mathematics

MDPI

Article

Digital k-Contractibility of an n-Times Iterated Connected Sum of Simple Closed k-Surfaces and Almost Fixed Point Property

Sang-Eon Han

Department of Mathematics Education, Institute of Pure and Applied Mathematics, Jeonbuk National University, Jeonju-City, Jeonbuk 54896, Korea; sehan@jbnu.ac.kr; Tel.: +82-63-270-4449

Received: 22 January 2020; Accepted: 26 February 2020; Published: 4 March 2020

Abstract: The paper firstly establishes the so-called *n-times iterated connected sum* of a simple closed k-surface in $\mathbb{Z}^3$, denoted by $\mathcal{C}_k^n$, $k \in \{6, 18, 26\}$. Secondly, for a simple closed 18-surface MSS_{18}, we prove that there are only two types of connected sums of it up to 18-isomorphism. Besides, given a simple closed 6-surface MSS_6, we prove that only one type of $MSS_6 \sharp MSS_6$ exists up to 6-isomorphism, where $\sharp$ means the digital connected sum operator. Thirdly, we prove the digital k-contractibility of $\mathcal{C}_k^n := \overbrace{MSS_k \sharp \cdots \sharp MSS_k}^{\text{n-times}}, k \in \{18, 26\}$, which leads to the simply k-connectedness of $\mathcal{C}_k^n$, $k \in \{18, 26\}$, $n \in \mathbb{N}$. Fourthly, we prove that $\mathcal{C}_6^2$ and $\mathcal{C}_k^n$ do not have the almost fixed point property (*AFPP*, for short), $k \in \{18, 26\}$. Finally, assume a closed k-surface $S_k (\subset \mathbb{Z}^3)$ which is $(k, \bar{k})$-isomorphic to (X, k) in the picture $(\mathbb{Z}^3, k, \bar{k}, X)$ and the set X is symmetric according to each of xy-, yz-, and xz-planes of $\mathbb{R}^3$. Then we prove that S_k does not have the *AFPP*. In this paper given a digital image (X, k) is assumed to be k-connected and its cardinality $\mid X \mid \geq 2$.

Keywords: digital image; digital topology; $(k, \bar{k})$-isomorphism; *FPP*; *AFPP*; digital k-contractibility; digital surface; digital connected sum; simple closed k-surface; (almost) fixed point property; iterated connected sum

AMS Classification: 57M05; 55P10; 57M10

1. Introduction

In $\mathbb{Z}^3$, the concept of closed k-surface was introduced in [1–3] and its digital topological characterizations were also studied in many papers including [4–10]. Many explorations of various properties of closed k-surfaces have been proceeded from the viewpoints of digital topology, digital geometry, and fixed point theory [1,2,4–6,9–16]. Despite the studies of the earlier works [5–7,17,18], given (digital) closed k-surfaces, we need to further study both the digital k-contractibility of n-times iterated connected sums of closed k-surfaces and the non-almost fixed point property of them. Besides, we need to find a condition determining if a digital image (X, k) in $\mathbb{Z}^n$ has the *AFPP*. This approach facilitates the studies of digital geometry and fixed point theory.

So far, there were several kinds of approaches to establish a digital k-surface [3,5–7,9]. In the present paper we will often use the symbol " :=" to define a new term, and given a digital image (X, k) is assumed to be k-connected and its cardinality $\mid X \mid \geq 2$. Since the digital surface theory is related to computer science, the present paper mainly deals with digital k-surfaces X in $\mathbb{Z}^3$. Hence, we need to consider a binary digital image structure $(X, k, \bar{k})$ in $\mathbb{Z}^3$, denoted by $P := (\mathbb{Z}^3, k, \bar{k}, X)$, where the

$\bar{k}$-adjacency is concerned with the set $\mathbb{Z}^3 \setminus X$. To be precise, in the case of the study of a closed k-surface $X \subset \mathbb{Z}^3$, we should assume X in the binary digital picture P. For instance,

$$P \in \{(\mathbb{Z}^3, k, \bar{k}, X) \mid (k, \bar{k}) \in \{(6,26), (18,6), (26,6)\}\}. \tag{1}$$

Let us now study a (digital) closed k-surface X with one of the above frames P of (1).

Given two closed k-surfaces S_k and S'_k in $\mathbb{Z}^n$, the concept of digital connected sum of them was firstly introduced in [5,7] by using several types of simple closed k-curves in $\mathbb{Z}^2, k \in \{4,8\}$ (see Section 4). Hereafter, we denote by S_k a (simple) closed k-surface in $\mathbb{Z}^3$ (for the details, see Definition 5). Indeed, when studying various properties of closed k-surfaces, some digital k-homotopic features of S_k such as the k-contractibility are very important in digital surface theory.

For convenience, let MSS_6 (resp. MSS_{18}) be the minimal simple closed 6-surface (resp. the minimal simple closed 18-surface) [6]. The present paper deals with the following queries.

(Q1) We may ask if it is possible to propose the simple closed 6-surface MSS_6 in the picture $(\mathbb{Z}^3, 6, 18, MSS_6)$ instead of $(\mathbb{Z}^3, 6, 26, MSS_6)$.
Hereafter, the operator "$\sharp$" means the digital connected sum (see Section 4 for the details).
(Q2) How many types of $MSS_6 \sharp MSS_6$ exist ?
Let $\mathcal{C}_6^n := \overbrace{MSS_6 \sharp \cdots \sharp MSS_6}^{n\text{-times}}$. Then we have the following queries:
(Q3) How can we formulate $\mathcal{C}_6^n, n \in \mathbb{N} \setminus \{1\}$?
Given an MSS_{18}, we may raise the following query.
(Q4) How many types of $MSS_{18} \sharp MSS_{18}$ exist ?
Let $\mathcal{C}_{18}^n := \overbrace{MSS_{18} \sharp \cdots \sharp MSS_{18}}^{n\text{-times}}$. Then we have the following questions:
(Q5) How can we formulate $\mathcal{C}_{18}^n, n \in \mathbb{N} \setminus \{1\}$?
(Q6) How about the almost fixed point property (*AFPP* for short) of $\mathcal{C}_6^n, n \in \mathbb{N}$?
(Q7) How about the *AFPP* of $\mathcal{C}_{18}^n, n \in \mathbb{N}$?
(Q8) What are some properties relating to the *AFPP* of a closed k-surface in $\mathbb{Z}^3$.

The rest of the paper is organized as follows: Section 2 refers to some notions involving a digital k-surface and a connected sum of two digital k-surfaces. Section 3 stresses some utilities of the minimal simple closed surfaces MSS_6, MSS_{18}, MSS'_{18}, and MSS'_{26} from the viewpoints of digital curve and digital surface theory. Section 4 shows several types of n-times iterated connected sums of the minimal simple closed 6-surfaces, e.g., $\mathcal{C}_6^3 := MSS_6 \sharp MSS_6 \sharp MSS_6$. Section 5 proves that there are only two types of connected sums $MSS_{18} \sharp MSS_{18}$ up to 18-isomorphism. Besides, in the case of $MSS_{18} \sharp MSS_{18} \neq MSS_{18}$, we prove that only one type of $\mathcal{C}_{18}^3 := MSS_{18} \sharp MSS_{18} \sharp MSS_{18}$ exists up to 18-isomorphism. Section 6 intensively explores the 18-contractibility of an *n-times iterated connected sum* of simple closed 18-surfaces $\mathcal{C}_{18}^n := \overbrace{MSS_{18} \sharp \cdots \sharp MSS_{18}}^{n\text{-times}}$. Section 7 proves that both $\mathcal{C}_6^2$ and $\mathcal{C}_k^n$ do not have the almost fixed point property, $k \in \{18, 26\}, n \in \mathbb{N}$. Thus, these approaches play important roles in digital topology, digital geometry, fixed point theory, and so on. Section 8 concludes the paper with some remarks.

2. Basic Notions Involving Digital k-Surfaces and Connected Sums of Closed k-Surfaces

Let us now recall some terminology from digital curve and digital surface theories. Let $\mathbb{N}$ and $\mathbb{Z}$ represent the sets of natural numbers and integers, respectively.

We call a set $X (\subset \mathbb{Z}^n)$ with a k-adjacency a digital image, denoted by (X, k) [4,5,7,9,10]. In particular, in digital surface theory, we are absolutely required to consider a closed k-surface (X, k) with a k-adjacency in a binary digital picture $(\mathbb{Z}^n, k, \bar{k}, X)$ [19,20], where $n \in \mathbb{N}$ and the $\bar{k}$-adjacency is concerned with the set $\mathbb{Z}^n \setminus X$. In order to study (X, k) in $\mathbb{Z}^n, n \geq 1$, we need the k-adjacency relations of $\mathbb{Z}^n$ which are generalizations of the commonly used k-adjacency of $\mathbb{Z}^2$, $k \in \{4, 8\}$, and k-adjacency of $\mathbb{Z}^3$, $k \in \{6, 18, 26\}$. As a generalization of this approach into those of $\mathbb{Z}^n$, a paper [17] firstly established the digital k-connectivity of $\mathbb{Z}^n$, as follows: We say that distinct points $p, q \in \mathbb{Z}^n$ are k-(or $k(t, n)$-)adjacent if they satisfy the following property [17] (for the details, see also [21,22]).

For a natural number t, $1 \leq t \leq n$, we say that distinct points

$$p = (p_1, p_2, \cdots, p_n) \text{ and } q = (q_1, q_2, \cdots, q_n) \in \mathbb{Z}^n,$$

$$\text{are } k(t, n)\text{-}(k\text{-, for short})\text{adjacent if} \tag{2}$$

$$\text{at most } t \text{ of their coordinates differs by } \pm 1, \text{ and all the others coincide.}$$

These $k(t, n)$-adjacency relations of $\mathbb{Z}^n$ are determined according to the number $t \in \mathbb{N}$ [17] (see also [21,22]). Using the statement of (2), the k-adjacency relations of $\mathbb{Z}^n$ are obtained [17] (see also [21,22]), as follows

$$k := k(t, n) = \sum_{i=1}^{t} 2^i C_i^n, \text{where } C_i^n = \frac{n!}{(n-i)!\, i!}. \tag{3}$$

For instance, [7,22]

$$(n, t, k) \in \left\{ \begin{array}{l} (3,1,6), (3,2,18), (3,3,26); \\ (4,1,8), (4,2,32), (4,3,64), (4,4,80); \\ (5,1,10), (5,2,50), (5,3,130), (5,4,210), (5,5,242). \end{array} \right\}$$

A digital image (X, k) in $\mathbb{Z}^n$ can indeed be considered to be a set $X(\subset \mathbb{Z}^n)$ with one of the k-adjacency relations of (3). Using the k-adjacency relations of $\mathbb{Z}^n$ of (3), we say that a digital k-neighborhood of p in $\mathbb{Z}^n$ is the set [20]

$$N_k(p) := \{q \mid p \text{ is } k\text{-adjacent to } q\} \cup \{p\}.$$

Furthermore, we often use the notation [19]

$$N_k^*(p) := N_k(p) \setminus \{p\}.$$

For $a, b \in \mathbb{Z}$ with $a \lneq b$, the set $[a, b]_{\mathbb{Z}} = \{n \in \mathbb{Z} \mid a \leq n \leq b\}$ with 2-adjacency is called a digital interval [19]. Let us now recall some terminology and notions [17,19] which are used in this paper.

- It is natural to say that a digital image (X, k) is k-disconnected if there are nonempty sets $X_1, X_2 \subset X$ such that $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$ and further, there are no points $x_1 \in X_1$ and $x_2 \in X_2$ such that x_1 and x_2 are k-adjacent.
- We say that a digital image (X, k) is k-connected (or k-path connected) if it is not k-disconnected. Owing to this approach, we see that a singleton subset of (X, k) is obviously k-connected.
- Given a k-connected digital image (X, k) whose cardinality is greater than 1, the so-called k-path with $l + 1$ elements in $\mathbb{Z}^n$ is assumed to be a finite sequence $(x_i)_{i \in [0,l]_{\mathbb{Z}}} \subset \mathbb{Z}^n$ such that x_i and x_j are k-adjacent if $|\, i - j\,| = 1$ [19]. Eventually, in the case that a digital image (X, k) is k-connected, for any distinct points such as x, y in (X, k), we see that there is a k-path $(x_i)_{i \in [0,l]_{\mathbb{Z}}} \subset X$ such that $x = x_0$ and $y = x_l$.
- For a digital image (X, k), the k-component of $x \in X$ is defined to be the maximal k-connected subset of (X, k) containing the point x [19].
- We say that a simple k-path means a finite set $(x_i)_{i \in [0,m]_{\mathbb{Z}}} \subset \mathbb{Z}^n$ such that x_i and x_j are k-adjacent if and only if $|\, i - j\,| = 1$ [19]. In the case of $x_0 = x$ and $x_m = y$, we denote the length of the simple k-path with $l_k(x, y) := m$.

