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Modified Inertial Hybrid and Shrinking Projection Algorithms for Solving Fixed Point Problems

Bing Tan [1], Shanshan Xu [2] and Songxiao Li [1,*]

[1] Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China; bingtan72@std.uestc.edu.cn

[2] School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China; xss0702@std.uestc.edu.cn

* Correspondence: jyulsx@163.com

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Abstract: In this paper, we introduce two modified inertial hybrid and shrinking projection algorithms for solving fixed point problems by combining the modified inertial Mann algorithm with the projection algorithm. We establish strong convergence theorems under certain suitable conditions. Finally, our algorithms are applied to convex feasibility problem, variational inequality problem, and location theory. The algorithms and results presented in this paper can summarize and unify corresponding results previously known in this field.

Keywords: conjugate gradient method; steepest descent method; hybrid projection; shrinking projection; inertial Mann; strongly convergence; nonexpansive mapping

MSC: 49J40; 47H05; 90C52

1. Introduction

Throughout this paper, let C denote a nonempty closed convex subset of real Hilbert spaces H with standard inner products $\langle \cdot, \cdot \rangle$ and induced norms $\| \cdot \|$. For all $x, y \in C$, there is $\|Tx - Ty\| \leq \|x - y\|$, and the mapping $T : C \to C$ is said to be nonexpansive. We use $\operatorname{Fix}(T) := \{x \in C : Tx = x\}$ to represent the set of fixed points of a mapping $T : C \to C$. The main purpose of this paper is to consider the following fixed point problem: Find $x^* \in C$, such that $T(x^*) = x^*$, where $T : C \to C$ is nonexpansive with $\operatorname{Fix}(T) \neq \emptyset$.

There are various specific applications for approximating fixed point problems with nonexpansive mappings, such as monotone variational inequalities, convex optimization problems, convex feasibility problems, and image restoration problems; see, e.g., [1–6]. It is well known that the Picard iteration method may not converge, and an effective way to overcome this difficulty is to use Mann iterative method, which generates sequences $\{x_n\}$ recursively:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n \,, \quad n \geq 0 \,, \tag{1}$$

the iterative sequence $\{x_n\}$ defined by (1) weakly converges to a fixed point of T when the condition $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = +\infty$ is satisfied, where $\{\alpha_n\} \subset (0, 1)$.

Many practical applications, for instance, quantum physics and image reconstruction, are in infinite dimensional spaces. To investigate these problems, norm convergence is usually preferable to weak convergence. Therefore, modifying the Mann iteration method to obtain strong convergence is an important research topic. For recent works, see [7–12] and the references therein. On the other hand, the Ishikawa iterative method can strongly converge to the fixed point of nonlinear mappings.

For more discussion, see [13–16]. In 2003, Nakajo and Takahashi [17] established strong convergence of the Mann iteration with the aid of projections. Indeed, they considered the following algorithm:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\, T x_n\,, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}\,, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \leq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n} x_0\,, \quad n \in N\,, \end{cases} \tag{2}$$

where $\{\alpha_n\} \subset [0,1)$, T is a nonexpansive mapping on C and $P_{C_n \cap Q_n}$ is the metric projection from C onto $C_n \cap Q_n$. This method is now referred to as the hybrid projection method. Inspired by Nakajo and Takahashi [17], Takahashi, Takeuchi, and Kubota [18] also proposed a projection-based method and obtained strong convergence results, which is now called the shrinking projection method. In recent years, many authors gained new algorithms based on projection method; see [10,18–23].

Generally, the Mann algorithm has a slow convergence rate. In recent years, there has been tremendous interest in developing the fast convergence of algorithms, especially for the inertial type extrapolation method, which was first proposed by Polyak in [24]. Recently, some researchers have constructed different fast iterative algorithms by means of inertial extrapolation techniques, for example, inertial Mann algorithm [25], inertial forward–backward splitting algorithm [26,27], inertial extragradient algorithm [28,29], inertial projection algorithm [30,31], and fast iterative shrinkage–thresholding algorithm (FISTA) [32]. The results of these algorithms and other related ones not only theoretically analyze the convergence properties of inertial type extrapolation algorithms, but also numerically demonstrate their computational performance on some data analysis and image processing problems.

In 2008, Mainge [25] proposed the following inertial Mann algorithm based on the idea of the Mann algorithm and inertial extrapolation:

$$\begin{cases} w_n = x_n + \delta_n (x_n - x_{n-1})\,, \\ x_{n+1} = (1-\eta_n)\, w_n + \eta_n T w_n\,, \quad n \geq 1\,. \end{cases} \tag{3}$$

It should be pointed out that the iteration sequence $\{x_n\}$ defined by (3) only obtains weak convergence results under the following assumptions:

(C1) $\delta_n \in [0,1)$ and $0 < \inf_{n\geq 1} \eta_n \leq \sup_{n \geq 1} \eta_n < 1$;

(C2) $\sum_{n=1}^{\infty} \delta_n \|x_n - x_{n-1}\|^2 < +\infty$.

