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Sharp Bounds on the Minimum M-Eigenvalue of Elasticity M-Tensors

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Abstract: The M-eigenvalue of elasticity M-tensors play important roles in nonlinear elastic material analysis. In this paper, we establish an upper bound and two sharp lower bounds for the minimum M-eigenvalue of elasticity M-tensors without irreducible conditions, which improve some existing results. Numerical examples are proposed to verify the efficiency of the obtained results.

Keywords: elasticity M-tensors; minimum M-eigenvalue; upper and lower bounds

MSC: 15A18; 15A42

1. Introduction

Tensor eigenvalue problems play an important role in numerical multilinear algebra [1–7], and they have a wide range in medical resonance [8], imaging spectral hypergraph theory [9], automatical control [10–13]. Particularly, the eigenvalue problem of the fourth-order elastic modulus tensor was dealt with by Love for the isotropic tensor [14] and for the anisotropic tensor [15–21]. A fourth-order real tensor $\mathcal{A} = (a_{ijkl})$ is called a partially symmetric tensor, denoted by $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$, if

$$a_{ijkl} = a_{jikl} = a_{ijlk}, \ i, j, k, l \in N = \{1, 2, \ldots, n\}. \quad (1)$$

A fourth-order partially symmetric tensor is useful in nonlinear elastic material analysis [3,16–27]. Ostrosablin [16] first constructed a complete system of eigentensors for the fourth rank tensor of elastic modulus, and Nikabadze [18] generalized these results and constructed a full system of eigentensors for a tensor of any even rank, as well as a complete system of eigentensor-columns for a tensor-block matrix of any even rank [22,23]. For example, a fourth-order partially symmetric tensor with $n = 2$ or 3, called the elasticity tensor, can be used in the two/three-dimensional field equations for a homogeneous compressible nonlinearly elastic material for static problems without body forces [27]. To identify the strong ellipticity in elastic mechanics, Han et al. [25] introduced M-eigenvalues of a fourth-order partially symmetric tensor. For $\lambda \in \mathbb{R}, x, y \in \mathbb{R}^n$, if

$$\begin{cases} \mathcal{A} \cdot xy^2 = \lambda x \\ \mathcal{A}x^2y\cdot = \lambda y \\ x^\top x = 1 \\ y^\top y = 1, \end{cases} \quad (2)$$

where $(\mathcal{A} \cdot xy^2)_i = \sum_{j,k,l \in [n]} a_{ijkl} x_j y_k y_l$, $(\mathcal{A}x^2y\cdot)_l = \sum_{i,j,k \in [n]} a_{ijkl} x_i x_j y_k$, then the scalar λ is called an M-eigenvalue of the tensor $\mathcal{A}$, and x and y are called left and right M-eigenvectors of $\mathcal{A}$ associated with the M-eigenvalue. Then the M-spectral radius of $\mathcal{A}$ is denoted by

$$\rho_M(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma_M(\mathcal{A})\}.$$

Recently, tensors with special structures, such as nonnegative tensors, M-tensors and H-tensors, are becoming the focus in recent research [2,24,28–31]. Some effective algorithms for finding eigenvalue and the corresponding eigenvector have been implemented [1,24,32–35]. For example, Bozorgmanesh et al. [32] propose an algorithm that can solve E-eigenvalue problem faster. However, it is very difficult for these algorithms to compute all M-eigenvalues or E-eigenvalues. Thus, some researchers turned to investigating eigenvalue inclusion sets [4,7,36–41]. Particularly, some bounds for the minimum H-eigenvalue of nonsingular M-tensors have been proposed [2,28,30,42,43]. Ding et al. [24] introduced a structured partially symmetric tensor named elasticity M-tensors and established important properties of elasticity M-tensors and nonsingular elasticity M-tensors.

Definition 1. *$\mathcal{A} \in \mathbb{E}_{4,n}$ is called an elasticity M-tensor if there exist a nonnegative tensor $\mathcal{B} \in \mathbb{E}_{4,n}$ and a real number $s \geq \rho_M(\mathcal{B})$ such that*

$$\mathcal{A} = s\mathcal{I}_M - \mathcal{B},$$

where $\rho_M(\mathcal{B})$ is the M-spectral radius and $\mathcal{I}_M = (e_{ijkl}) \in \mathbb{E}_{4,n}$ is called elasticity identity tensor with its entries

$$e_{ijkl} = \begin{cases} 1, & \text{if } i = j \text{ and } k = l \\ 0, & \text{otherwise,} \end{cases}$$

Furthermore, if $s > \rho(\mathcal{B})$, then we call $\mathcal{A}$ a nonsingular elasticity M-tensor.

Based on structural properties of elasticity M-tensors, He et al. [26] proposed some bounds for the minimum M-eigenvalue under irreducible conditions. However, some of information eigenvectors x on elasticity M-tensors is not fully mined, such as $\max_{i,j \in N, i \neq j} |x_i||x_j| \leq \frac{1}{2}$. Meanwhile, irreducibility is a relatively strict condition for elasticity M-tensor. Inspired by these observations, we want to present sharp bounds for the minimum M-eigenvalue of elasticity M-tensors by describing eigenvectors precisely without irreducible conditions, which improve existing results in [26].