- A simple closed k-curve (or simple k-cycle) with l elements in $\mathbb{Z}^n$, denoted by $SC_k^{n,l}$ [17,19], $l \geq 4, l \in \mathbb{N}_0 \setminus \{2\}$, $\mathbb{N}_0$ is the set of even natural numbers, means the finite set $(x_i)_{i \in [0,l-1]_{\mathbb{Z}}} \subset \mathbb{Z}^n$ such that x_i and x_j are k-adjacent if and only if $|\,i-j\,| = \pm 1 (mod\, l)$.
- For a digital image (X,k), a digital k-neighborhood of $x_0 \in X$ with radius ε is defined in X as the following subset [17] of X

$$N_k(x_0, \varepsilon) := \{x \in X \,|\, l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}, \tag{4}$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbb{N}$. For instance, for $X \subset \mathbb{Z}^n$, we obtain [17]

$$N_k(x, 1) = N_k(x) \cap X. \tag{5}$$

For a digital image (X,k), since X is a subset of $\mathbb{Z}^n$, if it is assumed as a subspace of the typical n-dimensional Euclidean topological space, it can naturally be a discrete topological subspace. However, as mentioned above, since a digital image (X,k) with the digital k-connectivity (see (3)) is a kind of a digital graph in $\mathbb{Z}^n$, the paper [17] already established another metric for (X,k). Eventually, the sets of (4) and (5) can be represented by using this metric on X derived from (X,k). The important thing is that this metric is different from the typical Euclidean metric. Indeed, a paper [17] firstly established the metric using the "length of a shortest simple k-path from x_0 to x" for two points x_0, x in (X,k). Owing to the length of a shortest k-path in (4), we prove that a k-connected digital image (X,k) can be considered to be a metric space, as follows:

Let us consider the map d_k on a k-connected (or k-path connected) digital image (X,k) defined by

$$d_k : (X,k) \times (X,k) \to \mathbb{N} \cup \{0\}$$

such that

$$d_k(x, x') := \begin{Bmatrix} l_k(x, x'), \text{ if } x \neq x'; \\ 0, \text{ if } x = x'. \end{Bmatrix} \tag{6}$$

Owing to (6), we can see that $d_k(x, x') \geq 1$ if $x \neq x'$ and further, we obviously see that the function d_k satisfies the metric axioms. Thus, we can represent the set $N_k(x_0, \varepsilon)$ of (4) in the following way

$$N_k(x_0, \varepsilon) = \{x \in X |\, d_k(x_0, x) \leq \varepsilon\}. \tag{7}$$

Consequently, we can represent the set of (5), as follows:

$$N_k(x_0, 1) = \{x \in X |\, d_k(x_0, x) \leq 1\}. \tag{8}$$

Rosenfeld [23] defined the notion of digital continuity of a map $f : (X, k_0) \to (Y, k_1)$ by saying that f maps every k_0-connected subset of (X, k_0) into a k_1-connected subset of (Y, k_1).

Motivated by this approach, using the set of (5) or (8), we can represent the digital continuity of a map between digital images by using a digital k-neighborhood (see Proposition 1 below). Due to this approach, we have strong advantages of calculating digital fundamental groups of digital images (X,k) in terms of the unique digital lifting theorem [17], the digital homotopy lifting theorem [24], a radius 2-(k_0, k_1)-isomorphism and its applications [24], the study of multiplicative properties for a digital fundamental group [25,26], a Cartesian product of the covering spaces [26], and so on, as follows:

Proposition 1. *[17,18] Let (X, k_0) and (Y, k_1) be digital images in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. A function $f : (X, k_0) \to (Y, k_1)$ is (digitally) (k_0, k_1)-continuous if and only if for every $x \in X$ $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

In Proposition 1, in the case $n_0 = n_1$ and $k := k_0 = k_1$, the map f is called a 'k-continuous' map. Since an n-dimensional digital image (X, k) is considered to be a set X in $\mathbb{Z}^n$ with one of the k-adjacency relations of (3) (or a digital k-graph [27]), regarding a classification of n-dimensional digital images, we prefer the term a (k_0, k_1)-isomorphism (or k-isomorphism) as in [27] (see also [18]) to a (k_0, k_1)-homeomorphism (or k-homeomorphism) as in [28].

Definition 1. *[27] (see also a (k_0, k_1)-homeomorphism in [28]) Consider two digital images (X, k_0) and (Y, k_1) in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Then a map $h : X \to Y$ is called a (k_0, k_1)-isomorphism if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous. Then we use the notation $X \approx_{(k_0,k_1)} Y$. Besides, in the case $k := k_0 = k_1$, we use the notation $X \approx_k Y$.*

The following notion of interior is often used in establishing a digital connected sum of digital closed k-surfaces.

Definition 2. *[5] Let $c^* := (x_0, x_1, \cdots, x_n)$ be a closed k-curve in $(\mathbb{Z}^2, k, \bar{k}, c^*)$. A point x of $\overline{c^*}$, the complement of c^* in $\mathbb{Z}^2$, is said to be interior to c^* if it belongs to the bounded $\bar{k}$-connected component of $\overline{c^*}$.*

The following digital images MSC_8^*, MSC_4^*, and $MSC_8'^*$ in $\mathbb{Z}^2$ [5,6,17] have essentially been used in establishing a connected sum and studying the digital fundamental group of a digital connected sum of closed k-surfaces. Thus we now recall them.

($\star$) $MSC_8^* := MSC_8 \cup Int(MSC_8)$ [6], where MSC_8 is a digital image 8-isomorphic to the digital image, $MSC_8 := SC_8^{2,6} := \{c_0 = (0,0), c_1 = (1,1), c_2 = (1,2), c_3 = (0,3), c_4 = (-1,2), c_5 = (-1,1)\}$.
($\star$) $MSC_4^* := MSC_4 \cup Int(MSC_4)$ [6], where MSC_4 is a digital image 4-isomorphic to the digital image, $MSC_4 := SC_4^{2,8} := \{v_0 = (0,0), v_1 = (1,0), v_2 = (2,0), v_3 = (2,1), v_4 = (2,2), v_5 = (1,2), v_6 = (0,2), v_7 = (0,1)\}$.
($\star$) $MSC_8'^* := MSC_8' \cup Int(MSC_8')$ [6], where MSC_8' is a digital image 8-isomorphic to the digital image, $MSC_8' := SC_8^{2,4} := \{w_0 = (0,0), w_1 = (1,1), w_2 = (0,2), w_3 = (-1,1)\}$.

Based on the pointed digital homotopy in [29] (see also [28]), the following notion of k-homotopy relative to a subset $A \subset X$ is often used in studying k-homotopic properties of digital images (X, k) in $\mathbb{Z}^n$. For a digital image (X, k) and $A \subset X$, we often call $((X, A), k)$ a digital image pair.

Definition 3. *[17,24,28] Let $((X, A), k_0)$ and (Y, k_1) be a digital image pair and a digital image in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Let $f, g : X \to Y$ be (k_0, k_1)-continuous functions. Suppose there exist $m \in \mathbb{N}$ and a function $H : X \times [0, m]_{\mathbb{Z}} \to Y$ such that*

- *for all $x \in X, H(x, 0) = f(x)$ and $H(x, m) = g(x)$;*
- *for all $x \in X$, the induced function $H_x : [0, m]_{\mathbb{Z}} \to Y$ given by $H_x(t) = H(x, t)$ for all $t \in [0, m]_{\mathbb{Z}}$ is $(2, k_1)$-continuous;*
- *for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $H_t : X \to Y$ given by $H_t(x) = H(x, t)$ for all $x \in X$ is (k_0, k_1)-continuous.*
  Then we say that H is a (k_0, k_1)-homotopy between f and g [28].
- *Furthermore, for all $t \in [0, m]_{\mathbb{Z}}$, assume that the induced map H_t on A is a constant which follows the prescribed function from A to Y [17] (see also [5]). To be precise, $H_t(x) = f(x) = g(x)$ for all $x \in A$ and for all $t \in [0, m]_{\mathbb{Z}}$.*

Then we call H a (k_0, k_1)-homotopy relative to A between f and g, and we say that f and g are (k_0, k_1)-homotopic relative to A in Y, $f \simeq_{(k_0,k_1)rel.A} g$ in symbols [17].

In Definition 3, if a k-continuous map $f : X \to X$ is k-homotopic to a certain constant map $c_{\{x_0\}}, x_0 \in X$, then we say that f is (pointed) *k-null homotopic* in (X, k) [28]. In Definition 3, if $A = \{x_0\} \subset X$, then we say that F is a pointed (k_0, k_1)-homotopy at $\{x_0\}$ [28]. When f and g are pointed (k_0, k_1)-homotopic in Y, we use the notation $f \simeq_{(k_0,k_1)} g$. In the case $k := k_0 = k_1$ and $n_0 = n_1$, f and g are said to be pointed k-homotopic in Y and we use the notation $f \simeq_k g$ and $f \in [g]$ which denotes the k-homotopy class of g. If, for some $x_0 \in X$, 1_X is k-homotopic to the constant map in the space X relative to $\{x_0\}$, then we say that (X, x_0) is *pointed k-contractible* [28]. Indeed, motivated by this approach, the notion of strong k-deformation retract was developed in [30].

Based on this k-homotopy, the notion of digital homotopy equivalence was firstly introduced in [31] (see also [32]), as follows:

Definition 4. *[31] (see also [32]) For two digital images (X, k) and (Y, k) in $\mathbb{Z}^n$, if there are k-continuous maps $h : X \to Y$ and $l : Y \to X$ such that the composite $l \circ h$ is k-homotopic to 1_X and the composite $h \circ l$ is k-homotopic to 1_Y, then the map $h : X \to Y$ is called a k-homotopy equivalence and is denoted by $X \simeq_{k \cdot h \cdot e} Y$. Besides, we say that (X, k) is k-homotopy equivalent to (Y, k). In the case that the identity map 1_X is k-homotopy equivalent to a certain constant map $c_{\{x_0\}}, x_0 \in X$, we say that (X, k) is k-contractible.*

In Definition 4, in the case $X \simeq_{k \cdot h \cdot e} Y$, we say that (X, k) is the same k-homotopy type as (Y, k). In view of Definitions 3 and 4, we obviously see that the pointed k-contractibility implies the k-contractibility, the converse does not hold. Let (X, k) be k-contractible. Then it is obvious that any k-loop in (X, k) is k-null homotopic in (X, k).

The digital k-fundamental group is induced from the pointed k-homotopy [28]. For a given digital image (X, k), by using several notions such as digital k-homotopy class [29], Khalimsky operation of two k-homotopy classes [29], trivial extension [28], the paper [28] defined the digital k-fundamental group, denoted by $\pi^k(X, x_0), x_0 \in X$. Indeed, in digital topology there are several kinds of digital fundamental groups [33]. In addition, we have the following: If X is pointed k-contractible, then $\pi^k(X, x_0)$ is a trivial group [28]. Hereafter, we shall assume that each digital image (X, k) is k-connected.

Using the unique digital lifting theorem [17] and the homotopy lifting theorem [24] in digital covering theory [4,17,18,25,26], for a non-k-contractible space $SC_k^{n,l}$, we obtain the following:

Theorem 1. *[17] For a non-k-contractible $SC_k^{n,l}$, $\pi_1^k(SC_k^{n,l})$ is an infinite cyclic group.*

Namely, for an $SC_k^{n,l}, l \geq 6$, it turns out that $\pi_1^k(SC_k^{n,l})$ is an infinite cyclic group. Regarding Theorem 1, we see that $SC_{3^n-1}^{n,4}$ has the trivial group, $n \geq 2$ [24,28] and further, $SC_4^{2,4}$ also has the trivial group because $SC_4^{2,4}$ is 4-contractible (see a certain idea from Example 1 below).