It should be noted that the condition (C2) is very strong, which prohibits execution of related algorithms. Recently, Bot and Csetnek [33] got rid of the condition (C2); for more details, see Theorem 5 in [33].

In 2014, Sakurai and Iiduka [34] introduced an algorithm to accelerate the Halpern fixed point algorithm in Hilbert spaces by means of conjugate gradient methods that can accelerate the convergence rate of the steepest descent method. Very recently, inspired by the work of Sakurai and Iiduka [34], Dong et al. [35] proposed a modified inertial Mann algorithm by combining the inertial method, the Picard algorithm and the conjugate gradient method. Their numerical results showed that the proposed algorithm has some advantages over other algorithms. Indeed, they obtained the following result:

Theorem 1. *Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Set $\mu \in (0,1], \eta > 0$ and $x_0, x_1 \in H$ arbitrarily and set $d_0 := (Tx_0 - x_0)/\eta$. Define a sequence $\{x_n\}$ by the following algorithm:*

$$\begin{cases} w_n = x_n + \delta_n (x_n - x_{n-1}), \\ d_{n+1} = \frac{1}{\eta}(Tw_n - w_n) + \psi_n d_n, \\ y_n = w_n + \eta d_{n+1}, \\ x_{n+1} = \mu \nu_n w_n + (1 - \mu\nu_n) y_n, n \geq 1. \end{cases} \tag{4}$$

The iterative sequence $\{x_n\}$ defined by (4) converges weakly to a point in $\mathrm{Fix}(T)$ under the following conditions:

(D1) *$\{\delta_n\} \subset [0, \delta]$ is nondecreasing with $\delta_1 = 0$ and $0 \leq \delta < 1$, $\sum_{n=1}^{\infty} \psi_n < \infty$;*

(D2) *Exists $\nu, \sigma, \varphi > 0$ such that $\varphi > \frac{\delta^2(1+\delta)+\delta\sigma}{1-\delta^2}$ and $0 < 1 - \mu\nu \leq 1 - \mu\nu_n \leq \frac{\varphi - \delta[\delta(1+\delta)+\delta\varphi+\sigma]}{\varphi[1+\delta(1+\delta)+\delta\varphi+\sigma]}$;*

(D3) *$\{w_n\}$ defined in (4) assume that $\{Tw_n - w_n\}$ is bounded and $\{Tw_n - y\}$ is bounded for any $y \in \mathrm{Fix}(T)$.*

Inspired and motivated by the above works, in this paper, based on the modified inertial Mann algorithm (4) and the projection algorithm (2), we propose two new modified inertial hybrid and shrinking projection algorithms, respectively. We obtain strong convergence results under some mild conditions. Finally, our algorithms are applied to a convex feasibility problem, a variational inequality problem, and location theory.

The structure of the paper is the following. Section 2 gives the mathematical preliminaries. Section 3 present modified inertial hybrid and shrinking projection algorithms for nonexpansive mappings in Hilbert spaces and analyzes their convergence. Section 4 gives some numerical experiments to compare the convergence behavior of our proposed algorithms with previously known algorithms. Section 5 concludes the paper with a brief summary.

2. Preliminaries

We use the notation $x_n \to x$ and $x_n \rightharpoonup x$ to denote the strong and weak convergence of a sequence $\{x_n\}$ to a point $x \in H$, respectively. Let $\omega_w\{x_n\} := \left\{x : \exists x_{n_j} \rightharpoonup x\right\}$ denote the weak w-limit set of $\{x_n\}$. For any $x, y \in H$ and $t \in R$, we have $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$.

For any $x \in H$, there is a unique nearest point $P_C x$ in C, such that $P_C(x) := \mathrm{argmin}_{y \in C} \|x - y\|$. P_C is called the metric projection of H onto C. $P_C x$ has the following characteristics:

$$P_C x \in C \quad \text{and} \quad \langle P_C x - x, P_C x - y \rangle \leq 0, \quad \forall y \in C. \tag{5}$$

From this characterization, the following inequality can be obtained

$$\|x - P_C x\|^2 + \|y - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, \forall y \in C. \tag{6}$$

We give some special cases with simple analytical solutions:

(1) The Euclidean projection of x_0 onto an Euclidean ball $B[c, r] = \{x : \|x - c\| \leq r\}$ is given by

$$P_{B[c,r]}(x) = c + \frac{r}{\max\{\|x - c\|, r\}}(x - c).$$

(2) The Euclidean projection of x_0 onto a box $\mathrm{Box}[\ell, u] = \{x : \ell \leq x \leq u\}$ is given by

$$P_{\mathrm{Box}[\ell,u]}(x)_i = \min\{\max\{x_i, \ell_i\}, u_i\}.$$

(3) The Euclidean projection of x_0 onto a halfspace $H_{a,b}^- = \{x : \langle a, x\rangle \le b\}$ is given by

$$P_{H_{a,b}^-}(x) = x - \frac{[\langle a, x\rangle - b]_+}{\|a\|^2} a \,.$$

Next we give some results that will be used in our main proof.