This paper is organized as follows. In Section 2, some preliminary results are recalled. In Section 3, we establish an upper bound and two sharp lower bounds for the minimum M-eigenvalue of elasticity M-tensors. Numerical examples are proposed to verify the efficiency of the obtained results.

2. Preliminaries

In this section, we firstly introduce some definitions and important properties of elasticity M-tensors [24,26,27].

Definition 2. *Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ be a square tensor, then $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ is called reducible if there exists a nonempty proper index subset $J \subset \{1, 2, \ldots, n\}$such that $a_{i_1 i_2 \ldots i_m} = 0, \forall\, i_1 \in J, \forall\, i_2, \ldots, i_m \notin J$. If $\mathcal{A}$ is not reducible, then $\mathcal{A}$ is irreducible.*

Lemma 1 (Theorem 1 of [27])**.** *M-eigenvalues always exist. If x and y are left and right M-eigenvectors of $\mathcal{A}$, associated with an M-eigenvalue λ, then then $\lambda = \mathcal{A}x^2y^2$.*

Lemma 2 (Lemma 2.3 of [26]). *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an irreducible elasticity tensor and $\tau_M(\mathcal{A})$ be the minimal M-eigenvalues of $\mathcal{A}$, then*

$$\tau_M(\mathcal{A}) \leq \min_{i,l \in N}\{a_{iill}\}.$$

Lemma 3 (Lemma 2.3 of [26]). *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an irreducible and elasticity M-tensors and $\tau_M(\mathcal{A})$ be the minimal M-eigenvalue of $\mathcal{A}$. Then $\tau_M(\mathcal{A}) \geq 0$ is an M-eigenvalue of $\mathcal{A}$ with positive eigenvectors.*

Lemma 4 (Theorem 4.1 of [24]). *The M-spectral radius of any nonnegative tensor in $\mathbb{E}_{4,n}$ is exactly its greatest M-eigenvalue. Furthermore, there is a pair of nonnegative M-eigenvectors corresponding to the M-spectral radius.*

In the following, we characterize M-eigenvectors of elasticity M-tensors without irreducibility conditions.

Lemma 5. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be a elasticity M-tensor and $\tau_M(\mathcal{A})$ be the minimal M-eigenvalue. Then, there is a nonnegative M-eigenvector corresponding to $\tau_M(\mathcal{A}) \geq 0$.*

Proof. Since $\mathcal{A}$ is a elasticity M-tensor, there exist a nonnegative tensor $\mathcal{B} \in \mathbb{E}_{4,n}$ and a real number $c \geq \rho_M(\mathcal{B})$ such that

$$\mathcal{A} = c\mathcal{I}_M - \mathcal{B},$$

where $\rho_M(\mathcal{B})$ is the greatest M-eigenvalue of $\mathcal{B}$ with nonnegative eigenvectors by Theorem 4.1 in [24]. Setting $\tau_M(\mathcal{A}) = c - \rho_M(\mathcal{B})$, we have $\tau_M(\mathcal{A}) \geq 0$. It follows from Proposition 2.2 in [24] that $\tau_M(\mathcal{A})$ and $\rho_M(\mathcal{B})$ have the same eigenvectors. It follows from Lemma 4 that there exists a nonnegative M-eigenvector corresponding to $\tau_M(\mathcal{A}) \geq 0$. Thus, the conclusion follows. □

3. Bounds for the Minimum *M*-Eigenvalue of Elasticity *M*-Tensors

In this section, we establish sharp bounds for $\tau_M(\mathcal{A})$. We begin our work by collecting the information of $\max\limits_{i,j \in N, i \neq j} x_i^2 x_j^2$.

Lemma 6. *For any $x \in \mathbb{R}^n$, if*

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1,$$

then

$$\max_{i,j \in N, i \neq j} x_i^2 x_j^2 \leq \frac{1}{4}.$$

Further, $\max\limits_{i,j \in N, i \neq j} x_i x_j \leq \frac{1}{2}$ for all $x_i, x_j \geq 0$.

Proof. Define

$$f(x_1, \cdots, x_n) = x_i^2 x_j^2 - \lambda(x_1^2 + x_2^2 + \cdots + x_n^2 - 1),$$

where λ denotes Lagrange multiplier. For all $i \neq j$, deriving the above equation x_i and x_j respectively, we get

$$\begin{cases} 2x_j^2 x_i = 2\lambda x_i, \\ 2x_i^2 x_j = 2\lambda x_j. \end{cases}$$

Hence, we obtain $x_i^2 = x_j^2$, $\lambda = \frac{1}{2}$. Particularly, set

$$x_i = \pm x_j = \pm\frac{\sqrt{2}}{2}, x_n = 0, n \neq i, j$$

with $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. So,

$$\max_{i,j\in N, i\neq j} x_i^2 x_j^2 \leq \frac{1}{4}.$$

Further,

$$\max_{i,j\in N, i\neq j} x_i x_j \leq \frac{1}{2}, \ \forall x_i, x_j \geq 0.$$

□

Remark 1. *For the right M-eigenvector* $y \in \mathbb{R}^n$*, we can establish similar conclusions that* $\max\limits_{i,j\in N, i\neq j} y_i y_j \leq \frac{1}{2}$ *for all* $y_i, y_j \geq 0$.