The following are proven in [5,7,17,18,28].

- $MSC_8 := SC_8^{2,6}$ is not 8-contractible and $MSC_4 := SC_4^{2,8}$ is not 4-contractible either [5,17].
- MSC_8' are 8-contractible [5,7,28].
- Due to Theorem 1, it turns out that $SC_k^{n,l}$ is not k-contractible if $l \geq 6$.

In particular, both the non-8-contractibility of MSC_8 and the non-4-contractibility of MSC_4 play important roles in formulating a connected sum of two closed k-surfaces (see Section 4 for the details).

Whereas $SC_6^{3,6}$ itself is not 6-contractible (see Theorem 1), identity map $1_{SC_6^{3,6}}$ is 6-null homotopic in $(I^3, 6)$, where $SC_6^{3,6} \subset I^3$. To be precise, we obtain the following:

Example 1. *Let us consider $SC_6^{3,6} := (c_i)_{i \in [0,5]_{\mathbb{Z}}}$ embedded in $(I^3, 6)$ (see Figure 1), where $c_0 := (0,0,0), c_1 := (0,0,1), c_2 := (0,1,1), c_3 := (-1,1,1,), c_4 := (-1,1,0), c_5 := (-1,0,0)$. It is obvious that $SC_6^{3,6}$ itself is* *not 6-contractible (see Theorem 1) because its 6-fundamental group is an infinite cyclic group [17]. However, identity map $1_{SC_6^{3,6}}$ is clearly 6-null homotopic in $(I^3, 6)$ (see Figure 1). To be specific, consider the map*

$$H : SC_6^{3,6} \times [0,3]_{\mathbb{Z}} \to (I^3, 6)$$

such that for $x \in SC_6^{3,6}$

$$\left\{\begin{array}{l} H(x,0) = x, i.e., H(x,0) = 1_{SC_6^{3,6}}(x); \\ H(x,1) = \{c_0, c_1, d_1, c_5\} \text{ by using the mappings} \\ c_0 \to c_0, \{c_1, c_2\} \to \{c_1\}, c_3 \to d_1 \text{ and } \{c_4, c_5\} \to \{c_5\}; \\ H(x,2) = \{c_0, c_5\} \text{ in terms of the mappings} \\ \{c_0, c_1, c_2\} \to \{c_0\} \text{ and } \{c_3, c_4, c_5\} \to \{c_5\}, \\ i.e., c_3 \to c_5 \text{ via } c_3 \to d_1 \to c_5; \text{and} \\ H(x,3) = \{c_0\}, x \in SC_6^{3,6}, i.e., H(x,3) = c_{\{c_0\}}(x). \end{array}\right\}$$

Then we see that the map H is a 6-homotopy making $1_{SC_6^{3,6}}$ 6-null homotopic in $(I^3, 6)$.

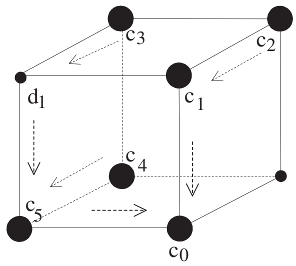

Figure 1. Configuration of the pointed 6-null homotopic of $1_{SC_6^{3,6}}$ in $(I^3, 6)$.

In view of Example 1, we observe that $SC_6^{3,6}$ is not 6-contractible in itself because its digital 6-fundamental group is an infinite cyclic group (see Theorem 1, for the details, see [4,17]).

Remark 1. *The digital image $(I^3, 6)$ is 6-contractible (see [34]).*

Hereafter, we denote the n-dimensional digital cube (or digital n-cube) with

$$I^n := \prod_{i=1}^{n} [x_i, x_i + 1]_{\mathbb{Z}} \subset \mathbb{Z}^n, n \in \mathbb{N}.$$

Based on the 6-contractibility of $(I^3, 6)$ (see [34]), using a similar method as the proof of it (see Remark 2 of [8]), it is obvious that (I^n, k) is pointed k-contractible for any k-adjacency of $\mathbb{Z}^n$, where the k-adjacency is that of (3) according to the dimension "n".

Let us now examine if a k-isomorphism preserves a k-homotopy between two k-continuous maps.

Theorem 2. *A k-isomorphism preserves a k-homotopy.*

Proof. Given two spaces $X := (X, k)$, $Y := (Y, k)$ in $\mathbb{Z}^n$, consider two k-continuous functions $f, g : X \to Y$, relating to a k-homotopy $F : X \times [a, b]_{\mathbb{Z}} \to Y$, i.e., $f \simeq_k g$. Besides, further assume two

k-isomorphisms $h_1 : X \to X'$ and $h_2 : Y \to Y'$, where (X', k) and (Y', k) are considered in $\mathbb{Z}^n$. Then, it is clear that the two composites

$$h_2 \circ f \circ h_1^{-1} \text{ and } h_2 \circ g \circ h_1^{-1}$$

are also k-continuous maps from X' to Y'. Based on the given k-homotopy and the two k-isomorphisms h_1 and h_2, we now define the new map

$$G := h_2 \circ F \circ h_1^{-1} : X' \times [a, b]_{\mathbb{Z}} \to Y'.$$

Then, we obtain the following:

(1) for all $x' \in X'$, $G(x', a) = h_2 \circ f \circ h_1^{-1}(x')$ and $G(x', b) = h_2 \circ g \circ h_1^{-1}(x')$;
(2) for all $x' \in X'$, the induced function $G_{x'} : [a, b]_{\mathbb{Z}} \to Y'$ defined by $G_{x'}(t) := G(x', t)$ for all $t \in [a, b]_{\mathbb{Z}}$ is k-continuous;
(3) for all $t \in [a, b]_{\mathbb{Z}}$, the induced function $G_t : X' \to Y'$ defined by $G_t(x') := G(x', t)$ for all $x' \in X'$ is k-continuous.

Thus we have a conclusion that G is a k-homotopy between $h_2 \circ f \circ h_1^{-1}$ and $h_2 \circ g \circ h_1^{-1}$. $\square$

Corollary 1. *A k-isomorphism preserves the k-contractibility.*

Proof. In Theorem 2, consider a k-contractible space (X, k) such that $X \simeq_{k \cdot h \cdot e} \{x_0\}$ for some point $x_0 \in X$. Then, after replacing f (*resp.* g) by 1_X (*resp.* the constant map $c_{\{x_0\}}$), we prove the assertion. $\square$

Corollary 2. *A k-isomorphism preserves the pointed k-contractibility.*

Proof. In Theorem 2 and Corollary 1, consider a pointed k-contractible space (X, k) such that 1_X is k-homotopic to the constant map in the space $\{x_0\}$ relative to $\{x_0\}$. After replacing f (*resp.* g) with 1_X (*resp.* the constant map $c_{\{x_0\}}$), we complete the proof. $\square$

Using a method similar to the proof of Theorem 2, we obtain the following:

Corollary 3. *A (k_0, k_1)-isomorphism preserves a (k_0, k_1)-homotopy equivalence.*

3. Utilities of the Minimal Simple Closed 6-, 18- and 26-Surfaces; MSS_6, MSS_{18}, MSS'_{18}, MSS'_{26}

This section stresses some utilities of the minimal simple closed 6-, 18-, 26-surfaces, e.g., MSS_6, MSS_{18}, MSS'_{18}, MSS'_{26} [6] from the viewpoints of digital surface and digital homotopy theory. Indeed, these models for simple closed k-surfaces play important roles in digital homotopy theory, digital surface theory, and fixed point theory. Furthermore, these have been used in formulating connected sums of some simple closed k-surfaces, $k \in \{6, 18, 26\}$ [5–7]. Besides, these were essentially used in proceeding with geometric realizations of digital k-surfaces [7,8].

In order to study closed k-surfaces in $\mathbb{Z}^n$, let us recall some terminology from digital surface theory, as follows: A point $x \in (X, k)$ is called a *k-corner* if x is k-adjacent to two and only two points y, $z \in X$ such that y and z are k-adjacent to each other [2]. The k-corner x is called *simple* if y, z are not k-corners and if x is the only point k-adjacent to both y, z. (X, k) is called a *generalized simple closed k-curve* if what is obtained by removing all simple k-corners of X is a simple closed k-curve [2,9]. For a k-connected digital image (X, k) in $X \subset \mathbb{Z}^3$, we recall [1,2,6]

$$|X|^x := N_{26}(x, 1) \setminus \{x\}. \tag{9}$$

In general, for a k-connected digital image (X, k) in $\mathbb{Z}^n$, $n \geq 3$, we can state [7]

$$|X|^x := N_{3^n-1}(x, 1) \setminus \{x\}. \tag{10}$$

Hereafter, for a k-surface in $\mathbb{Z}^n, n \in \mathbb{N} \setminus \{1,2\}$ [5,6], we call the set $|X|^x$ of (9) the *minimal* $(3^n - 1)$-*adjacency neighborhood* of x in X.

We say that two subsets, (A,k) and (B,k) of (X,k), are k-adjacent if $A \cap B = \emptyset$ and there are points $a \in A$ and $b \in B$ such that a and b are k-adjacent [19]. In particular, in the case that B is a singleton, say $B = \{x\}$, we say that A is k-adjacent to x.

Papers [5–7] introduced the notion of a closed k-surface in $\mathbb{Z}^n, n \geq 3$ and various properties of it. However, in the present paper, we will stress the study of closed k-surfaces in $\mathbb{Z}^3$ with the following approach in [3,9,10].

Definition 5. *[3,10] Let (X,k) be a digital image in $\mathbb{Z}^3$, and $\overline{X} := \mathbb{Z}^3 \setminus X$. Then, X is called a closed k-surface if it satisfies the following.*
(1) In the case $(k,\bar{k}) \in \{(26,6),(6,26)\}$, for each point $x \in X$,
(a) $|X|^x$ has exactly one k-component k-adjacent to x;
(b) $|\overline{X}|^x$ has exactly two $\bar{k}$-components which are $\bar{k}$-adjacent to x; we denote by C^{xx} and D^{xx} these two components; and
(c) for any point $y \in N_k(x) \cap X$ (or $N_k(x,1)$ in (X,k)), $N_{\bar{k}}(y) \cap C^{xx} \neq \phi$ and $N_{\bar{k}}(y) \cap D^{xx} \neq \phi$.
Furthermore, if a closed k-surface X does not have a simple k-point, then X is called simple.
(2) In the case $(k,\bar{k}) = (18,6)$,
(a) X is k-connected,
(b) for each point $x \in X$, $|X|^x$ is a generalized simple closed k-curve.
Furthermore, if the image $|X|^x$ is a simple closed k-curve, then the closed k-surface X is called simple.

Hereafter, we denote by MSS_k a *minimal simple closed k-surface* in $\mathbb{Z}^3$ (see Figure 2). Furthermore, we recall the following closed k-surfaces, $k \in \{6,18,26\}$ [5]:

Remark 2. *(1) $MSS_6 \approx_6 [-1,1]^3_{\mathbb{Z}} \setminus \{0_3\}$, where $0_3 := (0,0,0)$. Then, MSS_6 is the minimal simple closed 6-surface which is not 6-contractible (see Figure 2c). Namely, we obtain the digital picture $(\mathbb{Z}^3, 6, 26, MSS_6)$ according to (1).*
(2) $MSS'_{18} \approx_{18} \{p \in \mathbb{Z}^3 \mid d(p,0_3) = 1\}$, where d is the typical Euclidean distance in $\mathbb{R}^3$. Thus we obtain the digital picture $(\mathbb{Z}^3, 18, 6, MSS'_{18})$ according to (1).