Lemma 1. *[36] Let C be a nonempty closed convex subset of real Hilbert spaces H and let $T : C \to H$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Assume that $\{x_n\}$ be a sequence in C and $x \in H$ such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$ as $n \to \infty$, then $x \in \mathrm{Fix}(T)$.*

Lemma 2. *[37] Let C be a nonempty closed convex subset of real Hilbert spaces H. For any $x, y, z \in H$ and $a \in R$. $\left\{v \in C : \|y - v\|^2 \le \|x - v\|^2 + \langle z, v\rangle + a\right\}$ is convex and closed.*

Lemma 3. *[38] Let C be a nonempty closed convex subset of real Hilbert spaces H. Let $\{x_n\} \subset H$, $u \in H$ and $m = P_C u$. If $\omega_w \{x_n\} \subset C$ and satisfies the condition $\|x_n - u\| \le \|u - m\|$, $\forall n \in N$. Then $x_n \to m$.*

3. Modified Inertial Hybrid and Shrinking Projection Algorithms

In this section, we introduce two modified inertial hybrid and shrinking projection algorithms for nonexpansive mappings in Hilbert spaces using the ideas of the inertial method, the Picard algorithm, the conjugate gradient method, and the projection method. We prove the strong convergence of the algorithms under suitable conditions.

Theorem 2. *Let C be a bounded closed convex subset of real Hilbert spaces H and let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Assume that the following conditions are satisfied:*

$$\eta > 0, \delta_n \subset [\delta_1, \delta_2], \delta_1 \in (-\infty, 0], \delta_2 \in [0, \infty), \psi_n \subset [0, \infty), \lim_{n\to\infty} \psi_n = 0, \nu_n \subset (0, \nu], 0 < \nu < 1 \,.$$

Set $x_{-1}, x_0 \in H$ arbitrarily and set $d_0 := (Tx_0 - x_0)/\eta$. Define a sequence $\{x_n\}$ by the following:

$$\begin{cases} w_n = x_n + \delta_n(x_n - x_{n-1}) \,, \\ d_{n+1} = \frac{1}{\eta}(Tw_n - w_n) + \psi_n d_n \,, \\ y_n = w_n + \eta d_{n+1} \,, \\ z_n = \nu_n w_n + (1 - \nu_n) y_n \,, \\ C_n = \left\{z \in H : \|z_n - z\|^2 \le \|w_n - z\|^2 - \nu_n(1 - \nu_n)\|w_n - y_n\|^2 + \xi_n\right\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0\rangle \le 0\} \,, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \,, \quad n \ge 0 \,, \end{cases} \tag{7}$$

where the sequence $\{\xi_n\}$ is defined by $\xi_n := \eta\psi_n M_2 [\eta\psi_n M_2 + 2M_1]$, $M_1 := diam\ C = \sup_{x,y\in C} \|x - y\|$ and $M_2 := \max\left\{\max_{1\le k\le n_0} \|d_k\|, \frac{2}{\eta} M_1\right\}$, where n_0 satisfies $\psi_n \le \frac{1}{2}$ for all $n \ge n_0$. Then the iterative sequence $\{x_n\}$ defined by (7) converges to $P_{\mathrm{Fix}(T)} x_0$ in norm.

Proof. We divided our proof in three steps.
Step 1. To begin with, we need to show that $\mathrm{Fix}(T) \subset C_n \cap Q_n$. It is easy to check that C_n is convex by Lemma 2. Next we prove $\mathrm{Fix}(T) \subset C_n$ for all $n \ge 0$. Assume that $\|d_n\| \le M_1$ for some $n \ge n_0$. The triangle inequality ensures that

$$\|d_{n+1}\| = \|\frac{1}{\eta}(Tw_n - w_n) + \psi_n d_n\| \le \frac{1}{\eta}\|Tw_n - w_n\| + \psi_n\|d_n\| \le M_2 \,,$$

which implies that $\|d_n\| \leq M_2$ for all $n \geq 0$, that is, $\{d_n\}$ is bounded. Due to $w_n \in C$, we get that $\|w_n - p\| \leq M_1$ for all $u \in \mathrm{Fix}(T)$. From the definition of $\{y_n\}$ and nonexpansivity of T we obtain

$$\begin{aligned}\|y_n - u\| &= \|w_n + \eta\left(\frac{1}{\eta}(Tw_n - w_n) + \psi_n d_n\right) - u\| = \|Tw_n + \eta\psi_n d_n - u\| \\ &\leq \|w_n - u\| + \eta\psi_n M_2 .\end{aligned}$$