Without irreducible conditions, we propose a sharp upper bound for the minimum M-eigenvalue of elasticity M-tensors.

Theorem 1. *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ *be a elasticity M-tensor. Then,*

$$\tau_M(\mathcal{A}) \leq \min\{\min_{i,l\in N} a_{iill}, \frac{\sum\limits_{i\in N} S_i(\mathcal{A})}{n^2}\},$$

where $S_i(\mathcal{A}) = \sum\limits_{j,k,l\in N} a_{ijkl}$.

Proof. Let $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue of $\mathcal{A}$. It follows Lemma 1 that

$$\tau_M(\mathcal{A}) = \min_{x,y}\{f_{\mathcal{A}}(x,y) = \mathcal{A}x^2y^2 : x^\top x = 1 \text{ and } y^\top y = 1\}. \tag{3}$$

Setting a feasible solution of (3)

$$(\bar{x}, \bar{y}) = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}),$$

we obtain

$$\tau_M(\mathcal{A}) \leq f_{\mathcal{A}}(\bar{x}, \bar{y}) = \sum_{i,j\in N} \sum_{k,l\in N} \frac{a_{ijkl}}{n^2} = \frac{\sum\limits_{i\in[n]} S_i(\mathcal{A})}{n^2}. \tag{4}$$

From Lemma 2 and (4), it holds that

$$\tau_M(\mathcal{A}) \leq \min\{\min_{i,l\in N} a_{iill}, \frac{\sum\limits_{i\in N} S_i(\mathcal{A})}{n^2}\}.$$

□

Next, we propose sharp lower bounds for the minimum M-eigenvalue of elasticity M-tensors.

Theorem 2. *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ *be a elasticity M-tensor. Then*

$$\tau_M(\mathcal{A}) \geq \max\{\min_{i\in N}\{\alpha_i - G_i(\mathcal{A})\}, \min_{l\in N}\{\beta_l - M_l(\mathcal{A})\}\}$$

where

$$G_i(\mathcal{A}) = \omega_i(\mathcal{A}) - \frac{1}{2} r_i(\mathcal{A}),\ \alpha_i = \min_{l\in N}\{a_{iill}\},\ \omega_i(\mathcal{A}) = \max_{l\in N}(\alpha_i - a_{iill} - \sum_{\substack{j=1,\\ j\neq i}}^{n} a_{ijll}),\ r_i(\mathcal{A}) = \sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{ijkl},$$

$$M_l(\mathcal{A}) = m_l(\mathcal{A}) - \frac{1}{2}c_l(\mathcal{A}),\ \beta_l = \min_{i\in N}\{a_{iill}\},\ m_l(\mathcal{A}) = \max_{i\in N}(\beta_l - a_{iill} - \sum_{\substack{k=1,\\ k\neq l}}^{n} a_{iikl}),\ c_l(\mathcal{A}) = \sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijkl}.$$

Proof. Let $\tau_M(\mathcal{A})$ be the minimum M-eigenvalue of $\mathcal{A}$. By Lemma 5, there exist nonnegative left and right M-eigenvectors (x, y) corresponding to $\tau_M(\mathcal{A})$. On one hand, setting $x_p = \max_{k\in N}\{x_k\}$, by $x^\top x = 1$, one has $0 < |x_p| \leq 1$. Recalling the p-th equation of $\tau_M(\mathcal{A})x = \mathcal{A}xy^2$, we obtain

$$((a_{pp11}y_1^2 + \cdots + a_{ppnn}y_n^2) - \tau(\mathcal{A}))x_p = -\sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{pjkl}x_jy_ky_l - \sum_{\substack{j,l=1,\\ j\neq p}}^{n} a_{pjll}x_jy_l^2, \tag{5}$$

Setting $\alpha_p = \min_{l\in N}\{a_{ppll}\}$, by (5) and Lemma 6, one has

$$\begin{aligned}
(\alpha_p - \tau_M(\mathcal{A}))x_p &= \sum_{\substack{l=1,\\ j=p}}^{n} (\alpha_p - a_{ppll})y_l^2x_p - \sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{pjkl}x_jy_ky_l - \sum_{\substack{j,l=1,\\ j\neq p}}^{n} a_{pjll}x_jy_l^2 \\
&\leq \sum_{\substack{l=1,\\ j=p}}^{n} (\alpha_p - a_{ppll})y_l^2x_p - \sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{pjkl}x_py_ky_l - \sum_{\substack{j,l=1,\\ j\neq p}}^{n} a_{pjll}x_py_l^2 \\
&= \sum_{l=1}^{n}(\alpha_p - a_{ppll} - \sum_{\substack{j=1,\\ j\neq p}}^{n} a_{pjll})y_l^2x_p - \sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{pjkl}x_py_ky_l \\
&\leq \max_{l\in N}(\alpha_p - a_{ppll} - \sum_{\substack{j=1,\\ j\neq p}}^{n} a_{pjll})x_p - \frac{1}{2}\sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{pjkl}x_p.
\end{aligned} \tag{6}$$