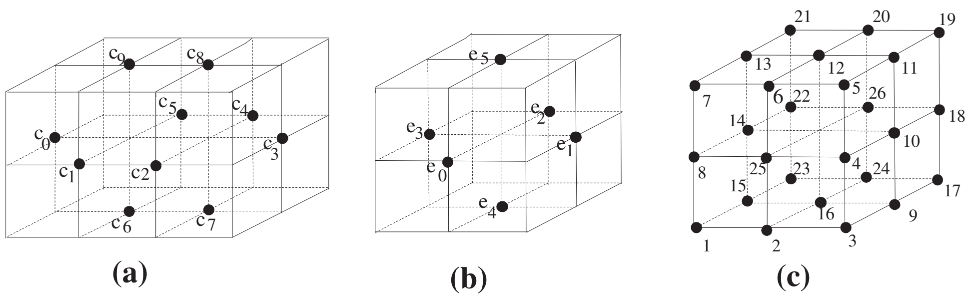

Figure 2. (**a**) MSS_{18} [5,6]: (**b**) $MSS'_{18} = MSS'_{26}$ [5,6]; (**c**) MSS_6 [5].

Papers [5,6] indeed stated that MSS'_{18} is 18-contractible and it is the minimal simple closed 18-surface. Besides, a paper [5] proved the simply 18-connectedness of MSS'_{18} and MSS_{18}. In addition, we see that MSS_6 is simply 6-connected [6,8].

Let us further recall two simple closed k-surface, $k \in \{18,26\}$, as follows:

- $MSS_{18} \approx_{18} (MSC_8 \times \{1\}) \cup (Int(MSC_8) \times \{0,2\})$ [5,6]. Thus we obtain the digital picture $(\mathbb{Z}^3, 18, 6, MSS_{18})$ according to (1).
- $MSS'_{26} := MSS'_{18}$ which is 26-contractible [5,6] and is the minimal simple closed 26-surface (see Figure 2b). Finally, we obtain the binary digital picture $(\mathbb{Z}^3, 26, 6, MSS'_{26})$ according to (1). Besides, we recall the following:

Remark 3. *[8] MSS_{18} is pointed 18-contractible.*

Proposition 2. *If given a digital image (X,k) is not k-connected, then it is not k-contractible.*

Proof. Owing to the second property of Definition 3, the assertion is proved. □

- (Correction) In the Figure 4c of [35], the given K-topological space (Z, κ_Z^2) should be referred to as "non-K-retractible" instead of "K-retractible".

4. Several Types of Models for $C_6^n := \overbrace{MSS_6 \sharp \cdots \sharp MSS_6}^{n\text{-times}}$

From now on we denote a (simple) closed k-surface in $\mathbb{Z}^3$ with S_k, $k \in \{6, 18, 26\}$, which will be used in this paper. In particular, we will mainly consider an S_k, $k \in \{6, 18, 26\}$ in the picture as referred to in (1), i.e.,

$$\{(\mathbb{Z}^3, 26, 6, S_{26}), (\mathbb{Z}^3, 18, 6, S_{18}), (\mathbb{Z}^3, 6, 26, S_6)\}. \tag{11}$$

Definition 6. *[5] In $\mathbb{Z}^3$, let S_{k_0} (resp. S_{k_1}) be a closed k_0-(resp. a closed k_1-)surface, where $k_0 = k_1 \in \{6, 18, 26\}$.*

- *Consider $A'_{k_0} \subset A_{k_0} \subset S_{k_0}$ and take $A_{k_0} \setminus A'_{k_0} \subset S_{k_0}$, where $A_{k_0} \approx_{(k_0,4)} MSC_4^*$ or $A_{k_0} \approx_{(k_0,8)} MSC_8^*$, or $A_{k_0} \approx_{(k_0,8)} MSC_8'^*$, and further, $A'_{k_0} \approx_{(k_0,4)} Int(MSC_4)$ or $A'_{k_0} \approx_{(k_0,8)} Int(MSC_8)$, or $A'_{k_0} \approx_{(k_0,8)} Int(MSC'_8)$, respectively.*
- *Let $f : A_{k_0} \to f(A_{k_0}) \subset S'_{k_1}$ be a (k_0, k_1)-isomorphism. Remove A'_{k_0} and $f(A'_{k_0})$ from S_{k_0} and S_{k_1}, respectively.*
- *Identify $A_{k_0} \setminus A'_{k_0}$ and $f(A_{k_0} \setminus A'_{k_0})$ by using the (k_0, k_1)-isomorphism f. Then, the quotient space $S'_{k_0} \cup S'_{k_1} / \sim$ is obtained by $i(x) \sim f(x) \in S'_{k_1}$ for $x \in A_{k_0} \setminus A'_{k_0}$ and is denoted by $S_{k_0} \sharp S_{k_1}$, where $S'_{k_0} = S_{k_0} \setminus A'_{k_0}$, $S'_{k_1} = S_{k_1} \setminus f(A'_{k_0})$, and the map $i : A_{k_0} \setminus A'_{k_0} \to S'_{k_0}$ is the inclusion map.*

Owing to Definition 6, $S_{k_0} \sharp S_{k_1}$ is obtained in $\mathbb{Z}^3$. Besides, the digital topological type of $S_{k_0} \sharp S_{k_1}$ absolutely depends on the choice of the subset $A_{k_0} \subset S_{k_0}$ [7]. Furthermore, the k-adjacency of $S_{k_0} \sharp S_{k_1}$ is required as follows:

Remark 4. *[5] In the quotient space $S_{k_0} \sharp S_{k_1} := S'_{k_0} \cup S'_{k_1} / \sim$, the subsets $A := S'_{k_0} \setminus (A_{k_0} \setminus A'_{k_0})$ and $B := S'_{k_1} \setminus f(A_{k_0} \setminus A'_{k_0})$ in $S_{k_0} \sharp S_{k_1}$ are assumed to be disjoint and there are no points $x \in A$ and $x' \in B$ such that x and x' are k-adjacent, where $k := k_0 = k_1$. Then, the digital image $(S_{k_0} \sharp S_{k_1}, k)$ is called a (digital) connected sum of S_{k_0} and S_{k_1}.*

As mentioned in Remark 4, the requirement involving the k-adjacency of $(S_{k_0} \sharp S_{k_1}, k)$ in $\mathbb{Z}^3$ plays an important role in studying connected sums of closed k_i-surfaces, $i \in \{0, 1\}, k = k_i$. Indeed, it turns out that [8] $(S_k \sharp S'_k, k)$ is also a closed k-surface in the picture $(\mathbb{Z}^3, k, \bar{k}, S_k \sharp S'_k)$, where S_k and S'_k are closed k-surfaces in the pictures $(\mathbb{Z}^3, k, \bar{k}, S_k)$ and $(\mathbb{Z}^3, k, \bar{k}, S'_k)$, respectively.

This section explores several methods of formulating the digital connected sums $MSS_6 \sharp MSS_6$, $MSS_{18} \sharp MSS_{18}$ and an n-times iterated connected sum of MSS_6 and that of MSS_{18}.

At the moment, let us recall the previously-mentioned queries in Section 1:

(Q1) After replacing $(6, 26)$ in Definition 5(1) with $(6, 18)$, we may ask if it is possible to propose the simple closed 6-surface MSS_6 in the picture $(\mathbb{Z}^3, 6, 18, MSS_6)$ instead of $(\mathbb{Z}^3, 6, 26, MSS_6)$.
This query is a reminder of the importance of the $\bar{k}$-adjacency of $\mathbb{Z}^3 \setminus S_k$ of a simple closed k-surface S_k in the picture $(\mathbb{Z}^3, k, \bar{k}, S_k)$.
(Q2) Given the MSS_6, how many models for $MSS_6 \sharp MSS_6$ exist?

Let $\mathcal{C}_6^n := \overbrace{MSS_6 \sharp \cdots \sharp MSS_6}^{n\text{-times}}$. Then we also have the following question:
(Q3) How can we formulate $\mathcal{C}_6^n, n \in \mathbb{N} \setminus \{1\}$?

To address these queries, we now study some properties of MSS_6 and $\mathcal{C}_6^n$. First of all, let us represent the question (Q1), as follows:

Unlike the three cases of (1), we may ask if there are other binary relations $(6, \bar{k})$ for MSS_6, $\bar{k} \in \{6, 18\}$.

Remark 5. *Regarding the question (Q1), we have a negative answer.*

Proof. Consider the point indicated by the number "3" in Figure 2c. Since the set $|\,\overline{MSS_6}\,|^3$ does not satisfy the properties of Definition 5(b) and (c), we cannot consider the picture $(\mathbb{Z}^3, 6, 18, MSS_6)$ for the simple closed 6-surface MSS_6.

Similarly, using a method similar to the above approach, we cannot take the picture $(\mathbb{Z}^3, 6, 6, MSS_6)$ for MSS_6. □

To address the above question (Q2), we have the following:

Lemma 1. *Given an MSS_6, the only one type of $MSS_6 \sharp MSS_6$ exists up to 6-isomorphism.*

Proof. In order to formulate $MSS_6 \sharp MSS_6$, we should follow Definition 6 and Remark 4. In this situation, it is obvious that we obtain six cases of $MSS_6 \sharp MSS_6$ (see one of the cases in Figure 3a) which are 6-isomorphic to each other. Regarding the establishment of a connected sum $MSS_6 \sharp MSS_6$, suppose some possibility of taking one of the points indicated by the numbers "8" or "7" in Figure 3a except the above-mentioned six points of MSS_6, e.g., the point p of Figure 3b. Then we have a contradiction to Remark 4. Hence we have the only one type of $MSS_6 \sharp MSS_6$ as suggested in Figure 3a up to 6-isomorphism. □

Regarding the question (Q3), we obtain the following:

Theorem 3. *In the case of $\mathcal{C}_6^n, n \in \mathbb{N} \setminus \{1, 2\}$, many types of models for $\mathcal{C}_6^n$ exist.*

Proof. Let us formulate $\mathcal{C}_6^3 := MSS_6 \sharp MSS_6 \sharp MSS_6$. As shown in Figure 3b, take a certain subset of MSS_6 which is $(6, 4)$-isomorphic to the set MSC_4^*, e.g., the set $(c_i)_{i \in [0,7]_{\mathbb{Z}}} \cup \{p\}$ in MSS_6 (Figure 3b). Depending on the choice of the corresponding part in $MSS_6 \sharp MSS_6$ (see Figure 3b), e.g., (1), (2), (3), and (4) in Figure 3b, we have different types of shapes for $\mathcal{C}_6^3 := MSS_6 \sharp MSS_6 \sharp MSS_6$. To be precise, if we follow Case (1) in Figure 3b, after deleting the two points p and d_{10} in Figure 3b, we obtain $\mathcal{C}_6^3$ by identifying the two sets $\{c_i \,|\, i \in [0, 7]_{\mathbb{Z}}\}$ and $\{d_{27}, d_{11}, d_{21}, d_{22}, d_{23}, d_9, d_{29}, d_{28}\}$ (see the method of Definition 6).

If we follow Case (2) in Figure 3b, after deleting the two points p and d_{13} in Figure 3b, we obtain $\mathcal{C}_6^3$ by identifying the two sets $\{c_i \,|\, i \in [0, 7]_{\mathbb{Z}}\}$ and $\{d_{12}, d_{19}, d_{20}, d_{15}, d_{14}, d_{23}, d_{22}, d_{21}\}$ (see the method of Definition 6).