Therefore,

$$\begin{aligned}\|z_n - u\|^2 &= \|\nu_n(w_n - u) + (1-\nu_n)(y_n - u)\|^2 \\ &= \nu_n\|w_n - u\|^2 + (1-\nu_n)\|y_n - u\|^2 - \nu_n(1-\nu_n)\|w_n - y_n\|^2 \\ &\leq \|w_n - u\|^2 + 2\eta\psi_n M_2\|w_n - u\| + (\eta\psi_n M_2)^2 - \nu_n(1-\nu_n)\|w_n - y_n\|^2 \\ &\leq \|w_n - u\|^2 - \nu_n(1-\nu_n)\|w_n - y_n\|^2 + \xi_n ,\end{aligned}$$

where $\xi_n = \eta\psi_n M_2[\eta\psi_n M_2 + 2M_1]$. Thus, we have $u \in C_n$ for all $n \geq 0$ and hence $\mathrm{Fix}(T) \subset C_n$ for all $n \geq 0$. On the other hand, it is easy to see that $\mathrm{Fix}(T) \subset C = Q_0$ when $n = 0$. Suppose that $\mathrm{Fix}(T) \subset Q_{n-1}$, by combining the fact that $x_n = P_{C_{n-1}\cap Q_{n-1}}x_0$ and (5) we obtain $\langle x_n - z, x_n - x_0\rangle \leq 0$ for any $z \in C_{n-1}\cap Q_{n-1}$. According to the induction assumption we have $\mathrm{Fix}(T) \subset C_{n-1}\cap Q_{n-1}$, and it follows from the definition of Q_n that $\mathrm{Fix}(T) \subset Q_n$. Therefore, we get $\mathrm{Fix}(T) \subset C_n \cap Q_n$ for all $n \geq 0$.

Step 2. We prove that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Combining the definition of Q_n and $\mathrm{Fix}(T) \subset Q_n$, we obtain

$$\|x_n - x_0\| \leq \|u - x_0\| , \quad \text{for all } u \in \mathrm{Fix}(T) .$$

We note that $\{x_n\}$ is bounded and

$$\|x_n - x_0\| \leq \|x^* - x_0\| , \quad \text{where } x^* = P_{\mathrm{Fix}(T)}x_0 . \tag{8}$$

The fact $x_{n+1} \in Q_n$, we have $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$, which means that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. Using (6), one sees that

$$\|x_n - x_{n+1}\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 ,$$

which implies that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Next, by the definition of w_n, we have

$$\|w_n - x_n\| = |\delta_n|\,\|x_n - x_{n-1}\| \leq \delta_2\|x_n - x_{n-1}\| \to 0\,(n \to \infty) ,$$

which further yields that

$$\|w_n - x_{n+1}\| \leq \|w_n - x_n\| + \|x_n - x_{n+1}\| \to 0\,(n \to \infty) .$$

Step 3. It remains to show that $x_n \to x^*$, where $x^* = P_{\mathrm{Fix}(T)}x_0$. From $x_{n+1} \in C_n$ we get

$$\|z_n - x_{n+1}\|^2 \leq \|w_n - x_{n+1}\|^2 - \nu_n(1-\nu_n)\|w_n - y_n\|^2 + \xi_n .$$

Therefore,

$$\|z_n - x_{n+1}\| \leq \|w_n - x_{n+1}\| + \sqrt{\xi_n} .$$

On the other hand, since $z_n = \nu_n w_n + (1-\nu_n)\,Tw_n + (1-\nu_n)\,\eta\psi_n d_n$ and $\nu_n \le \nu$, we obtain

$$\begin{aligned}\|Tw_n - w_n\| &= \frac{1}{1-\nu_n}\|z_n - w_n - (1-\nu_n)\,\eta\psi_n d_n\| \\ &\le \frac{1}{1-\nu}\|z_n - w_n\| + \eta\psi_n\|d_n\| \\ &\le \frac{1}{1-\nu}\left(\|z_n - x_{n+1}\| + \|w_n - x_{n+1}\|\right) + \eta\psi_n M_2 \\ &\le \frac{1}{1-\nu}\left(2\|w_n - x_{n+1}\| + \sqrt{\xi_n}\right) + \eta\psi_n M_2 \to 0\,(n\to\infty)\,.\end{aligned}$$

Consequently,

$$\begin{aligned}\|Tx_n - x_n\| &\le \|Tx_n - Tw_n\| + \|Tw_n - w_n\| + \|w_n - x_n\| \\ &\le 2\|w_n - x_n\| + \|Tw_n - w_n\| \to 0\,(n\to\infty)\,.\end{aligned} \tag{9}$$