It follows from (6) and definition of ω_p that

$$(\alpha_p - \tau_M(\mathcal{A}))x_p \leq (\omega_p(\mathcal{A}) - \frac{1}{2}r_p(\mathcal{A}))x_p,$$

which implies

$$\tau_M(\mathcal{A}) \geq \alpha_p - \omega_p(\mathcal{A}) + \frac{1}{2}r_p(\mathcal{A}). \tag{7}$$

On the other hand, setting $y_t = \max_{k\in N}\{y_k\}$, from the t-th equation of $\tau_M(\mathcal{A})y = \mathcal{A}x^2y$, we obtain

$$((a_{11tt}y_1^2 + \cdots + a_{nntt}y_n^2) - \tau(\mathcal{A}))y_t = -\sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijkt}x_ix_jy_k - \sum_{\substack{i,k=1,\\ k\neq t}}^{n} a_{iikt}x_i^2y_k. \tag{8}$$

Letting $\beta_t = \min_{i\in N}\{a_{iitt}\}$, by (8) and Lemma 6, we have

$$\begin{aligned}(\beta_t - \tau_M(\mathcal{A}))y_t = & \sum_{\substack{i=1,\\ k=t}}^{n} (\beta_t - a_{iitt})x_i^2 y_t - \sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijkt}x_i x_j y_k - \sum_{\substack{i,k=1,\\ k\neq t}}^{n} a_{iikt}x_i^2 y_k \\ \leq & \sum_{\substack{i=1,\\ k=t}}^{n} (\beta_t - a_{iitt})x_i^2 y_t - \sum_{\substack{i,k=1,\\ k\neq t}}^{n} a_{iikt}y_t x_i^2 - \sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijkt}x_i x_j y_t \\ = & \sum_{l=1}^{n}(\beta_t - a_{iitt} - \sum_{\substack{k=1,\\ k\neq t}}^{n} a_{iikt})x_i^2 y_t - \sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijkt}x_i x_j y_t \\ \leq & \max_{i\in N}(\beta_t - a_{iitt} - \sum_{\substack{k=1,\\ k\neq t}}^{n} a_{iikt})y_t - \frac{1}{2}\sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijkt}y_t. \end{aligned} \tag{9}$$

It follows from (9) and definition of m_t that

$$(\beta_t - \tau_M(\mathcal{A}))y_t \leq (m_t(\mathcal{A}) - \frac{1}{2}c_t(\mathcal{A}))y_t,$$

which shows

$$\tau_M(\mathcal{A}) \geq \beta_t - m_t(\mathcal{A}) + \frac{1}{2}c_t(\mathcal{A}). \tag{10}$$

From (7) and (10), the result follows. □

Now, we are at a position to prove that the bound in Theorem 2 is tighter than that of Theorem 3.1 of [26].

Corollary 1. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be a elasticity M-tensor. Then*

$$\max\{\min_{i\in N}\{\alpha_i - G_i(\mathcal{A})\}, \min_{l\in N}\{\beta_l - M_l(\mathcal{A})\}\} \geq \max\{\min_{i\in N}\{\alpha_i - R_i(\mathcal{A})\}, \min_{l\in N}\{\beta_l - C_l(\mathcal{A})\}\}.$$

Proof. On one hand, it follows from Theorem 3.1 of [26] that

$$\alpha_i - R_i(\mathcal{A}) = \alpha_i - \gamma_i(\mathcal{A}) - r_i(\mathcal{A}) = \alpha_i - \max_{l\in N}\{\sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ijll}|\} - \sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} |a_{ijkl}|.$$

Since $\alpha_i - a_{iill} \leq 0$ and $\sum_{\substack{j=1,\\ j\neq i}}^{n} a_{ijll} \leq 0$, we can verify

$$\max_{l\in N}\{\sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ijll}|\} \geq \max_{l\in N}(\alpha_i - a_{iill} - \sum_{\substack{j=1,\\ j\neq i}}^{n} a_{ijll}),\ \frac{1}{2}\sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{ijkl} \geq -\sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} |a_{ijkl}|,$$

which shows

$$\alpha_i - G_i(\mathcal{A}) \geq \alpha_i - R_i(\mathcal{A}), \forall i \in N. \tag{11}$$

On the other hand, it follows from Theorem 3.1 of [26] that

$$\beta_l - C_l(\mathcal{A}) = \beta_l - \delta_l(\mathcal{A}) - \frac{1}{2}c_l(\mathcal{A}) = \beta_l - \max_{i\in N}(\beta_l - a_{iill} - \sum_{\substack{k=1,\\ k\neq l}}^{n} a_{ijll}) + \frac{1}{2}\sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijkl}.$$

Following the similar arguments in the proof of (11), we obtain

$$\beta_l - \mathcal{M}_l(\mathcal{A}) \geq \beta_l - \mathcal{C}_l(\mathcal{A}), \forall l \in N. \tag{12}$$

It follows from (11) and (12) that

$$\max\{\min_{i\in N}\{\alpha_i - G_i(\mathcal{A})\}, \min_{l\in N}\{\beta_l - M_l(\mathcal{A})\}\} \geq \max\{\min_{i\in N}\{\alpha_i - R_i(\mathcal{A})\}, \min_{l\in N}\{\beta_l - C_l(\mathcal{A})\}\}.$$

□

Choosing x_q as a component of x with the second largest modulus, we obtain another sharp lower bound for $\tau_M(\mathcal{A})$.