Using a method similar to these two approaches, after following Cases (3) and (4), we can also obtain $\mathcal{C}_6^3$. Then we observe some different shapes between the $\mathcal{C}_6^3$ established via (2) and those formulated via (1) or (3). As a generalization of $\mathcal{C}_6^3$, we obviously obtain several types of models for $\mathcal{C}_6^n, n \in \mathbb{N} \setminus \{1, 2\}$. □

Motivated by Theorem 1 of [8], we obtain the following:

Remark 6. *[7] Given a closed 6-surface S_6 in the picture $(\mathbb{Z}^3, 6, 26, S_6)$, we obtain that $S_6 \sharp MSS_6$ is a simple closed 6-surface in the picture $(\mathbb{Z}^3, 6, 26, S_6 \sharp MSS_6)$.*

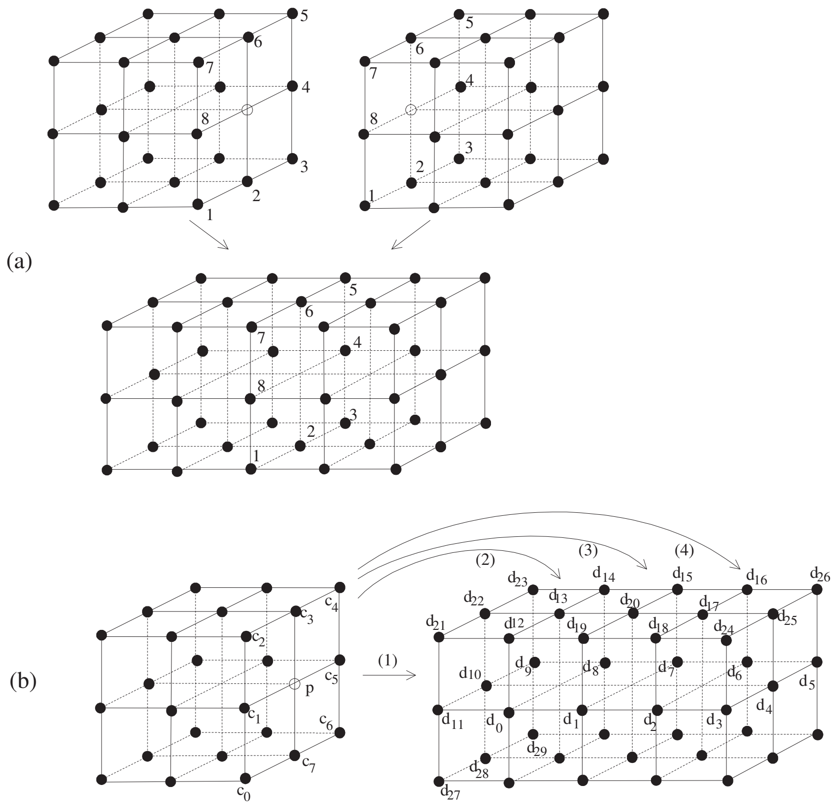

Figure 3. (**a**) Process of constructing $MSS_6 \sharp MSS_6$ [5]; (**b**) Configuration of $C_6^3 := MSS_6 \sharp MSS_6 \sharp MSS_6$.

5. Existence of Only Two Types of $C_{18}^n := \overbrace{MSS_{18} \sharp \cdots \sharp MSS_{18}}^{n\textbf{-times}}, n \geq 2$

This section proves an existence of only two types of $C_{18}^n := \overbrace{MSS_{18} \sharp \cdots \sharp MSS_{18}}^{n\text{-times}}, n \geq 2$. When establishing C_{18}^n, we assume $C_{18}^n := C_{18}^{n-1} \sharp MSS_{18}, n \geq 2$. Before studying $C_{18}^n, n \geq 2$, we now investigate some properties of MSS_{18} involving a choice of a suitable digital picture for MSS_{18}.

Remark 7. *Using a similar method as that of Remark 5, we obtain the following:*
(1) The set MSS_{18} cannot be a simple closed 18-surface in the picture $(\mathbb{Z}^3, 18, 18, MSS_{18})$.
(2) The set MSS'_{26} cannot be a simple closed 26-surface in the picture $(\mathbb{Z}^3, 26, 18, MSS'_{26})$.

Based on the digital connected sums of MSS_6, MSS_{18}, MSS'_{18}, and MSS'_{26} introduced [5], in order to study them more systematically, we need to address the following query.

(Q4) Given an MSS_{18}, how many types of $MSS_{18} \sharp MSS_{18}$ exist ?

Let $C_{18}^n := \overbrace{MSS_{18} \sharp \cdots \sharp MSS_{18}}^{n\text{-times}}$. Then we have the following question:

(Q5) How can we formulate $C_{18}^n, n \in \mathbb{N} \setminus \{1\}$?

Based on the establishment of $MSS_{18} \sharp MSS_{18}$ in [5,7], we need to address the query of (Q4), as follows:

Theorem 4. *Given an MSS_{18}, we obtain the following:*
(1) Only two types of $MSS_{18} \sharp MSS_{18}$ exist up to 18-isomorphism.
(2) In the case of $C_{18}^n, n \in \mathbb{N} \setminus \{1, 2\}$, only two methods are admissible in establishing C_{18}^n up to 18-isomorphism.

Proof. First of all, we need to ask if there is a certain possibility of taking a set $A_{18}(\subset MSS_{18})$ which is respectively $(18,4)$- and $(18,8)$-isomorphic to MSC_4^* and MSC_8^*, or $MSC_8'^*$ (see Definition 6). Then we can recognize that there are only six subsets in MSS_{18} satisfying this requirement, such as (see the set in Figure 4a)

$$\left\{\begin{array}{l} (1)\{c_0, c_1, c_2, c_3, c_4, c_5, c_8, c_9\}, \{c_i \mid i \in [0,7]_{\mathbb{Z}}\}, \\ (2)\{c_0, c_1, c_2, c_3, c_6, c_7, c_8, c_9\}, \{c_0, c_6, c_7, c_3, c_4, c_5, c_8, c_9\}, \\ (3)\{c_2, c_7, c_4, c_8, c_3\}, \{c_1, c_6, c_5, c_9, c_0\}. \end{array}\right\} \tag{12}$$

According to these considerations of (12), we now consider two cases, as follows:

(Case 1) Based on the cases of (12) (1)–(2), in the case that we follow the method suggested in Figure 4a, we obtain $MSS_{18}\sharp MSS_{18} = MSS_{18}$ [5]. Eventually, if we take this process for obtaining $\mathcal{C}_{18}^n$, then we have $\mathcal{C}_{18}^n = MSS_{18}$.
(Case 2) Based on the cases of (12) (3), according to the method suggested in Figure 4b, i.e., in the case $MSS_{18}\sharp MSS_{18} \neq MSS_{18}$, we now prove that there is only one type of $MSS_{18}\sharp MSS_{18}$ up to 18-isomorphism. To be precise, after identifying two sets denoted by the set $\{1,2,3,4\}$ of MSS_{18} (see Figure 4b), we obtain $MSS_{18}\sharp MSS_{18}$. Hence, we have only one way to proceed to $MSS_{18}\sharp MSS_{18}$ as proposed in Figure 4b up to 18-isomorphism. Eventually, we uniquely obtain $\mathcal{C}_{18}^n$ in terms of $\mathcal{C}_{18}^n := \mathcal{C}_{18}^{n-1}\sharp MSS_{18}$. □

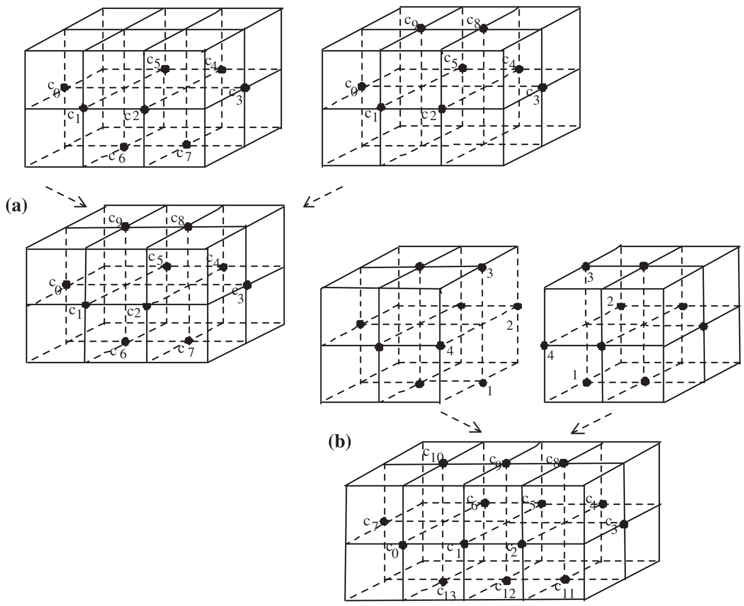

Figure 4. Explanation of the only two types of $MSS_{18}\sharp MSS_{18}$ in terms of the processes via (**a**) or (**b**) [5].

Remark 8. *When constructing $MSS_{18}\sharp MSS_{18}'$, we only take the part suggested in (12) (3) so that we obtain $MSS_{18}\sharp MSS_{18}' = MSS_{18}$ [5].*

As mentioned in [5], we obtain the following:

Corollary 4. *(1) For a simple closed 18-surface S_{18}, $MSS_{18}'\sharp S_{18}$ is a simple closed 18-surface.*
(2) $MSS_{18}'\sharp MSS_{18}' = MSS_{18}'$ [7].

6. Digital 18-Contractibility of $\mathcal{C}^n_{18}$ and Simply k-Connectedness of $\mathcal{C}^n_k$, $k \in \{6, 18, 26\}$

This section explores the digital 18-contractibility of $\mathcal{C}^n_{18}$ and the simply k-connectedness of $\mathcal{C}^n_k$, $k \in \{6, 18, 26\}$. Hereafter, we consider the process $\mathcal{C}^n_{18} := \mathcal{C}^{n-1}_{18} \sharp MSS_{18}$ and assume the case $MSS'_{18} \sharp MSS_{18} \neq MSS_{18}$. As stated in the proof of Theorem 4, we obtain the following:

Lemma 2. *In case $\mathcal{C}^2_{18} := MSS_{18} \sharp MSS_{18} \neq MSS_{18}$, $\mathcal{C}^3_{18} = \mathcal{C}^2_{18} \sharp MSS_{18}$ uniquely exists up to 18-isomorphism.*

Definition 7. *[17] For a k-connected digital image (X, k), if $\pi^k_1(X)$ trivial, then we say that (X, k) is simply k-connected.*

Lemma 3. *[4–6,8] Each of $\pi^6_1(MSS_6)$, $\pi^{18}_1(MSS_{18})$, $\pi^{18}_1(MSS'_{18})$, and $\pi^{26}_1(MSS'_{26})$ is trivial.*

Proof. First of all, we see that the 6-fundamental group of MSS_6 is a trivial group [8]. Next, we see that each of MSS_{18} and MSS'_{18} is 18-contractible and further, MSS'_{26} is 26-contractible, the proof is completed. □

Proposition 3. *A simple closed 6-surface S_6 is simply 6-connected.*

Proof. It is obvious that S_6 is 6-connected. Using a trivial extension of a 6-loop on S_6, we see that any 6-loop on S_6 is 6-null homotopic in S_6 so that $\pi^6_1(S_6)$ is trivial, which completes the proof. □

Indeed, in [5] we stated the simple closed k-surface structure of a connected sum of two simple closed k-surfaces (see Theorem 5.4 of [5]).

Corollary 5. *[8] Given two simple closed k-surfaces S_k and S'_k in $\mathbb{Z}^3$, $S_k \sharp S'_k$ is a simple closed k-surface in $\mathbb{Z}^3$.*

Theorem 5. *The n-times of connected sums of MSS_6, $\mathcal{C}^n_6 := \overbrace{MSS_6 \sharp \cdots \sharp MSS_6}^{n\text{-times}}$, is simply 6-connected.*

Proof. For convenience, for $\mathcal{C}^n_6 := \overbrace{MSS_6 \sharp \cdots \sharp MSS_6}^{n\text{-times}}$, using a method similar to the proof of the triviality of $\pi^6_1(MSS_6)$, since any 6-loop on $\mathcal{C}^n_6$ is proved to be 6-null homotopic in $\mathcal{C}^n_6$ by using a trivial extension, we obtain that $\pi^6_1(\mathcal{C}^n_6)$ is trivial. Besides, since $\mathcal{C}^n_6$ is 6-connected, the proof is completed. □

Since MSS_6 is not 6-contractible, we obtain the following:

Remark 9. *The connected sum $\mathcal{C}^n_6$ is not 6-contractible.*

Let us now prove the 18-contractibility of $\mathcal{C}^n_{18}$, $n \in \mathbb{N}$, as follows:

Theorem 6. *The n-times of connected sums of MSS_{18}, $\mathcal{C}^n_{18} := \overbrace{MSS_{18} \sharp \cdots \sharp MSS_{18}}^{n\text{-times}}$, is 18-contractible.*

Before proving the assertion, as mentioned in (Case 1) of Theorem 4, at the moment we may only deal with the case $MSS_{18} \sharp MSS_{18} \neq MSS_{18}$ because MSS_{18} is 18-contractible (see the 18-homotopy of (9) of [8] and Figure 2b of [8]).