In view of (9) and Lemma 1, it follows that every weak limit point of $\{x_n\}$ is a fixed point of T. i.e., $\omega_w\{x_n\} \subset \mathrm{Fix}(T)$. By means of Lemma 3 and the inequality (8), we get that $\{x_n\}$ converges to $P_{\mathrm{Fix}(T)}x_0$ in norm. The proof is complete. $\square$

Theorem 3. *Let C be a bounded closed convex subset of real Hilbert spaces H and let $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$*. Assume that the following conditions are satisfied:*

$$\eta > 0, \delta_n \subset [\delta_1, \delta_2]\,, \delta_1 \in (-\infty, 0], \delta_2 \in [0,\infty), \psi_n \subset [0,\infty), \lim_{n\to\infty}\psi_n = 0, \nu_n \subset (0,\nu], 0 < \nu < 1\,.$$

Set $x_{-1}, x_0 \in H$ arbitrarily and set $d_0 := (Tx_0 - x_0)/\eta$. Define a sequence $\{x_n\}$ by the following:

$$\begin{cases} w_n = x_n + \delta_n(x_n - x_{n-1})\,, \\ d_{n+1} = \frac{1}{\eta}\,(Tw_n - w_n) + \psi_n d_n\,, \\ y_n = w_n + \eta d_{n+1}\,, \\ z_n = \nu_n w_n + (1-\nu_n)\,y_n\,, \\ C_{n+1} = \left\{z \in C_n : \|z_n - z\|^2 \le \|w_n - z\|^2 - \nu_n\,(1-\nu_n)\,\|w_n - y_n\|^2 + \xi_n\right\}, \\ x_{n+1} = P_{C_{n+1}}x_0\,, \quad n \ge 0\,. \end{cases} \tag{10}$$

where the sequence $\{\xi_n\}$ is defined by $\xi_n := \eta\psi_n M_2\,[\eta\psi_n M_2 + 2M_1]$, $M_1 :=$ diam $C = \sup_{x,y\in C}\|x-y\|$ and $M_2 := \max\left\{\max_{1\le k\le n_0}\|d_k\|\,, \frac{2}{\eta}M_1\right\}$, where n_0 satisfies $\psi_n \le \frac{1}{2}$ for all $n \ge n_0$. Then the iterative sequence $\{x_n\}$ defined by (10) *converges to $P_{\mathrm{Fix}(T)}x_0$ in the norm.*

Proof. We divided our proof in three steps.
Step 1. Our first goal is to show that $\mathrm{Fix}(T) \subset C_{n+1}$ for all $n \ge 0$. According to Step 1 in Theorem 2, for all $u \in \mathrm{Fix}(T)$, we obtain

$$\|z_n - u\|^2 \le \|w_n - u\|^2 - \nu_n\,(1-\nu_n)\,\|w_n - y_n\|^2 + \xi_n\,.$$

Therefore, $u \in C_{n+1}$ for each $n \ge 0$ and hence $\mathrm{Fix}(T) \subset C_{n+1} \subset C_n$.
Step 2. As mentioned above, the next thing to do in the proof is show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Using the fact that $x_n = P_{C_n}x_0$ and $\mathrm{Fix}(T) \subset C_n$, we have

$$\|x_n - x_0\| \le \|u - x_0\|\,, \quad \text{for all } u \in \mathrm{Fix}(T)\,.$$

It follows that $\{x_n\}$ is bounded, in addition, we note that

$$\|x_n - x_0\| \le \|x^* - x_0\|\,, \quad \text{where } x^* = P_{\mathrm{Fix}(T)}x_0\,. \tag{11}$$

On the other hand, since $x_{n+1} \in C_n$, we obtain $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$, which implies that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. In view of (6), we have

$$\|x_n - x_{n+1}\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 ,$$

which further implies that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Also, we have $\lim_{n\to\infty} \|w_n - x_n\| = 0$ and $\lim_{n\to\infty} \|w_n - x_{n+1}\| = 0$.

Step 3. Finally, we have to show that $x_n \to x^*$, where $x^* = P_{\mathrm{Fix}(T)} x_0$. The remainder of the argument is analogous to that in Theorem 2 and is left to the reader. □

Remark 1. *We remark here that the modified inertial hybrid projection algorithm* (7) *(in short, MIHPA) and the modified inertial shrinking projection algorithm* (10) *(in short, MISPA) contain some previously known results. When $\delta_n = 0$ and $\psi_n = 0$, the MIHPA becomes the hybrid projection algorithm (in short, HPA) proposed by Nakajo and Takahashi [17] and the MISPA becomes the shrinking projection algorithm (in short, SPA) proposed by Takahashi, Takeuchi, and Kubota [18]. When $\delta_n = 0$ and $\psi_n \neq 0$, the MIHPA becomes the modified hybrid projection algorithm (in short, MHPA) proposed by Dong et al. [35], the MISPA becomes the modified shrinking projection algorithm (in short, MSPA).*

4. Numerical Experiments

In this section, we provide three numerical applications to demonstrate the computational performance of our proposed algorithms and compare them with some existing ones. All the programs are performed in MATLAB2018a on a personal computer Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz 1.800 GHz, RAM 8.00 GB.