Theorem 3. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be a elasticity M-tensor. Then*

$$\tau_M(\mathcal{A}) \geq \max\{\phi_1(\mathcal{A}), \phi_2(\mathcal{A})\},$$

where

$$\begin{aligned}
\phi_1(\mathcal{A}) &= \min_{i,v\in N, v\neq i}\{\tfrac{1}{2}\{\alpha_i + \tfrac{1}{2}r_i(\mathcal{A}) + \alpha_v - \omega_v^i(\mathcal{A}) - \Delta_{i,v}^{\frac{1}{2}}(\mathcal{A})\}, \alpha_i + \tfrac{1}{2}r_i(\mathcal{A}), \alpha_v - \omega_v^i(\mathcal{A})\},\\
\phi_2(\mathcal{A}) &= \min_{u,l\in N, u\neq l}\{\tfrac{1}{2}\{\beta_l + \tfrac{1}{2}c_l(\mathcal{A}) + \beta_u - m_u^l(\mathcal{A}) - \theta_{l,u}^{\frac{1}{2}}(\mathcal{A})\}, \beta_l + \tfrac{1}{2}c_l(\mathcal{A}), \beta_u - m_u^l(\mathcal{A})\},\\
\Delta_{i,v}(\mathcal{A}) &= (\alpha_i + \tfrac{1}{2}r_i(\mathcal{A}) - \alpha_v + \omega_v^i(\mathcal{A}))^2 + 4\omega_i(\mathcal{A})(\gamma_v^i(\mathcal{A}) - \tfrac{1}{2}r_v(\mathcal{A})),\\
\theta_{l,u}(\mathcal{A}) &= (\beta_l + \tfrac{1}{2}c_l(\mathcal{A}) - \beta_u + m_u^l(\mathcal{A}))^2 + 4m_l(\mathcal{A})(\delta_u^l(\mathcal{A}) - \tfrac{1}{2}c_u(\mathcal{A})),\\
\gamma_v^i(\mathcal{A}) &= \max_{l\in N}(-a_{vill}), \omega_v^i(\mathcal{A}) = \max_{l\in N}(\alpha_v - a_{vvll} - \sum_{j=1;j\neq v,i}^{n} a_{vjll}),\\
\delta_u^l(\mathcal{A}) &= \max_{i\in N}(-a_{iilu}), m_u^l(\mathcal{A}) = \max_{i\in N}(\beta_u - a_{iiuu} - \sum_{k=1;k\neq u,l}^{n} a_{iikl}).
\end{aligned}$$

Proof. Let $\tau_M(\mathcal{A})$ be the minimal M-eigenvalue of $\mathcal{A}$. By Lemma 5, there exist nonnegative left and right M-eigenvectors (x, y) corresponding to $\tau_M(\mathcal{A})$. On one hand, set $x_p \geq x_q \geq \max_{k\in N, k\neq p,q}\{x_k\}$. By the p-th equation of $\tau_M(\mathcal{A})x = \mathcal{A}xy^2$, one has

$$((a_{pp11}y_1^2 + \cdots + a_{ppnn}y_n^2) - \tau_M(\mathcal{A}))x_p = -\sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{pjkl}x_jy_ky_l - \sum_{\substack{j,l=1,\\ j\neq p}}^{n} a_{pjll}x_jy_l^2. \tag{13}$$

Setting $\alpha_p = \min_{l\in N}\{a_{ppll}\}$, from (13) and Lemma 6, we obtain

$$\begin{aligned}
(\alpha_p - \tau_M(\mathcal{A}))x_p &= \sum_{\substack{l=1,\\ j=p}}^{n} (\alpha_p - a_{ppll})y_l^2 x_p - \sum_{\substack{j,l=1,\\ j\neq p}}^{n} a_{pjll}x_jy_l^2 - \sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{pjkl}x_jy_ky_l\\
&\leq \sum_{\substack{l=1,\\ j=p}}^{n} (\alpha_p - a_{ppll})y_l^2 x_q - \sum_{\substack{j,l=1,\\ j\neq p}}^{n} a_{pjll}x_qy_l^2 - \sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{pjkl}x_py_ky_l\\
&= \sum_{l=1}^{n}(\alpha_p - a_{ppll} - \sum_{\substack{j=1,\\ j\neq p}}^{n} a_{pjll})y_l^2 x_q - \sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{pjkl}x_py_ky_l\\
&\leq \max_{l\in N}(\alpha_p - a_{ppll} - \sum_{\substack{j=1,\\ j\neq p}}^{n} a_{pjll})x_q - \frac{1}{2}\sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{pjkl}x_p,
\end{aligned}$$

which implies

$$(\alpha_p - \tau_M(\mathcal{A}) + \frac{1}{2}r_p(\mathcal{A}))x_p \leq \omega_p(\mathcal{A})x_q. \tag{14}$$