Proof. Let us prove the assertion using the mathematical induction.
(Step 1) A paper [8] proved that $\mathcal{C}^1_{18} := MSS_{18}$ is 18-contractible (Remark 3 or the 18-homotopy of (9) proposed at the just above of Remark 2 of [8]).
(Step 2) For any $n \in \mathbb{N}$, assume that $\mathcal{C}^n_{18}$ is 18-contractible.

Let us now prove that $\mathcal{C}_{18}^{n+1}$ is 18-contractible. Owing to the 18-contractibility of $\mathcal{C}_{18}^{n}$, for some $m \in \mathbb{N}$, we may assume an 18-homotopy

$$H : \mathcal{C}_{18}^{n} \times [0, m]_{\mathbb{Z}} \to \mathcal{C}_{18}^{n} \tag{13}$$

supporting

$$1_{\mathcal{C}_{18}^{n}} \simeq_{18} c_{\{x_0\}}$$

for a certain point $x_0 \in \mathcal{C}_{18}^{n}$.
As usual, let

$$\mathcal{C}_{18}^{n+1} := \mathcal{C}_{18}^{n} \sharp MSS_{18}. \tag{14}$$

At the moment we should assume that the point x_0 is not deleted in the process of (14). Then we now establish a map

$$H' : \mathcal{C}_{18}^{n+1} \times [0, m+m']_{\mathbb{Z}} \to \mathcal{C}_{18}^{n+1}, m' \geq 1 \tag{15}$$

such that the restriction of H' of (15) to the set $B := \mathcal{C}_{18}^{n+1} \setminus MSS_{18}$ is equal to the 18-homotopy H of (13) on B, where this MSS_{18} is that of (14). Besides, we may assume $x_0 \in B$ and the singleton $\{x_0\}$ is that of (13). We now need only consider the remaining part $\mathcal{C}_{18}^{n+1} \setminus \mathcal{C}_{18}^{n}$ (see the right part of the dotted arrow of Figure 5b). Using a method of the 18-contractibility of MSS_{18} combined with the given 18-homotopy H of (13) (see Figure 5b), we finally have an 18-homotopy H' on $\mathcal{C}_{18}^{n+1}$ as in (15) supporting

$$1_{\mathcal{C}_{18}^{n+1}} \simeq_{18} c_{\{x_0\}}$$

for a the point $x_0 \in \mathcal{C}_{18}^{n+1}$ (see the right part of Figure 5b shown by using the bold dotted arrow or the dotted ones). $\square$

To explain the process of the proof of Theorem 6.7, motivated by the 18-contractibility of MSS_{18} (see Lemma 1 and Figure 2 of [8]), we now consider the following:

Corollary 6. *$\mathcal{C}_{18}^{2}$ is 18-contractible.*

Proof. Let us consider the map (see Figure 6)

$$H : \mathcal{C}_{18}^{2} \times [0, 4]_{\mathbb{Z}} \to \mathcal{C}_{18}^{2} \tag{16}$$

defined by

$$H(x, 0) = 1_{\mathcal{C}_{18}^{2}}(x), x \in \mathcal{C}_{18}^{2}.$$

$$H(x,1) = \begin{Bmatrix} 5, x \in \{1,5\}; \\ 12, x \in \{4,11,12\}; \\ 6, x \in \{2,6\}; \\ 13, x \in \{10,13\}; \\ 7, x \in \{3,7\}; \text{and} \\ 14, x \in \{8,9,14\}. \end{Bmatrix}$$

$$H(x,2) = \begin{Bmatrix} 12, x \in \{1,4,5,11,12\} \\ 13, x \in \{2,6,10,13\}; \text{and} \\ 14, x \in \{3,7,8,9,14\}. \end{Bmatrix}$$

$$H(x,3) = \begin{Bmatrix} 12, x \in \{1,2,4,5,6,10,11,12,13\}; \text{and} \\ 13, x \in \{3,7,8,9,14\}. \end{Bmatrix}$$

$$H(x,4) = c_{\{12\}}(x), x \in \mathcal{C}^2_{18}.$$

Then the map of (16) is an 18-homotopy making $\mathcal{C}^2_{18}$ 18-contractible, i.e., $1_{\mathcal{C}^2_{18}} \simeq_{18} c_{\{12\}}$. $\square$

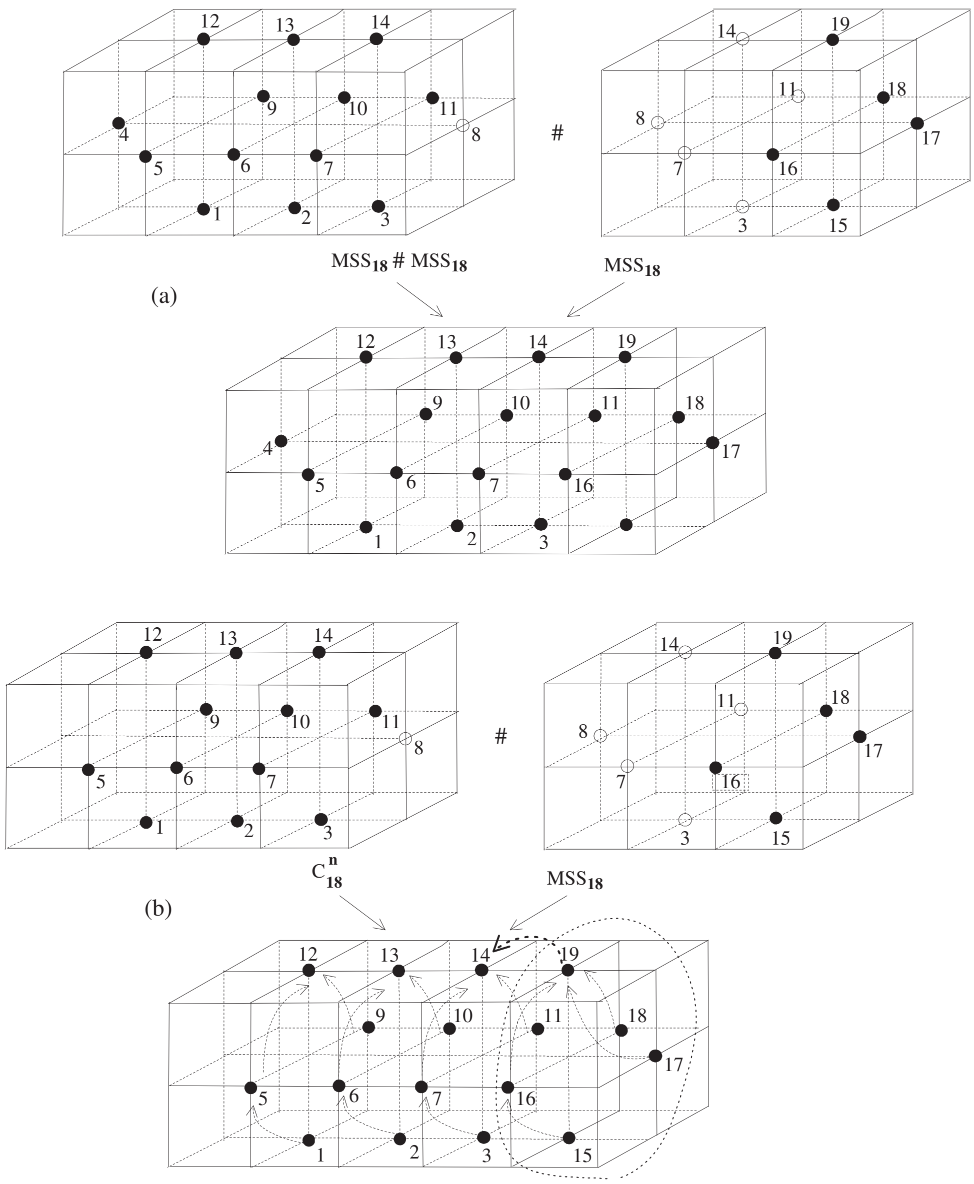

Figure 5. (**a**) Explanation of the process of establishing $\mathcal{C}^3_{18} := \mathcal{C}^2_{18} \sharp MSS_{18}$. (**b**) Configuration of an 18-homotopy $H : \mathcal{C}^{n+1}_{18} \times [0, m+m']_{\mathbb{Z}} \to \mathcal{C}^{n+1}_{18}, m' \geq 1$.

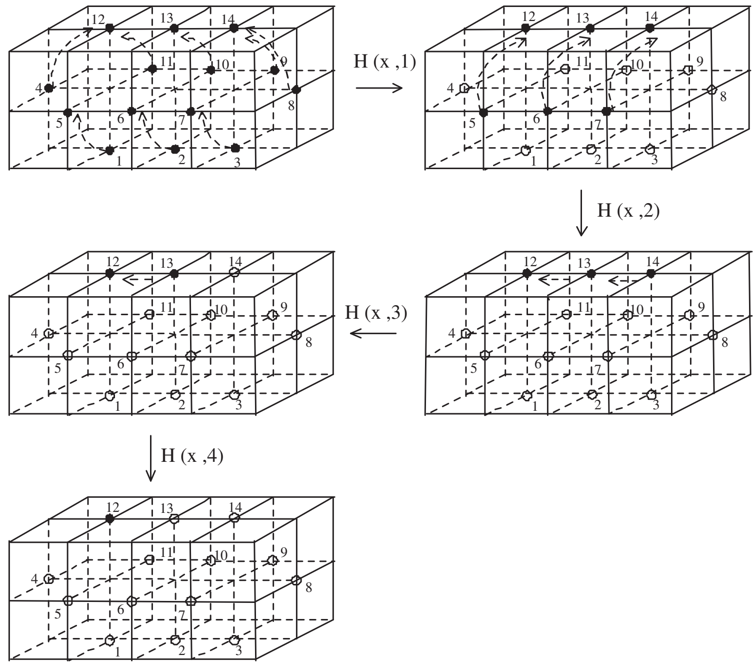

Figure 6. Configuration of the 18-homotopy of (16) involving the 18-contractibility of $\mathcal{C}^2_{18}$ (see the proof of Corollary 6).

Corollary 7. *The n-times of connected sum of* MSS'_{26}*, denoted by* $\mathcal{C}^n_{26}$*, is* 26*-contractible.*

Proof. Since there is only one type of $MSS'_{26}\sharp MSS'_{26} = MSS'_{26}$, $\mathcal{C}^n_{26}$ is equal to MSS'_{26} which is 26-contractible, the proof is completed. □

7. Non-almost Fixed Point Property of $\mathcal{C}^n_k, k \in \{6, 18\}$

This section investigates if each of $\mathcal{C}^2_6$ and $\mathcal{C}^n_{18}$ has the AFPP. In order to address the problems proposed with (Q6)–(Q8), let us now recall the category of digital topological spaces and further, the fixed point property and the almost fixed point property from the viewpoint of digital topology.

- We denote by DTC the category consisting of two data: The set of digital images (X, k) as $Ob(DTC)$ and the set of (k_0, k_1)-continuous maps between every pair of digital images (X, k_0) and (Y, k_1) in $Ob(DTC)$ as $Mor(DTC)$ [18].
- We say that a digital image (X, k) in $\mathbb{Z}^n$ has the fixed point property (for short FPP) [23] if for every k-continuous map $f : (X, k) \to (X, k)$ there is a point $x \in X$ such that $f(x) = x$.

Due to the study of the non-FPP of a digital picture (or digital image) in [23](see Theorem 4.1 of [23]), it is clear that only the digital image (or a digital picture) (X, k) with $|X| = 1$ has the FPP because a singleton set obviously has the FPP in DTC. Thus we need to recall the following (see Theorem 4.1 of [23] and Remark 4.3 of [34]):

Remark 10. *[23,34] Only a digital image* (X, k) *with* $|X| = 1$ *has the FPP.*

This property is obviously a certain implication of Theorems 3.3 and 4.1 of [23]. For the convenience of readers, we now confirm the assertion more precisely.

