu}}{1-\vartheta})e^{\nu\overline{\triangle}+\lambda}e^{\frac{-\lambda t}{\triangle}}\widetilde{E}[\|\xi\|^{p}]. \tag{46}$$

This completes the proof. □

Theorem 6. *Let* $p \geq 2$ *and* $\widehat{\triangle} = \inf_{k\geq 0}\{\triangle_k\} > 0$. *Assume that the conditions of Theorem 5 hold. If there exist constants* $L_1 > 0$, $L_2 > 0$, $M_k > 0, k = 1, 2, \cdots$, *such that for all* $t \geq 0$, $i, j = 1, 2, \cdots, l$, $k = 1, 2, \cdots$, $x, y \in R^n$, *(35), (36) hold, then the zero solution of system (34) is quasi sure exponentially stable.*

Corollary 1. *Let $p \geq 2$. Assume that the following conditions are true:*
(i) there exist constants $\gamma_1, \gamma_2 \geq 0$ such that for all $t \geq 0$, $x, y \in R^n$

$$\langle x^T, f(t,x,y)\rangle \leq \gamma_1|x|^2 + \gamma_2|y|^2; \tag{47}$$

(ii) there exist constants $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \geq 0$ such that for all $t \geq 0$, $x, y \in R^n$

$$G(\langle x^T, g(t,x,y)\rangle) \leq \kappa_1|x|^2 + \kappa_2|y|^2 \tag{48}$$

and

$$G(\langle \rho(t,x,y), \rho(t,x,y)\rangle) \leq \kappa_3|x|^2 + \kappa_4|y|^2; \tag{49}$$

(iii) there exist constants $\eta_k \geq 0, k = 1, 2, \cdots$ such that $|x(t_k) + I_k(t_k, x(t_k))|^p \leq \eta_k |x(t_k)|^p$;
(iv) there exist constant δ such that $\ln \eta_k \leq \delta\Delta_{k-1}$; and
(v) $[p(\gamma_1+\kappa_1) + p(p-1)\kappa_3 + (p-2)(\gamma_2+\kappa_2) + (p-1)(p-2)\kappa_4] + 2\beta[\gamma_2+\kappa_2+(p-1)\kappa_4] + \delta < 0$.

Then, the zero solution of system (34) is p-th moment exponentially stable with pth moment exponent λ, where $\beta = \sup_{1\leq k<+\infty}\{\beta_k\}$, $\beta_k = \max\{e^{\delta\triangle_{k-1}}, e^{-\delta\triangle_{k-1}}\}$, λ is the unique solution of $\lambda + [p(\gamma_1+\kappa_1) + p(p-1)\kappa_3 + (p-2)(\gamma_2+\kappa_2) + (p-1)(p-2)\kappa_4] + 2\beta[\gamma_2+\kappa_2+(p-1)\kappa_4]e^{\lambda\tau} + \delta = 0$. Moreover, If $\widehat{\triangle} = \inf_{k\geq 0}\{\triangle_k\} > 0$ and there exist constants $L_1 > 0$, $L_2 > 0$, $M_k > 0, k = 1, 2, \cdots$, such that, for all $t \geq 0$, $i, j = 1, 2, \cdots, l$, $k = 1, 2, \cdots$, $x, y \in R^n$, (35), (36) hold, then the zero solution of system (34) is quasi sure exponentially stable.

Proof. Set $V(t, x(t)) = |x(t)|^p$. For $t \in (t_{k-1}, t_k]$

$$\begin{aligned}\mathcal{L}V(t,x(t)) =& \langle V_x(t,x(t)), f(t,x(t),x(t-\tau(t)))\rangle + G(\langle V_x(t,x(t)), g(t,x(t),x(t-\tau(t)))\rangle \\ &+ \langle V_{xx}(t,x(t))\rho(t,x(t),x(t-\tau(t))), \rho(t,x(t),x(t-\tau(t)))\rangle) \\ &= \langle p|x(t)|^{p-2}x^T(t), f(t,x(t),x(t-\tau(t)))\rangle + G(\langle p|x(t)|^{p-2}x^T(t), g(t,x(t),x(t-\tau(t)))\rangle) \\ &+ G(\langle p(p-1)|x(t)|^{p-2}\rho(t,x(t),x(t-\tau(t))), \rho(t,x(t),x(t-\tau(t)))\rangle) \\ &\leq [p(\gamma_1+\kappa_1) + p(p-1)\kappa_3]|x(t)|^p + [p(\gamma_2+\kappa_2) + p(p-1)\kappa_4|x(t)|^{p-2}|x(t-\tau(t))|^2.\end{aligned} \tag{50}$$