From the q-th equation of $\tau_M(\mathcal{A})x = \mathcal{A}xy^2$ and $\alpha_q = \min\limits_{l\in N}\{a_{qqll}\}$, we yield

$$\begin{aligned}
(\alpha_q - \tau_M(\mathcal{A}))x_q &= \sum_{\substack{l=1,\\ j=q}}^{n} (\alpha_q - a_{qqll})y_l^2 x_q - \sum_{\substack{j,l=1,\\ j\neq q}}^{n} a_{qjll}x_j y_l^2 - \sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{qjkl}x_j y_k y_l \\
&\leq \sum_{\substack{l=1,\\ j=q}}^{n} (\alpha_q - a_{qqll})y_l^2 x_q - \sum_{\substack{j,l=1,\\ j\neq q,p}}^{n} a_{qjll}x_q y_l^2 - \sum_{\substack{l=1,\\ j=p}}^{n} a_{qpll}x_p y_l^2 - \sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{qjkl}x_p y_k y_l \\
&= \sum_{l=1}^{n}(\alpha_q - a_{qqll} - \sum_{\substack{j=1,\\ j\neq q,p}}^{n} a_{qjll})y_l^2 x_q - \sum_{\substack{l=1,\\ j=p}}^{n} a_{qpll}x_p y_l^2 - \sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{qjkl}x_p y_k y_l \\
&\leq \max_{l\in N}(\alpha_q - a_{qqll} - \sum_{\substack{j=1,\\ j\neq q,p}}^{n} a_{qjll})x_q + \max_{l\in N}(-a_{qpll})x_p - \frac{1}{2}\sum_{\substack{j,k,l=1,\\ k\neq l}}^{n} a_{qjkl}x_p,
\end{aligned} \tag{15}$$

which implies

$$(\alpha_q - \tau_M(\mathcal{A}) - \omega_q^p(\mathcal{A}))x_q \leq (\gamma_q^p(\mathcal{A}) - \frac{1}{2}r_q(\mathcal{A}))x_p. \tag{16}$$

When $\alpha_p - \tau_M(\mathcal{A}) + \frac{1}{2}r_p(\mathcal{A}) \geq 0$ or $\alpha_q - \tau_M(\mathcal{A}) - \omega_q^p(\mathcal{A}) \geq 0$, multiplying inequalities (14) with (16), one has

$$(\alpha_p - \tau_M(\mathcal{A}) + \frac{1}{2}r_p(\mathcal{A}))(\alpha_q - \tau_M(\mathcal{A}) - \omega_q^p(\mathcal{A})) \leq \omega_p(\mathcal{A})(\gamma_q^p(\mathcal{A}) - \frac{1}{2}r_q(\mathcal{A})).$$

Then, solving for $\tau_M(\mathcal{A})$, we have

$$\tau_M(\mathcal{A}) \geq \frac{1}{2}(\alpha_p + \frac{1}{2}r_p(\mathcal{A}) + \alpha_q - \omega_q^p(\mathcal{A}) - \Delta_{p,q}^{\frac{1}{2}}(\mathcal{A})), \tag{17}$$

where $\Delta_{p,q}(\mathcal{A}) = (\alpha_p + \frac{1}{2}r_p(\mathcal{A}) - \alpha_q + \omega_q^p(\mathcal{A}))^2 + 4\omega_p(\mathcal{A})(\gamma_q^p(\mathcal{A}) - \frac{1}{2}r_q(\mathcal{A}))$.

When $\alpha_p - \tau_M(\mathcal{A}) + \frac{1}{2}r_p(\mathcal{A}) < 0$ and $\alpha_q - \tau_M(\mathcal{A}) - \omega_q^p(\mathcal{A}) < 0$, one has

$$\tau_M(\mathcal{A}) > \alpha_p + \frac{1}{2}r_p(\mathcal{A}) \text{ and } \tau_M(\mathcal{A}) > \alpha_q - \omega_q^p(\mathcal{A}). \tag{18}$$

It follows from (17) and (18) that

$$\tau_M(\mathcal{A}) \geq \min_{p,q\in N, q\neq p}\{\frac{1}{2}\{\alpha_p + \frac{1}{2}r_p(\mathcal{A}) + \alpha_q - \omega_q^p(\mathcal{A}) - \Delta_{p,q}^{\frac{1}{2}}(\mathcal{A})\}, \alpha_p + \frac{1}{2}r_p(\mathcal{A}), \alpha_q - \omega_q^p(\mathcal{A})\}.$$

On the other hand, set $y_t \geq y_s \geq \max\limits_{k\in N, k\neq t}\{y_k\}$. From the t-th equation of $\tau_M(\mathcal{A})y = \mathcal{A}x^2y$, it follows that

$$((a_{11tt}x_1^2 + \cdots + a_{nntt}x_n^2) - \tau_M(\mathcal{A}))y_t = -\sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijkt}x_i x_j y_k - \sum_{\substack{i,k=1,\\ k\neq t}}^{n} a_{iikt}x_i^2 y_k. \tag{19}$$