Example 1. *As an example, we consider the convex feasibility problem, for any nonempty closed convex set $C_i \subset R^N$ $(i = 0, 1, \dots, m)$, we find $x^* \in C := \bigcap_{i=0}^{m} C_i$, where one supposes that $C \neq \emptyset$. A mapping $T : R^N \to R^N$ is defined by $T := P_0 \left(\frac{1}{m} \sum_{i=1}^{m} P_i \right)$, where $P_i = P_{C_i}$ stands for the metric projection onto C_i. It follows from P_i being nonexpansive that the mapping T is also nonexpansive. Furthermore, we note that $\mathrm{Fix}(T) = \mathrm{Fix}\,(P_0) \bigcap_{i=1}^{m} \mathrm{Fix}\,(P_i) = C_0 \bigcap_{i=1}^{m} C_i = C$. In this experiment, we set C_i as a closed ball with center $c_i \in R^N$ and radius $r_i > 0$. Thus P_i can be computed with*

$$P_i(x) := \begin{cases} c_i + \frac{r_i}{\|c_i - x\|}(x - c_i), & \text{if } \|c_i - x\| > r_i\,; \\ x, & \text{if } \|c_i - x\| \leq r_i\,. \end{cases}$$

Choose $r_i = 1$ $(i = 0, 1, \dots, m)$, $\mathbf{c}_0 = [0, 0, \dots, 0]$, $\mathbf{c}_1 = [1, 0, \dots, 0]$, and $\mathbf{c}_2 = [-1, 0, \dots, 0]$. $\mathbf{c}_i$ is randomly selected from $(-1/\sqrt{N}, 1/\sqrt{N})^N$ $(i = 3, \dots, m)$. We have $\mathrm{Fix}(T) = \{\mathbf{0}\}$ from the special choice of $\mathbf{c}_1, \mathbf{c}_2$ and r_1, r_2. In Algorithms (7) *and* (10)*, setting $m = 30$, $N = 30$, $\eta = 1$, $\psi_n = \frac{1}{100(n+1)^2}$, $\nu_n = 0.1$. When the iteration error $E_n = \|x_n - Tx_n\|_2 < 10^{-2}$ is satisfied, the iteration stops. We test our algorithms under different inertial parameters and initial values. Results are shown in Table 1, where "Iter." represents the number of iterations.*

Table 1. Computational results for Example 1.

Algorithm	Initial Value	δ_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
MIHPA	rand(N,1)	Iter.	223	248	**218**	239	283	245	258	249	248	247
MISPA			127	137	148	159	169	163	167	187	186	190
MIHPA	ones(N,1)	Iter.	327	**315**	407	354	342	356	377	391	348	349
MISPA			174	189	181	199	217	208	279	250	243	256
MIHPA	10rand(N,1)	Iter.	1057	1377	1522	1494	1307	1119	1261	1098	**1005**	1070
MISPA			549	570	704	698	845	852	987	856	1003	975
MIHPA	−10rand(N,1)	Iter.	445	**410**	574	504	657	716	729	730	659	682
MISPA			316	**313**	350	416	423	386	427	392	516	556

Example 2. *Our another example is to consider the following variational inequality problem (in short, VI). For any nonempty closed convex set $C \subset R^N$,*

$$\text{find } x^* \in C \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{12}$$

where $f : R^N \to R^N$ is a mapping. Take $\mathrm{VI}(C, f)$ denote the solution of VI (12). $T : R^N \to R^N$ is defined by $T := P_C(I - \gamma f)$, where $0 < \gamma < 2/L$, and L is the Lipschitz constant of the mapping f. In [39], Xu showed that T is an averaged mapping, i.e., T can be seen as the average of an identity mapping I and a nonexpansive mapping. It follows that $\mathrm{Fix}(T) = \mathrm{VI}(C, f)$, we can solve VI (12) by finding the fixed point of T. Taking $f : R^2 \to R^2$ as follows:

$$f(x, y) = (2x + 2y + \sin(x), -2x + 2y + \sin(y)), \quad \forall x, y \in R.$$

The feasible set C is given by $C = \{x \in R^2 | -10\mathbf{e} \leq x \leq 10\mathbf{e}\}$, where $\mathbf{e} = (1,1)^{\mathsf{T}}$. It is not hard to check that f is Lipschitz continuous with constant $L = \sqrt{26}$ and 1-strongly monotone [40]. Therefore, VI (12) has a unique solution $\mathbf{x}^* = (0,0)^{\mathsf{T}}$.