In view of inequality

$$xy \leq \frac{x^c}{c} + \frac{x^d}{d}, x \geq 0, y \geq 0, c > 1, d > 1, \frac{1}{c} + \frac{1}{d} = 1, \tag{51}$$

we have

$$|x(t)|^{p-2}|x(t-\tau(t))|^2 \leq \frac{p-2}{p}|x(t)|^p + \frac{2}{p}|x(t-\tau(t))|^p. \tag{52}$$

Substituting (52) into (55), we have

$$\begin{aligned}\mathcal{L}V(t,x(t)) \leq& [p(\gamma_1+\kappa_1) + p(p-1)\kappa_3 + (p-2)(\gamma_2+\kappa_2) + (p-1)(p-2)\kappa_4]|x(t)|^p \\ &+ [2(\gamma_2+\kappa_2) + 2(p-1)\kappa_4]|x(t-\tau(t))|^p \\ &= [p(\gamma_1+\kappa_1) + p(p-1)\kappa_3 + (p-2)(\gamma_2+\kappa_2) + (p-1)(p-2)\kappa_4]V(t,x(t)) \\ &+ 2[\gamma_2+\kappa_2+(p-1)\kappa_4]V(t-\tau(t), x(t-\tau(t))).\end{aligned} \tag{53}$$

For $t = t_k$, it yields that

$$V(t_k^+, x(t_k) + I_k(t_k, x(t_k))) \leq \eta_k V(t_k, x(t_k)). \tag{54}$$

Thus, according to Theorem 3 and 4, the conclusion hold. □

Corollary 2. *Assume that $p \geq 2$ and the following conditions hold:*
(i) there exist constants $\gamma_1, \gamma_2 \geq 0$ such that for $t \geq 0$, $x, y \in R^n$

$$\langle x^T, f(t,x,y)\rangle \leq \gamma_1|x|^2 + \gamma_2|y|^2;$$

(ii) there exist constants $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \geq 0$ such that for all $t \geq 0$, $x, y \in R^n$

$$G(\langle x^T, g(t,x,y)\rangle) \leq \kappa_1|x|^2 + \kappa_2|y|^2$$

and

$$G(\langle \rho(t,x,y), \rho(t,x,y)\rangle) \leq \kappa_3|x|^2 + \kappa_4|y|^2;$$

(iii) there exist constants $\eta_k \geq 0, k = 1,2,\cdots$ such that $|x(t_k) + I_k(t_k, x(t_k))|^p \leq \eta_k|x(t_k)|^p$;
(iv) $\triangle_{k-1} \geq \tau$, for $k = 1,2,\cdots$; amd
(v) there exist a constant $\lambda > 0$ such that for $k = 1,2,\cdots$,

$$\begin{aligned}&\ln[\eta_k + \tfrac{2\gamma_2+2\kappa_2+2(p-1)\kappa_4}{1-\vartheta}] + [p(\gamma_1+\kappa_1) + p(p-1)\kappa_3 + (p-2)(\gamma_2+\kappa_2)\\ &+(p-1)(p-2)\kappa_4 + \tfrac{2\gamma_2+2\kappa_2+2(p-1)\kappa_4}{1-\vartheta}]\triangle_{k-1} \leq -\lambda.\end{aligned}$$

Them, the zero solution of system (34) is p-th moment exponentially stable with pth moment exponent $\frac{\lambda}{\triangle}$. Moreover, If $\widehat{\triangle} = \inf_{k\geq 0}\{\triangle_k\} > 0$ and there exist constants $L_1 > 0$, $L_2 > 0$, $M_k > 0, k = 1,2,\cdots$, such that, for all $t \geq 0$, $i,j = 1,2,\cdots,l$, $k = 1,2,\cdots$, $x,y \in R^n$, (35), (36) hold, then the zero solution of system (34) is quasi sure exponentially stable.