Proof. To wit the assertion, when establishing the notion of *AFPP* in [23] (see the bottom of the page 179 of [23]), we obviously find that Rosenfeld [23] stated two theorems such as Theorems 3.3 and 4.1 of [23] relating to the above assertion. More precisely, as mentioned in the above part (see the part just below Section 4 of [23]), a paper [23] finally mentioned the AFPP of an n-dimensional digital picture $(I^n, 3^n - 1)$ or a general picture $(X, 3^n - 1)$ in $\mathbb{Z}^n$. For instance, for the case of $([a,b]_{\mathbb{Z}}, 2), a \neq b$, Rosenfeld [23] proved the AFPP of it (see Theorem 3.3 of [23]). To be precise, for any 2-continuous self-map f of $([a,b]_{\mathbb{Z}}, 2)$, it turns out that $([a,b]_{\mathbb{Z}}, 2)$ has the AFPP instead of the FPP. Then, Theorem 3.3 implies that not every 2-continuous self-map f of $([a,b]_{\mathbb{Z}}, 2)$ support the FPP of it. However, the assertion supports the *AFPP* of $([a,b]_{\mathbb{Z}}, 2)$ instead of the *FPP*. Obviously, take a point $x \in [a,b]_{\mathbb{Z}}$ and $N_2(x,1) \subset [a,b]_{\mathbb{Z}}$. Then consider any point $x'(\neq x) \in N_2(x,1)$ and further, according to Theorem 3.3 of [23], consider a self-map f of $([a,b]_{\mathbb{Z}}, 2)$ defined by $f(t) = x$ for all $t(\neq x) \in [a,b]_{\mathbb{Z}}$, and $f(x) = x'$. Then, the map f is obviously 2-continuous and f implies that $([a,b]_{\mathbb{Z}}, 2)$ does not have the *FPP*. As a good example, consider a simple digital interval $([0,1]_{\mathbb{Z}}, 2)$ and consider the self-map f of it, say $f(0) = 1$ and $f(1) = 0$ which supports Theorem 3.3 of [23], which implies the *AFPP* of it instead of the *FPP*. Similarly, as mentioned in the beginning part of Section 4 of [23], the paper [23] proved that the n-dimensional case $(I^n, 3^n - 1)$ or a general picture $(X, 3^n - 1)$ in $\mathbb{Z}^n$ (see Theorem 4.1 of [23]) has the *AFPP* instead of the *FPP*. Eventually, with the same method as above, for any general digital image (X, k) in $\mathbb{Z}^n$, we confirm the assertion of Remark 10. □

Owing to Remark 10, it turns out that the study of the *FPP* in *DTC* is very trivial. Henceforth, Rosenfeld [23] firstly studied the almost fixed point property for digital images. Hence we need to stress the *AFPP* in *DTC*.

- We say that a digital image (X, k) in $\mathbb{Z}^n$ has the *almost fixed point property* (for short *AFPP*) [23] if for every k-continuous self-map f of (X, k), there is a point $x \in X$ such that $f(x) = x$ or $f(x)$ is k-adjacent to x.

Furthermore, a paper [8] proved that each of MSS_{18} and MSS'_{18} does not have the *AFPP* (see Theorem 7 below). Thus the study of the *AFPP* of $C^n_k, n \in \mathbb{N} \setminus \{1\}, k \in \{6, 18\}$ remains. Let us now address this issue.

Theorem 7. *[8] (1) MSS_{18} does not have the AFPP.*
(2) MSS'_{18} does not have the AFPP.

For $C^n_6 := \overbrace{MSS_6 \sharp \cdots \sharp MSS_6}^{n\text{-times}}$ and $C^n_{18} := \overbrace{MSS_{18} \sharp \cdots \sharp MSS_{18}}^{n\text{-times}}$, motivated by Theorem 7, we may impose the following queries involving the *AFPP* of C^n_6 and C^n_{18}.
(Q6) How about the *AFPP* of $C^n_6, n \in \mathbb{N}$?
(Q7) How about the *AFPP* of $C^n_{18}, n \in \mathbb{N}$?
To address these two queries, we first prove the non-*AFPP* of MSS_6, as follows:

Lemma 4. *MSS_6 does not have the AFPP.*

Proof. Consider the set MSS_6 in Figure 7a(1). Then, let f be a self-map of MSS_6 which is the composite of the three times reflections of MSS_6 according to the three xy-, yz-, and xz-planes in $\mathbb{R}^3$ (see the image of the map f on the set MSS_6 of Figure 7a(2)). Whereas the map f of Figure 7a is obviously a 6-continuous self-bijection of MSS_6, it does not support the *AFPP* of MSS_6. □

Theorem 8. *The digital image C^2_6 in the binary picture $(\mathbb{Z}^3, 6, 26, C^2_6)$ does not have the AFPP.*

Before proving the assertion, due to Lemma 1, we recall that C^2_6 uniquely exists up to 6-isomorphism.

Proof. Consider the set C^2_6 in Figure 7a(2). Then assume a self-map g of C^2_6 which is the composite of the three times reflections of MSS_6 according to the three xy-, yz-, and xz-planes in $\mathbb{R}^3$ (see the image of the map g of C_6^2 in Figure 7a(2)). Whereas the map g is obviously a 6-continuous bijection, it does not support the *AFPP* of C_6^2. $\square$

Corollary 8. *Let C_6^n be assumed as the set formulated via the method suggested in Figure 3b(1). The image C_6^n in the binary picture $(\mathbb{Z}^3, 6, 26, C_6^n)$ does not have the AFPP.*

As a generalization of the non-*AFPP* of MSS_{18} referred to in Theorem 7, we obtain the following:

Theorem 9. *The digital image C_{18}^n in the binary picture $(\mathbb{Z}^3, 18, 6, C_{18}^n)$ does not have the AFPP.*

Proof. (Case 1) In case $MSS_{18} \sharp MSS_{18} = MSS_{18}$, we observe that $C_{18}^n = MSS_{18}$. To be specific, by Theorem 7, we obtain $C_{18}^n := C_{18}^{n-1} \sharp MSS_{18}$ does not have the *AFPP* in *DTC*.

(Case 2) In case $MSS_{18} \sharp MSS_{18} \neq MSS_{18}$, let us now prove the non-*AFPP* of C_{18}^n. With the hypothesis, by Theorem 4, we see that C_{18}^n has the shape suggested in Figure 7c (just an example for C_{18}^2 in Figure 7c). Then, let h be a self-map of C_{18}^n which is the composite of the three times reflections of C_{18}^n according to the xy-, yz-, and xz-planes in $\mathbb{R}^3$. Whereas the map h is obviously an 18-continuous map, it does not support the *AFPP* of C_{18}^n. $\square$

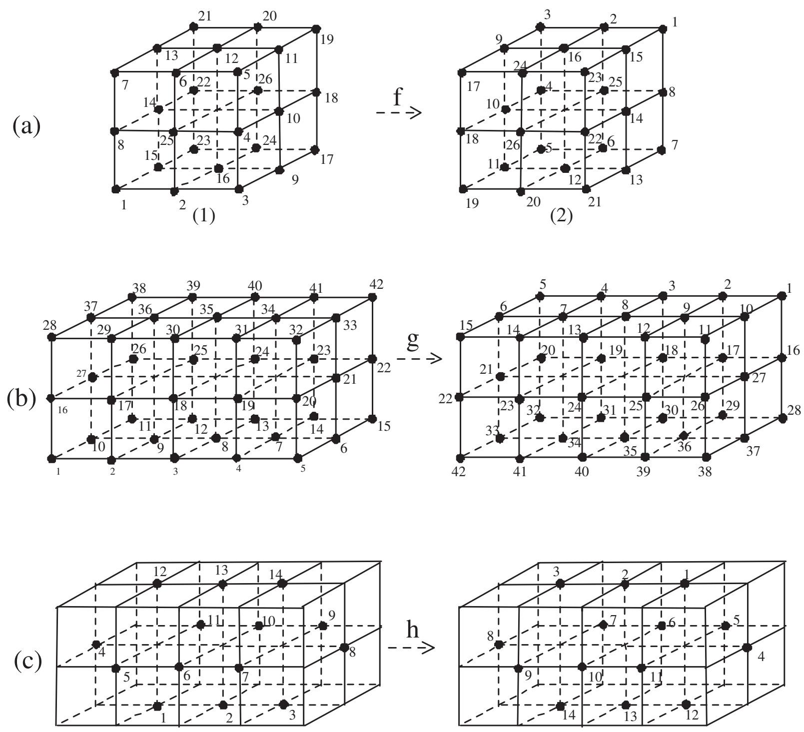

Figure 7. (**a**) Configuration of the *AFPP* of MSS_6. (**b**) Configuration of the non-*AFPP* of $C_6^2 := MSS_6 \sharp MSS_6$. (**c**) In case $MSS_{18} \sharp MSS_{18} \neq MSS_{18}$, configuration of the non-*AFPP* of C_{18}^2.

In order to generalize Theorem 9, we need the following notion which is stronger than the isomorphism of Definition 1.

Definition 8. *We say that a closed k-surface S_k in the picture $(\mathbb{Z}^3, k, \bar{k}, S_k)$ is $(k, \bar{k})$-isomorphic to (X, k) in the picture $(\mathbb{Z}^3, k, \bar{k}, X)$, $k \in \{6, 18, 26\}$ if*
(1) $S_k (\subset \mathbb{Z}^3)$ is k-isomorphic to (X, k) and
(2) $(\mathbb{Z}^3 \setminus S_k, \bar{k})$ is $\bar{k}$-isomorphic to $(\mathbb{Z}^3 \setminus X, \bar{k})$.

Remark 11. *Comparing the isomorphism of Definition 1 and that of Definition 8, we observe that they are different.*

As a generalization of Theorems 8 and 9, and Corollary 8, we obtain the following:

Proposition 4. *Consider a (simple) closed k-surface S_k in $(\mathbb{Z}^3, k, \bar{k}, S_k)$, $k \in \{6, 18, 26\}$ with the binary relations of (11). If it is $(k, \bar{k})$-isomorphic to (X, k) in the picture $(\mathbb{Z}^3, k, \bar{k}, X)$ and the set X is symmetric according to each of xy-, yz-, and xz-planes of $\mathbb{R}^3$, then S_k does not have the AFPP.*

Proof. With the hypothesis, we proceed with the following several steps for proving the assertion. For convenience we may assume $S_k := \{s_i \mid i \in [1, m]_{\mathbb{Z}}\}$ for some $m \in \mathbb{Z}$.

(Step 1) Take a $(k, \bar{k})$-isomorphism h from S_k to (X, k) in the given digital pictures (see Figure 8), where $X := \{x_i \mid i \in [1, m]_{\mathbb{Z}}, x_i := h(s_i)\}$. Namely, we may assume a $(k, \bar{k})$-isomorphism $h : S_k \to (X, k)$ defined by $h(s_i) = x_i, i \in [1, m]_{\mathbb{Z}}$.

(Step 2) Given the set (X, k), proceed to the composite of the three times of different reflections of (X, k) according to the certain xy-, yz-, and xz-planes in $\mathbb{R}^3$ which is a k-continuous bijection (or a k-isomorphism). Then we denote the composite with the self-map f of (X, k). For convenience, put $f(x_i) = x_j$, $i, j \in [1, m]_{\mathbb{Z}}$ and we see $i \neq j$.

(Step 3) We denote the digital image being proceeded with (Step 2) with (X', k), i.e., $f(X) := X' := \{x_j \mid x_j = f(x_i) \mid j \in [1, m]_{\mathbb{Z}}\}$. Then we see that the k-isomorphism f supports the non-*AFPP* (see the proof of Theorem 8). Indeed, although the set X' is equal to the set X, the subscript of each of all elements is completely changed from x_i to x_j, $i \neq j$.