We use the Algorithm (7) (MIHPA), the Algorithm (10) (MISPA), the modified hybrid projection algorithm (MHPA), the modified shrinking projection algorithm (MSPA), the hybrid projection algorithm (HPA), and the shrinking projection algorithm (SPA) to solve Example 2. Setting $\gamma = 0.9/\sqrt{26}$, $\eta = 1$, $\psi_n = \frac{1}{100(n+1)^2}$, $\nu_n = 0$ (we consider that T is an average mapping). The initial values are randomly generated by the MATLAB function rand(2,1). We use $E_n = \|x_n - x^*\|_2$ to denote the iteration error of algorithms, and the maximum iteration 300 as the stopping criterion. Results are reported in Table 2, where "Iter." denotes the number of iterations.

Table 2. Computational results for Example 2.

Iter.	HPA		SPA		MHPA		MSPA		MIHPA		MISPA	
	x_n	E_n	x_n	E_n	x_n	E_n	x_n	E_n	x_n	E_n	x_n	E_n
1	(0.2944,0.8061)	0.8582	(0.2944,0.8061)	0.8582	(0.4607,0.8706)	0.9850	(0.4607,0.8706)	0.9850	(0.4607,0.8706)	0.9850	(0.4607,0.8706)	0.9850
50	(0.0049,0.0164)	0.0171	(0.0000,0.0001)	0.0001	(0.0142,0.0357)	0.0384	(0.0142,0.0264)	0.0300	(0.0094,0.0357)	0.0369	(0.0142,0.0278)	0.0312
100	(0.0006,0.0017)	0.0018	(0.0000,0.0000)	0.0000	(0.0116,0.0110)	0.0159	(0.0072,0.0133)	0.0151	(0.0096,0.0144)	0.0173	(0.0067,0.0137)	0.0153
200	(-0.0003,0.0013)	0.0014	(0.0000,0.0000)	0.0000	(0.0059,0.0053)	0.0080	(0.0034,0.0068)	0.0076	(0.0061,0.0047)	0.0077	(0.0036,0.0060)	0.0070
300	(0.0007,0.0003)	**0.0008**	(0.0000,0.0000)	**0.0000**	(0.0043,0.0030)	**0.0053**	(0.0021,0.0045)	**0.0049**	(0.0045,0.0038)	**0.0058**	(0.0018,0.0053)	**0.0056**

Example 3. *The Fermat–Weber problem is a famous model in location theory. It is can be formulated mathematically as the problem of finding $\mathbf{x} \in R^n$ that solves*

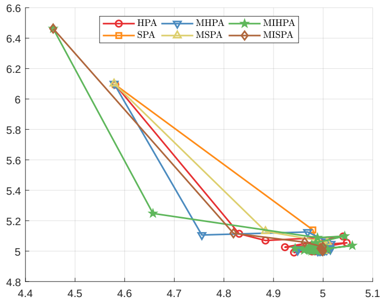
$$\min_{\mathbf{x}} \left\{ f(\mathbf{x}) := \sum_{i=1}^{m} \omega_i \|\mathbf{x} - \mathbf{a}_i\|_2 \right\}, \tag{13}$$

where $\omega_i > 0$ are given weights and $\mathbf{a}_i \in R^n$ are anchor points. It is easy to check that the objective function f in (13) is convex and coercive. Therefore, the problem has a nonempty solution set. It should be noted that f is not differentiable at the anchor points. The most famous method to solve the problem (13) is the Weiszfeld algorithm; see [41] for more discussion. Weiszfeld proposed the following fixed point algorithm: $\mathbf{x}_{n+1} = T(\mathbf{x}_n), n \in N$. The mapping $T : R^n \backslash \mathbf{A} \longmapsto R^n$ is defined by $T(\mathbf{x}) := \frac{1}{\sum_{i=1}^m \frac{\omega_i}{\|\mathbf{x} - \mathbf{a}_i\|}} \sum_{i=1}^m \frac{\omega_i \mathbf{a}_i}{\|\mathbf{x} - \mathbf{a}_i\|}$, where $\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$. We consider a small example with $n = 2, m = 4$ anchor points,

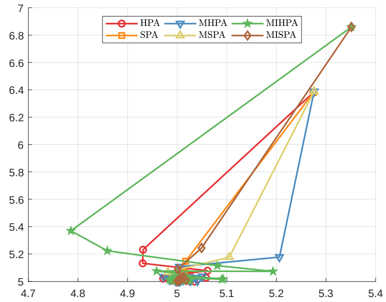
$$\mathbf{a}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 10 \\ 0 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} 0 \\ 10 \end{pmatrix}, \mathbf{a}_4 = \begin{pmatrix} 10 \\ 10 \end{pmatrix},$$

and $\omega_i = 1$ for all i. It follows from the special selection of anchor points $\mathbf{a}_i$ $(i = 1, 2, 3, 4)$ that the optimal value of (13) is $\mathbf{x}^* = (5,5)^{\mathsf{T}}$.