Proof. Set $V(t) = |x(t)|^p$. Similar to the proof of Corollary 1, we see that (53) and (54) hold. Consequently, the conclusions follow from Theorem 5. □

5. Example

Consider the following impulsive stochastic delayed systems driven by G-Brownian motion

$$\begin{cases} dx(t) = f(t,x(t),x(t-\frac{1}{4})) + g_{ij}(t,x(t),x(t-\frac{1}{4})d\langle B^i,B^j\rangle_t \\ \qquad +\rho_j(t,x(t),x(t-\frac{1}{4})dB_t^j, t \geq 0, t \neq t_k, i,j = 1,2, \\ \triangle x(t_k) = -0.5x(t_k), \end{cases} \tag{55}$$

where $x(t) = (x_1(t), x_2(t))^T$, $t_k = 0.05k$,

$$f(t,x(t),x(t-\frac{1}{4})) = \begin{pmatrix} -0.5x_1(t) + 1.2x_1(t-\frac{1}{4}) - 0.6x_2(t-\frac{1}{4}) \\ -0.5x_2(t) + 0.5x_1(t-\frac{1}{4}) + 0.2x_2(t-\frac{1}{4}) \end{pmatrix},$$

$$g_{11}(t,x(t),x(t-\frac{1}{4})) = \begin{pmatrix} -0.5x_1(t) + 0.2x_1(t-\frac{1}{4}) \\ -0.4x_2(t) + 0.3x_2(t-\frac{1}{4}) \end{pmatrix},$$

$$g_{12}(t,x(t),x(t-\frac{1}{4})) = \begin{pmatrix} -0.5x_2(t) + 0.2x_1(t-\frac{1}{4}) \\ -0.4x_1(t) + 0.3x_2(t-\frac{1}{4}) \end{pmatrix},$$

$$g_{21}(t,x(t),x(t-\frac{1}{4}))=\begin{pmatrix}-0.5x_2(t)-0.2x_1(t-\frac{1}{4})\\ -0.6x_1(t)-0.3x_2(t-\frac{1}{4})\end{pmatrix},$$

$$g_{22}(t,x(t),x(t-\frac{1}{4}))=\begin{pmatrix}0.3x_1(t)+0.1x_1(t-\frac{1}{4})\\ 0.3x_2(t)-0.2x_2(t-\frac{1}{4})\end{pmatrix},$$

$$\rho_1(t,x(t),x(t-\frac{1}{4}))=\begin{pmatrix}0\\ 0\end{pmatrix},$$

$$\rho_2(t,x(t),x(t-\frac{1}{4}))=\begin{pmatrix}-0.2x_1(t)+0.4x_2(t-\frac{1}{4})\\ 0.3x_2(t)-0.5x_1(t-\frac{1}{4})\end{pmatrix},$$

$$\Upsilon=\{\Xi=\begin{pmatrix}\gamma_{11}\gamma_{12}\\ \gamma_{21}\gamma_{22}\end{pmatrix}:\gamma_{11}\in[\frac{1}{3},\frac{1}{2}],\gamma_{12}\in[\frac{1}{3},\frac{1}{2}],\gamma_{22}=[\frac{1}{8},\frac{1}{4}]\}$$

Set $V(t,x(t))=x_1^2(t)+x_2^2(t)$. By simple calculation, we conclude that

$$\langle V_x(t,x(t)),f(t,x(t),x(t-\frac{1}{4}))\rangle\leq 1.3|x(t)|^2+1.7|x(t-\frac{1}{4})|^2,$$

$$\mathcal{D}=\langle V_x(t,x(t)),g(t,x(t),x(t-\frac{1}{4}))\rangle=\begin{pmatrix}D_1 0\\ 0 D_2\end{pmatrix},$$

$$\mathcal{E}=\langle V_{xx}(t,x(t))\rho(t,x(t),x(t-\frac{1}{4})),\rho(t,x(t),x(t-\frac{1}{4}))\rangle=\begin{pmatrix}00\\ 0E_1\end{pmatrix},$$

