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Review

On Row Sequences of Hermite–Padé Approximation and Its Generalizations

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Abstract: Hermite–Padé approximation has been a mainstay of approximation theory since the concept was introduced by Charles Hermite in his proof of the transcendence of e in 1873. This subject occupies a large place in the literature and it has applications in different subjects. Most of the studies of Hermite–Padé approximation have mainly concentrated on diagonal sequences. Recently, there were some significant contributions in the direction of row sequences of Type II Hermite–Padé approximation. Moreover, various generalizations of Type II Hermite–Padé approximation were introduced and studied on row sequences. The purpose of this paper is to reflect the current state of the study of Type II Hermite–Padé approximation and its generalizations on row sequences. In particular, we focus on the relationship between the convergence of zeros of the common denominators of such approximants and singularities of the vector of approximated functions. Some conjectures concerning these studies are posed.

Keywords: Montessus de Ballore theorem; orthogonal polynomials; Faber polynomials; simultaneous Padé approximation; Hermite–Padé approximation; multipoint Padé approximation; inverse type results

1. Introduction

Charles Hermite [1] was the first who introduced the idea of Hermite–Padé approximation. Particularly, he used it for systems of exponential functions to prove that e is transcendental. A formal study of Hermite–Padé approximation for general systems of functions was initiated by Mahler [2] (see also the papers by his students, Coates and Jager [3,4], for important results in this regard). There are basically two different types of Hermite–Padé approximation, namely Types I and II. However, the combination of both (called mixed type) is possible. In this paper, we primarily consider Type II.

Hermite–Padé approximation and its relatives have applications in various areas, for example, in number theory (see [1,5–9]), numerical analysis (see [10–18]), multiple orthogonal polynomials (see [18–21]), linear algebraic equations (see [22]), nonlinear dynamical systems (see [23]), Brownian motion (see [24]), in random matrices (see [19,25]), Gibbs phenomenon (see [26]), and Lie algebra solution of differential equations (see [27]). In addition to the proof of the transcendence of e, Hermite–Padé approximation was used in various irrationality and transcendence proofs of important numbers (see, e.g., [1,5–8]). Moreover, one can say that the theory of multiple orthogonal polynomials originated from Hermite–Padé approximation (see, e.g., Section 2.2 of [28] for the explanation).

An optimal choice of the coefficients of the denominators and numerators of Hermite–Padé approximants makes it an important tool to study analytic continuation of functions and localization of their singularities. We can make this point more clearly by considering classical Padé approximation (scalar Hermite–Padé approximation) stated as follows.

Given a formal Taylor series at the origin

$$F(z) = \sum_{k=0}^{\infty} f_k z^k, \tag{1}$$

for any integers $n, m \geq 0$, we can find polynomials $P \in \mathbb{P}_n$ and polynomials $Q \in \mathbb{P}_m$, $Q \not\equiv 0$, such that

$$(QF - P)(z) = \mathcal{O}(z^{n+m+1}), \qquad \text{as} \quad z \to 0$$

($\mathbb{P}_n$ is the set of all polynomials of degree at most n). The rational function

$$R_{n,m} := \frac{P}{Q} = \frac{P_{n,m}}{Q_{n,m}} \tag{2}$$

is uniquely defined and is called the (n, m) *classical Padé approximant of* F. The polynomials $P_{n,m}$ and $Q_{n,m}$ in Equation (2) are selected so that $Q_{n,m}$ is monic and $\gcd(P_{n,m}, Q_{n,m}) = 1$. For a function F as in Equation (1), we denote by $R_0(F)$ the radius of the largest open disk at the origin to which F can be extended analytically and by $R_m(F)$ the radius of the largest open disk at the origin to which F can be extended so that F has at most m poles counting multiplicities. Set

$$\mathbb{B}_R := \{z \in \mathbb{C} : |z| < R\}.$$

By $Q_m(F)$, we denote the monic polynomial whose zeros are the poles of F in $\mathbb{B}_{R_m(F)}$ counting multiplicities. The set of all distinct zeros of $Q_m(F)$ is denoted by $\mathcal{P}_m(F)$.