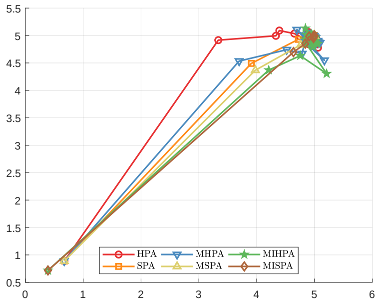
We use the same algorithms as in Example 2, and our parameter settings are as follows, setting $\eta = 1$, $\psi_n = \frac{1}{100(n+1)^2}$, $\nu_n = 0.1$. We use $E_n = \|x_n - x^*\|_2 < 10^{-4}$ or maximum iteration 300 as the stopping *criterion. The initial values are randomly generated by the MATLAB function 10rand(2,1). Figures 1 and 2 show the convergence behavior of iterative sequence $\{x_n\}$ and iteration error E_n, respectively.*



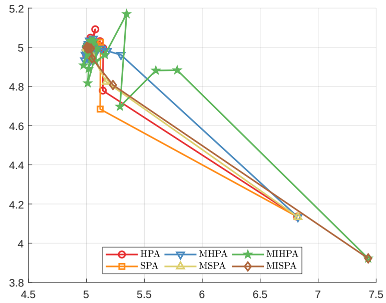
(**a**) Case I



(**b**) Case II

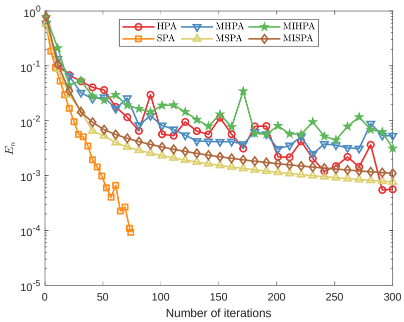


(**c**) Case III

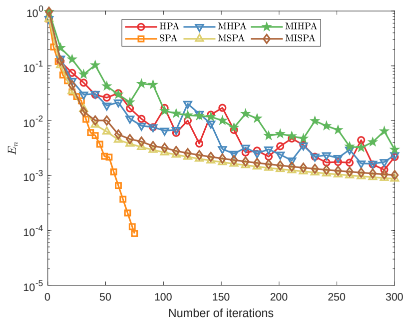


(**d**) Case IV

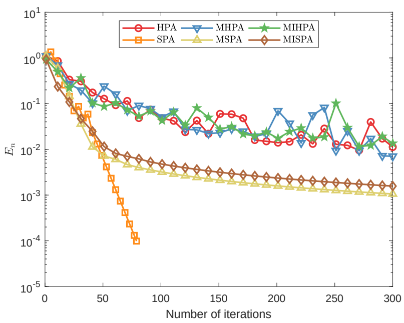
Figure 1. Convergence process at different initial values for Example 3.



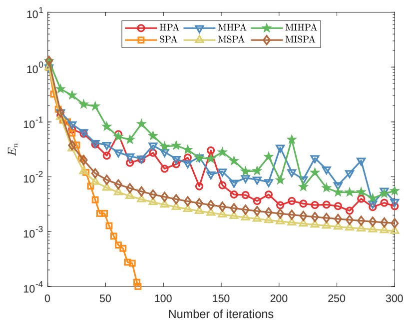
(**a**) Case I



(**b**) Case II



(**c**) Case III



(**d**) Case IV

Figure 2. Convergence behavior of iteration error $\{E_n\}$ for Example 3.

Remark 2. *From Examples 1–3, we know that our proposed algorithms are effective and easy to implement. Moreover, initial values do not affect the computational performance of our algorithms. However, it should be mentioned that the MIHPA algorithm, the MISPA algorithm, the MHPA algorithm, and the MSPA algorithm will slow down the speed and accuracy of the HPA algorithm and the SPA algorithm. The acceleration may be eliminated by the projection onto the set C_n and Q_n and C_{n+1}.*

5. Conclusions

In this paper, we proposed two modified inertial hybrid and shrinking projection algorithms based on the inertial method, the Picard algorithm, the conjugate gradient method, and the projection method. We could then work with the strong convergence theorems under suitable conditions. However, numerical experiments showed that our algorithms cannot accelerate some previously known algorithms.

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