(Step 4) After assigning each element $s_i \in S_k$ with $s_j \in S_k$ such that

$$s_j := h^{-1} \circ f^{-1}(x_i), i, j \in [1, m]_{\mathbb{Z}},$$

we obtain the set $S'_k := \{s_j \mid j \in [1, m]_{\mathbb{Z}}\}$. Indeed, although $S'_k = S_k$ as a set, we see that each element $s_i \in S_k$ is changed into another element $s_j \in S_k$. Consider the map $h' : (X(= X'), k) \to S_k(= S'_k)$ defined by

$$h'(x_j) = s_j \in S'_k = S_k, j \in [1, m]_{\mathbb{Z}}.$$

(Step 5) We finally obtain the composite of h, f, and h'(see Figure 8), i.e.,

$$h' \circ f \circ h : S_k \to S_k, \tag{17}$$

such that

$$h' \circ f \circ h(s_i) = h'(f(h(s_i))) = h'(f(x_i)) = h'(x_j) = s_j.$$

Finally, we see that the composite $h' \circ f \circ h$ is a certain k-continuous bijection (or a k-isomorphism) of S_k which does not support the *AFPP* of S_k. □

$$\begin{tikzcd}
S_k \arrow[r, dashed, "h"] \arrow[d, dashed] & (X, k) \arrow[d, dashed, "f"'] \\
S_k & (X, k) \arrow[l, dashed, "h'"']
\end{tikzcd}$$

Figure 8. Explanation of the composite $h' \circ f \circ h$.

Remark 12. *Proposition 4 includes the assertions of Theorems 7, 8, 9, and Lemma 4.*

8. Conclusions and Further Work

After formulating $\mathcal{C}_k^n$, $k \in \{6, 18, 26\}$, the present paper proved that there are only two types of connected sums $MSS_{18}\sharp MSS_{18}$ up to 18-isomorphism, only one type of $MSS_6\sharp MSS_6$ up to 6-isomorphism and further, several types of connected sums $\mathcal{C}_6^3 := MSS_6\sharp MSS_6\sharp MSS_6$. Furthermore, it turns out that there are several types of connected sums for $\mathcal{C}_6^3 := MSS_6\sharp MSS_6\sharp MSS_6$. Besides, in case $MSS_{18}\sharp MSS_{18} \neq MSS_{18}$ up to 18-isomorphism, we proved that $\mathcal{C}_{18}^n := \overbrace{MSS_{18}\sharp \cdots \sharp MSS_{18}}^{\text{n-times}}$ uniquely exists up to 18-isomorphism. In addition, we proved the digital k-contractibility of $\mathcal{C}_k^n := \overbrace{MSS_k\sharp \cdots \sharp MSS_k}^{\text{n-times}}, k \in \{18, 26\}$ and further, the simply k-connectedness of $\mathcal{C}_k^n$, $k \in \{6, 18, 26\}$, $n \in \mathbb{N}$. Finally, we explored the non-*AFPP* of each of $\mathcal{C}_6^2$, $\mathcal{C}_{18}^n$ and $\mathcal{C}_{26}^n$. In view of several homotopic properties of MSS_6, MSS_{18}, MSS'_{18}, and MSS'_{26} and further, the non-*AFPP* of them and their connected sums, we obtain the following:

As a further work, based on Proposition 4, we need to further study the *AFPP* of $\mathcal{C}_6^n, n \in \mathbb{N} \setminus \{1, 2\}$ according to the processes associated with Figure 3b(2), (3), and (4). As mentioned above, some homotopic features of the models MSS_6, MSS_{18}, MSS'_{18}, MSS'_{26} play important roles in digital topology and digital geometry because each of them can be considered to be the typical sphere-like model in Euclidean topology. Hence, the features referred to in Figure 9 facilitate studying many objects involving *AFPP* for digital images. Furthermore, the notion of digital connected sum also plays a crucial role in digital geometry because it can contribute to formulating another surface from two given surfaces. Besides, using the new topological structures in [36], we can study the *FPP* and *AFPP* of S_k as subspaces of the newly-established topological structures. Finally, considering the geometric realization of a digital k-surface with an *SST*-structure in [37], we can deal with them from the viewpoint of computational geometry. In addition, after establishing a certain cone metric on a digital image [38–42], we need to further compare the current digital metric spaces using a length of simple k-path with cone metric spaces.

Digital closed k-surfaces	Digital k-contractibility	Simply k-connected	AFPP
$\mathbf{MSS_6}$	Non-6-contractibility	Simply 6-connected	NO
$\mathbf{MSS_{18}}$	18-contractibility	Simply 18-connected	NO
$\mathbf{MSS'_{18}}$	18-contractibility	Simply 18-connected	NO
$\mathbf{MSS'_{26}}$	26-contractibility	Simply 26-connected	NO
$\mathbf{C_6^2}$	Non-6-contractibility	Simply 6-connected	NO
$\mathbf{C_{18}^n}$	18-contractibility	Simply 18-connected	NO

Figure 9. Digital topological properties of the non-*AFPP* of the minimal simple closed k-surfaces MSS_6, MSS_{18}, MSS'_{18}, MSS'_{26}, $\mathcal{C}_6^2$, and $\mathcal{C}_{18}^n$.

Funding: The author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2019R1I1A3A03059103). In addition, this research was supported by "Research Base Construction Fund Support Program funded by Jeonbuk National University in 2019".

Conflicts of Interest: The author declares no conflict of interest.

References

1. Bertrand, G. Simple points, topological numbers and geodesic neighborhoods in cubic grids. *Pattern Recognit. Lett.* **1994**, *15*, 1003–1011. [CrossRef]
2. Bertrand, G.; Malgouyres, M. Some topological properties of discrete surfaces. *J. Math. Imaging Vis.* **1999**, *20*, 207–221. [CrossRef]
3. Morgenthaler, D.G.; Rosenfeld, A. Surfaces in three dimensional digital images. *Inf. Control.* **1981**, *51*, 227–247. [CrossRef]
4. Han, S.-E. *Discrete Homotopy of a Closed k-Surface*; LNCS 4040; Springer: Berlin, Germany, 2006; pp. 214–225.
5. Han, S.-E. Connected sum of digital closed surfaces. *Inf. Sci.* **2006**, *176*, 332–348. [CrossRef]
6. Han, S.-E. Minimal simple closed 18-surfaces and a topological preservation of 3D surfaces. *Inf. Sci.* **2006**, *176*, 120–134. [CrossRef]
7. Han, S.-E. Digital fundamental group and Euler characteristic of a connected sum of digital closed surfaces. *Inf. Sci.* **2007**, *177*, 3314–3326. [CrossRef]
8. Han, S.-E. Fixed point theory for digital k-surfaces and some remarks on the Euler characteristics of digital closed surfaces. *Mathematics* **2019**, *7*, 1244. [CrossRef]
9. Malgouyres, R.; Bertran, G.D. A new local property of strong n-surfaces. *Pattern Recognit. Lett.* **1999**, *20*, 417–428. [CrossRef]
10. Malgouyres, R.; Lenoir, A. Topology preservation within digital surfaces. *Graph. Model.* **2000**, *62*, 71–84. [CrossRef]
11. Favacchio, G.; Guardo, E. The minimal free resolution of fat almost complete intersections in $\mathbb{P}^1 \times \mathbb{P}^1$. *Can. J. Math.* **2017**, *69*, 1274–1291. [CrossRef]
12. Favacchio, G.; Migliore, J. Multiprojective spaces and the arithmetically Cohen-Macaulay property. *Math. Proc. Camb. Philos. Soc.* **2019**, *166*, 583–597. [CrossRef]
13. Guardo, E.; Tuyl, A.V. ACM sets of points in multiprojective space. *Collect. Math.* **2008**, *59*, 191–213. [CrossRef]
14. Massey, W.S. *Algebraic Topology*; Springer: New York, NY, USA, 1977.
15. Rosenfeld, A.; Klette, R. Digital geometry. *Inf. Sci.* **2002**, *148*, 123–127. [CrossRef]
16. Han, S.-E. Remarks on the preservation of the almost fixed point property involoving serveral types of digitizations. *Mathematics* **2019**, *7*, 954. [CrossRef]
17. Han, S.-E. Non-product property of the digital fundamental group. *Inf. Sci.* **2005**, *171*, 73–91. [CrossRef]
18. Han, S.-E. Equivalent (k_0, k_1)-covering and generalized digital lifting. *Inf. Sci.* **2008**, *178*, 550–561. [CrossRef]
19. Kong, T.Y.; Rosenfeld, A. Digital topology: Introduction and survey. *Comput. Vis. Graph. Image Process.* **1989**, *48*, 357–393. [CrossRef]
20. Rosenfeld, A. Digital topology. *Am. Math. Mon.* **1979**, *86*, 76–87. [CrossRef]
21. Han, S.-E. The k-homotopic thinning and a torus-like digital image in $\mathbb{Z}^n$. *J. Math. Imaging Vis.* **2008**, *31*, 1–16. [CrossRef]
22. Han, S.-E. Estimation of the complexity of a digital image from the viewpoint of fixed point theory. *Appl. Math. Comput.* **2019**, *347*, 236–248. [CrossRef]
23. Rosenfeld, A. Continuous functions on digital pictures. *Pattern Recognit. Lett.* **1986**, *4*, 177–184. [CrossRef]
24. Han, S.-E. Digital coverings and their applications. *J. Appl. Math. Comput.* **2005**, *18*, 487–495.
25. Han, S.-E. Cartesian product of the universal covering property. *Acta Appl. Math.* **2009**, *108*, 363–383. [CrossRef]
26. Han, S.-E. Multiplicative property of the digital fundamental group. *Acta Appl. Math.* **2010**, *110*, 921–944. [CrossRef]
27. Han, S.-E. On the simplicial complex stemmed from a digital graph. *Honam Math. J.* **2005**, *27*, 115–129.

28. Boxer, L. A classical construction for the digital fundamental group. *J. Math. Imaging Vis.* **1999**, *10*, 51–62. [CrossRef]
29. Khalimsky, E. Motion, deformation, and homotopy in finite spaces. In Proceedings of the IEEE International Conferences on Systems, Man, and Cybernetics, Alexandria, VA, USA, 20–23 October 1987; pp. 227–234.
30. Han, S.-E. Strong k-deformation retract and its applications. *J. Korean Math. Soc.* **2007**, *44*, 1479–1503. [CrossRef]
31. Han, S.-E. On the classification of the digital images up to a digital homotopy equivalence. *J. Comput. Commun. Res.* **2000**, *10*, 194–207.
32. Han, S.-E.; Park, B.G. Digital Graph (k_0, k_1)-Homotopy Equivalence and Its Applications. 2003. Available online: http://atlas-conferences.com/c/a/k/b/35.htm (accessed on 20 July 2003).
33. Han, S.-E. Comparison among digital fundamental groups and its applications. *Inf. Sci.* **2008**, *178*, 2091–2104. [CrossRef]
34. Han, S.-E. Fixed point theorems for digital images. *Honam Math. J.* **2015**, *37*, 595–608. [CrossRef]
35. Kang, J.-M.; Han, S.-E.; Lee, S. The fixed point property of non-retractible topological spaces. *Mathematics* **2019**, *7*, 879. [CrossRef]
36. Han, S.-E.; Jafari, S.; Kang, J.M. Topologies on $\mathbb{Z}^n$ which are not homeomorphic to the n-dimensional Khalimsky topological space. *Mathematics* **2019**, *7*, 1072 . [CrossRef]
37. Han, S.-E. Jordan surface theorem for simple closed SST-surfaces. *Topol. Its Appl.* **2020**, *272*, 106953. [CrossRef]
38. Aleksić, S.; Kadelburg, Z.; Mitrović, Z.D.; Radenović, S. A new survey; Cone metric spaces. *J. Int. Math. Virtual Inst.* **2019**, *9*, 93–121.
39. Ćirić, L. *Some Recent Results in Metircal Fixed Point Theory*; University of Belgrade: Beograd, Serbia, 2003.
40. Vuorinen, V.M.V. On quasiconformal maps with identigy boudanary values. *Trans. Am. Math. Soc.* **2011**, *363*, 2467–2479.
41. Radenović, S.; Rhoader, B.E. Fixed point theorem for two non-self mappings in cone metric spaces. *Comput. Math. Appl.* **2009**, *57*, 1701–1707. [CrossRef]
42. Todorčević, V. *Harmonic Quasiconformal Mapings and Hyperbolic Type Metrics*; Springer Natur: Cham, Switzerland, 2019.