Letting $\beta_t = \min_{i\in N}\{a_{iitt}\}$, by (19) and Lemma 6, we obtain

$$\begin{aligned}(\beta_t - \tau_M(\mathcal{A}))y_t &= \sum_{\substack{i=1,\\ k=t}}^{n} (\beta_t - a_{iitt})x_i^2 y_t - \sum_{\substack{i,k=1,\\ k\neq t}}^{n} a_{iikt}x_i^2 y_k - \sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijkt}x_i x_j y_k \\
&\leq \sum_{\substack{i=1,\\ k=t}}^{n} (\beta_t - a_{iitt})x_i^2 y_s - \sum_{\substack{i,k=1,\\ k\neq t}}^{n} a_{iikt}x_i^2 y_s - \sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijkt}x_i x_j y_t \\
&= \sum_{i=1}^{n}(\beta_t - a_{iitt} - \sum_{\substack{k=1,\\ k\neq t}}^{n} a_{iikt})x_i^2 y_s - \sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijkt}x_i x_j y_t \\
&\leq \max_{i\in N}(\beta_t - a_{iitt} - \sum_{\substack{k=1,\\ k\neq t}}^{n} a_{iikt})y_s - \frac{1}{2}\sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijkt}y_t. \end{aligned} \tag{20}$$

Using (20), we yield

$$(\beta_t - \tau_M(\mathcal{A}) + \frac{1}{2}c_t(\mathcal{A}))y_t \leq m_t(\mathcal{A})y_s. \tag{21}$$

Recalling s-th equation of $\tau(\mathcal{A})y = \mathcal{A}x^2y$ and $\beta_s = \min_{i\in N}\{a_{iiss}\}$, we have

$$\begin{aligned}(\beta_s - \tau_M(\mathcal{A}))y_s &= \sum_{\substack{i=1,\\ k=s}}^{n} (\beta_s - a_{iiss})x_i^2 y_s - \sum_{\substack{i,k=1,\\ k\neq s}}^{n} a_{iiks}x_i^2 y_k - \sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijks}x_i x_j y_k \\
&\leq \sum_{\substack{i=1,\\ k=s}}^{n} (\beta_s - a_{iiss})x_i^2 y_t - \sum_{\substack{i,k=1,\\ k\neq s,t}}^{n} a_{iiks}x_i^2 y_s - \sum_{\substack{i=1,\\ k=t}}^{2} a_{iits}x_i^2 y_t - \sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijks}x_i x_j y_t \\
&= \sum_{i=1}^{n}(\beta_s - a_{iiss} - \sum_{\substack{k=1,\\ k\neq s,t}}^{n} a_{iiks})x_i^2 y_s - \sum_{\substack{i=1,\\ k=t}}^{n} a_{iits}x_i^2 y_t - \sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijks}x_i x_j y_t \\
&\leq \max_{i\in N}(\beta_s - a_{iiss} - \sum_{\substack{k=1,\\ k\neq s,t}}^{n} a_{iiks})y_t + \max_{i\in N}(-a_{iits})y_t - \frac{1}{2}\sum_{\substack{i,j,k=1,\\ i\neq j}}^{n} a_{ijks}y_t,\end{aligned}$$

which implies

$$(\beta_s - \tau_M(\mathcal{A}) - m_s^t(\mathcal{A}))y_s \leq (\delta_s^t(\mathcal{A}) - \frac{1}{2}c_s(\mathcal{A}))y_t. \tag{22}$$

When $\beta_t - \tau_M(\mathcal{A}) + \frac{1}{2}c_t(\mathcal{A}) \geq 0$ or $\beta_s - \tau_M(\mathcal{A}) - m_s^t(\mathcal{A}) \geq 0$, multiplying inequalities (21) with (22), one has

$$(\beta_t - \tau_M(\mathcal{A}) + \frac{1}{2}c_t(\mathcal{A}))(\beta_s - \tau_M(\mathcal{A}) - m_s^t(\mathcal{A})) \leq m_t(\mathcal{A})(\delta_s^t(\mathcal{A}) - \frac{1}{2}c_s(\mathcal{A})).$$

Then, solving for $\tau(\mathcal{A})$, we obtain

$$\tau_M(\mathcal{A}) \geq \frac{1}{2}\{\beta_t + \frac{1}{2}c_t(\mathcal{A}) + \beta_s - m_s^t(\mathcal{A}) - \theta_{t,s}^{\frac{1}{2}}(\mathcal{A})\}, \tag{23}$$

where $\theta_{t,s}(\mathcal{A}) = (\beta_t + \frac{1}{2}c_t(\mathcal{A}) - \beta_s + m_s^t(\mathcal{A}))^2 + 4m_t(\mathcal{A})(\delta_s^t(\mathcal{A}) - \frac{1}{2}c_s(\mathcal{A}))$.

When $\beta_t - \tau_M(\mathcal{A}) + \frac{1}{2}c_t(\mathcal{A}) < 0$ and $\beta_s - \tau_M(\mathcal{A}) - m_s^t(\mathcal{A}) < 0$, then we have

$$\tau_M(\mathcal{A}) > \beta_t + \frac{1}{2}c_t(\mathcal{A}) \; and \; \tau_M(\mathcal{A}) > \beta_s - m_s^t(\mathcal{A}). \tag{24}$$

From (23) and (24), it holds that

$$\tau_M(\mathcal{A}) \geq \min_{t,s\in N, s\neq t}\{\frac{1}{2}\{\beta_t + \frac{1}{2}c_t(\mathcal{A}) + \beta_s - m_s^t(\mathcal{A}) - \theta_{t,s}^{\frac{1}{2}}(\mathcal{A})\}, \beta_t + \frac{1}{2}c_t(\mathcal{A}), \beta_s - m_s^t(\mathcal{A})\}.$$

Thus, the desired result holds. □

In the following, we use Example 3.1 of [26] to show the superiority of our results.