where $D_1=2x_1^2(t)+0.8x_1(t)x_1(t-\frac{1}{4})-1.6x_2^2(t)+1.2x_2(t)x_2(t-\frac{1}{4})$, $D_2=1.2x_1^2(t)+0.4x_1(t)x_1(t-\frac{1}{4})+1.2x_2^2(t)-0.8x_2(t)x_2(t-\frac{1}{4})$, $E_1=2[-0.2x_1(t)+0.4x_2(t-\frac{1}{4})]^2+2[0.3x_2(t)-0.5x_1(t-\frac{1}{4})]^2$. Furthermore, we have

$$G(\mathcal{D})=\frac{1}{2}\sup_{\Xi\in\Upsilon}tr(\mathcal{D}\Xi)\leq 0.775|x(t)|^2+0.2|x(t-\frac{1}{4})|^2,$$

$$G(\mathcal{E})=\frac{1}{2}\sup_{\Xi\in\Upsilon}tr(\mathcal{E}\Xi)\leq 0.24|x(t)|^2+0.4|x(t-\frac{1}{4})|^2.$$

Taking $p=2$, $\gamma_1=1.3$, $\gamma_2=1.7$, $\kappa_1=0.775$, $\kappa_2=0.2$, $\kappa_3=0.24$, $\kappa_4=0.4$, $\eta_k=0.25$, $\delta=-27.73$, $\beta=4$, it yields

$$[p(\gamma_1+\kappa_1)+p(p-1)\kappa_3+(p-2)(\gamma_2+\kappa_2)+(p-1)(p-2)\kappa_4]+2\beta[\gamma_2+\kappa_2+(p-1)\kappa_4]+\delta=-4.7<0$$

Based on Corollary 1, the zero solution of system (55) can achieve stability in the mean square.

6. Conclusions

This paper studies the stability of IGSFDSs. By using G-Lyapunov method, some sufficient conditions of stability are obtained. These new results are employed to IGSDDSs. Meanwhile, delay-dependent method is developed to investigate the stability of IGSDDSs. The future research topics would be extending these results to neural networks and multi-agent systems.

Author Contributions: Conceptualization, L.P. and Y.R.; methodology, L.P.; validation, Y.R., J.C. and L.P.; formal analysis, J.C. and L.P.; investigation, L.P. and Y.R.; writing—original draft preparation, J.C. and L.P.;

and writing—review and editing, Y.R. and J.C. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Natural Science Foundation of China (No. 11871076), the Natural Science Foundation of Guangdong Province (Nos. 2015A030310425 and 2019A1515011031), and Talent project of Lingnan Normal University (No. ZL2036).