Example 1. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,2}$ be an elasticity M-tensor, whose entries are*

$$a_{ijkl} = \begin{cases} a_{1111} = a_{1122} = 4.1, a_{2211} = a_{2222} = 5, \\ a_{1112} = a_{1121} = -1, a_{2212} = a_{2221} = -1, \\ a_{2111} = a_{1211} = -1, a_{1222} = a_{2122} = -1, \\ a_{ijkl} = 0, otherwise. \end{cases}$$

The bounds via different estimations given in the literature are shown in Table 1:

Table 1. Bound estimations of the minimum M-eigenvalue with different methods

References	Interval
Lemma 2 and Theorem 3.1 of [26]	$1.10 \leq \tau_M(\mathcal{A}) \leq 4.10$
Lemma 2 and Theorem 3.2 of [26]	$1.29 \leq \tau_M(\mathcal{A}) \leq 4.10$
Theorem 1 and Theorem 2	$2.10 \leq \tau_M(\mathcal{A}) \leq 2.55$
Theorem 1 and Theorem 3	$2.35 \leq \tau_M(\mathcal{A}) \leq 2.55$

By computations, we obtain that the minimum M-eigenvalue and corresponding with left and right M-eigenvectors are$(\tau_M(\mathcal{A}), \bar{x}, \bar{y}) = (2.4534, (0.8398, 0.5430), (0.7071, 0.7071))$. It is easy to see that the results given in Theorems 3.1–3.3 are sharper than some existing results [26]. It is noted that Theorems 2 and 3 have their own advantages. Theorem 3 can estimate the lower bound of the minimum M-eigenvalue more accurately, but the calculation of Theorem 2 is simpler.

Ding et al. [24] pointed out that a tensor is M-positive if and only if its smallest M-eigenvalue is positive. In the following, the results given in Theorems 2 and 3 can exactly check the positiveness of the elasticity M-tensor $\mathcal{A}$.

Example 2. *Consider the elasticity M-tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,2}$ defined by*

$$a_{ijkl} = \begin{cases} a_{1111} = 3.1, a_{1122} = 4.1, a_{2211} = 5, a_{2222} = 6, \\ a_{1112} = a_{1121} = -1, a_{2212} = a_{2221} = -1, \\ a_{2111} = a_{1211} = -1, a_{1222} = a_{2122} = -2, \\ a_{1221} = a_{1212} = a_{2121} = a_{2112} = -0.5. \end{cases}$$

The bounds via different estimations given in the literature are shown in Table 2.

Table 2. Bound estimations of the minimum M-eigenvalue and testing the M-positive definiteness

References	Interval
Lemma 2 and Theorem 3.1 of [26]	$-1.90 \leq \tau_M(\mathcal{A}) \leq 3.10$
Lemma 2 and Theorem 3.2 of [26]	$-1.45 \leq \tau_M(\mathcal{A}) \leq 3.10$
Theorems 1 and 2	$0.60 \leq \tau_M(\mathcal{A}) \leq 1.55$
Theorems 1 and 3	$0.77 \leq \tau_M(\mathcal{A}) \leq 1.55$

By computations, we obtain that the minimum M-eigenvalue and corresponding with left and right M-eigenvectors are $(\tau_M(\mathcal{A}), \bar{x}, \bar{y}) = (1.3350, (0.8462, 0.5329), (0.7190, 0.6951))$.

From Theorems 2 and 3, we obtain $0.77 \leq \tau_M(\mathcal{A}) \leq 1.55$, which shows that $\mathcal{A}$ is M-positive definite. However, the existing results of [26] cannot identify the M-positiveness of $\mathcal{A}$.

For the medium-sized tensors, we show the validity of the estimations by our theorems.

Example 3. *All testing elasticity M-tensors $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ are generated as follows: $a_{iijj} = n^2 + 4i + 4j$ and other elements are generated randomly in $[-0.5, 0]$ by MATLAB R2014a, where n denotes variable dimension. For different dimensions elasticity M-tensors, the values presented in the table are the average values of* 10 *examples. The bounds via different estimations given in the literature are shown in Table 3.*

Table 3. Comparison estimations of the minimum M-eigenvalue with random elasticity M-tensors

References	$n = 10$ Bounds	$n = 20$ Bounds	$n = 30$ Bounds
Lemma 2.2 and Theorem 3.1 of [9]	[−129.8, 105]	[−1561.9, 408]	[−5660.2, 908]
Lemma 2.2 and Theorem 3.2 of [9]	[−102.4, 105]	[−1324.4, 408]	[−4975.5, 908]
Theorems 1 and 2	[−8.9, 102.6]	[−530.4, 383.9]	[−2173.7, 799.7]
Theorems 1 and 3	[3.8, 102.6]	[−419.3, 383.9]	[−1956.8, 799.7]

4. Conclusions

In this paper, we exactly characterized the information of eigenvectors without irreducible conditions. Further, we proposed a new upper bound and two sharp lower bounds for the minimum M-eigenvalue of elasticity M-tensors by establishing new eigenvalue inequality. Numerical examples were proposed to verify the efficiency of the obtained results.

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