Acknowledgments: The authors thank the associate editor and the referees for their very helpful comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Mao, X. *Stochastic Differential Equations and Applications*; Ellis Horwood: Chichester, UK, 1997.
2. Mao, X. Razumikhin-type theorems on exponential stability of stochastic functional differential equations. *Stoch. Process. Their Appl.* **1996**, *65*, 233–250. [CrossRef]
3. Peng, S. G-Expectation, G-Brownian Motion and Related Stochastic Calculus of ItÔ Type. *Stoch. Anal. Appl.* **2007**, *2*, 541–567.
4. Peng, S. Nonlinear expectations and stochastic calculus under uncertainty. *arXiv* **2010**, arXiv:1002.4546v1.
5. Peng, S. Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations. *Sci. China Ser. A* **2009**, *52*, 1391–1411. [CrossRef]
6. Denis, L.; Hu, M.; Peng, S. Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths. *Potential Anal.* **2011**, *34*, 139–161. [CrossRef]
7. Li, X.; Peng, S. Stopping times and related Itô calculus with G-Brownian motion. *Stoch. Process. Their Appl.* **2011**, *121*, 1492–1508. [CrossRef]
8. Peng, S. Multi-dimension G-Brownian motion and related stochastic calculus under G-expectation. *Stoch. Process. Their Appl.* **2008**, *118*, 2223–2253. [CrossRef]
9. Gao, F. Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion. *Stoch. Process. Their Appl.* **2009**, *119*, 3356–3382. [CrossRef]
10. Li, X.; Lin, X.; Lin, Y. Lyapunov-type conditions and stochastic differential equations driven by G-Brownian motion. *J. Math. Anal. Appl.* **2016**, *439*, 235–255. [CrossRef]
11. Ren, Y.; Yin, W.; Sakthivel, R. Stabilization of stochastic differential equations driven by G-Brownian motion with feedback control based on discrete-time state observation. *Automatica* **2018**, *95*, 146–151. [CrossRef]
12. Yin, W.; Cao, J.; Ren, Y. Quasi-sure exponential stabilization of stochastic systems induced by G-Brownian motion with discrete time feedback control. *J. Math. Anal. Appl.* **2019**, *474*, 276–289. [CrossRef]
13. Lin, Y. Stochastic differential equations driven by G-Brownian motion with reflecting boundary conditions. *Electron. J. Probbab.* **2013**, *18*, 1–23. [CrossRef]
14. Lakshmikantham, V.; Bainov, D.; Simeonov, P. *Theory of Impulsive Differential Equations*; World Scientific: Singapore, 1989.
15. Yang, T. *Impulsive Control Theory*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2001.
16. Liu, X.; Wang, Q. The method of Lyapunov functionals and exponential stability of impulsive systems with time delay. *Nonlinear Anal.* **2007**, *66*, 1465–1484. [CrossRef]
17. Guan, Z.; Liu, Z.; Feng, G.; Wang, Y. Synchronization of complex dynamical networks with time-varying delays via impulsive distributed control. *IEEE Trans. Circuits Syst. I* **2010**, *57*, 2182–2195. [CrossRef]
18. Liu, B.; Hill, D. Impulsive consensus for complex dynamical networks with nonidentical nodes and coupling time-delays. *SIAM J. Control Optim.* **2011**, *49*, 315–338. [CrossRef]
19. Liu, B. Stability of solutions for stochastic impulsive systems via comparison approach. *IEEE Trans. Autom. Control* **2008**, *53*, 2128–2133. [CrossRef]
20. Cheng, P.; Deng, F. Global exponential stability of impulsive stochastic functional differential systems. *Statist. Probab. Lett.* **2010**, *80*, 1854–1862. [CrossRef]
21. Peng, S.; Jia, B. Some criteria on pth moment stability of impulsive stochastic functional differential equations. *Statist. Probab. Lett.* **2010**, *80*, 1085–1092. [CrossRef]
22. Pan, L.; Cao, J. Exponential stability of impulsive stochastic functional differential equations. *J. Math. Anal. Appl.* **2011**, *382*, 672–685. [CrossRef]
23. Ren, W.; Xiong, J. Stability analysis of impulsive stochastic nonlinear systems. *IEEE Trans. Autom. Control* **2017**, *62*, 4791–4797. [CrossRef]

24. Ren, Y.; Jia, X.; Sakthivel, R. The pth moment stability of solutions to impulsive stochastic differential equations driven by G-Brownian motion. *Anal. Appl.* **2016**. [CrossRef]
25. Ren, Y.; Jia, X.; Hu, L. Exponential stability of solutions to impulsive stochastic differential equations driven by G-Brownian motion. *Discrete Contin. Dyn. Syst. Ser. B* **2017**, *20*, 2157–2169. [CrossRef]
26. Zhang, R.; Zeng, D.; Park, J.; Zhong, S.; Liu, Y.; Zhou, X. New approaches to stability analysis for time-varying delay systems. *J. Frankl. Inst.* **2019**, *356*, 4174–4189. [CrossRef]
27. Zhang, X.; Han, Q.; Wang, J. Admissible delay upper bounds for global asymptotic stability of neural networks with time-varying delays. *IEEE Trans. Neural Netw.* **2018**, *29*, 5319–5329. [CrossRef] [PubMed]
28. Zhang, X.; Han, Q.; Seuret, A.; Gouaisbaut, F.; He, Y. Overview of recent advances in stability of linear systems with time-varying delays. *IET Control Theory Appl.* **2019**, *13*, 1–16. [CrossRef]