Analytic continuation and locations of poles including their multiplicities of $F(z) = \sum_{k=0}^{\infty} f_k z^k$ in $\mathbb{B}_{R_m(F)}$ when F has exactly m poles in $\mathbb{B}_{R_m(F)}$ can be completely described by the convergences of $\{R_{n,m}\}_{n\geq 0}$ and $\{Q_{n,m}\}_{n\geq 0}$ in the following theorem (see [29,30]).

Theorem 1 (Montessus de Ballore–Gonchar's Theorem)**.** *Let F be defined as in Equation (1) and fix $m \in \mathbb{N}$. Then, the following statements are equivalent:*

(a) *$R_0(F) > 0$ and F has exactly m poles in $\mathbb{B}_{R_m(F)}$ counting multiplicities.*

(b) *There is a polynomial Q_m of degree m, $Q_m(0) \neq 0$, such that the sequence of $\{Q_{n,m}\}_{n\geq 0}$ satisfies*

$$\limsup_{n\to\infty} \|Q_m - Q_{n,m}\|^{1/n} = \theta < 1,$$

where $\|\cdot\|$ denotes the coefficient norm in the space of all polynomials.

Moreover, if either (a) or (b) holds, then $Q_m = Q_m(F)$,

$$\theta = \frac{\max\{|\lambda| : \lambda \in \mathcal{P}_m(F)\}}{R_m(F)}, \tag{3}$$

and

$$\limsup_{n\to\infty} \|F - R_{n,m}\|_K^{1/n} = \frac{\|z\|_K}{R_m(F)}, \tag{4}$$

where K is any compact subset of $\mathbb{B}_{R_m(F)} \setminus \mathcal{P}_m(F)$ and $\|\cdot\|_K$ denotes the sup-norm on K.

In [29], Montessus de Ballore proved that (a) implies (b) with $Q_m = Q_m(F)$ and the inequalities with the sign "$\leq$" instead of "=" in Equations (3) and (4). This part of the above theorem is commonly known as Montessus de Ballore's theorem. The implication (b)$\Rightarrow$(a) with the inequalities with the sign "$\geq$" instead of "=" in Equations (3) and (4) was proved by Gonchar in [30]. In this current paper, we refer to this part as the inverse statement of Montessus de Ballore–Gonchar's Theorem. We also note that the sequence $\{R_{n,m}(z)\}_{n\geq 0}$ diverges at every point z, $|z| > R_m(F)$ (see Section 7 of [31]).

Later, Gonchar [31] studied the rate of attraction of each individual pole of F in $\mathbb{B}_{R_m(F)}$ to zeros of $Q_{n,m}$. He introduced several indicators describing the asymptotic behavior of the zeros of $Q_{n,m}$ to a point $\lambda \in \mathbb{C} \setminus \{0\}$. Set

$$|z - w|_1 := \min\{1, |z - w|\}, \qquad z, w \in \mathbb{C}.$$

Fix $m \in \mathbb{N}$ and a function F defined as in Equation (1). Let

$$\mathcal{P}_{n,m} := \{\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,m_n}\}, \qquad m_n \leq m, \qquad n \geq 0,$$

denote the collection of zeros of $Q_{n,m}$ repeated according to their multiplicity and enumerated in nondecreasing distance to the point λ. The first indicator is defined by

$$\Delta(\lambda) := \limsup_{n\to\infty} \prod_{j=1}^{m_n} |\lambda_{n,j} - \lambda|_1^{1/n} = \limsup_{n\to\infty} \prod_{|\lambda_{n,j}-\lambda|<1} |\lambda_{n,j} - \lambda|^{1/n}.$$

It is easy to check that $0 \leq \Delta(\lambda) \leq 1$ under the convention when $m_n = 0$ or $|\lambda_{n,j} - \lambda| \geq 1$ for all $j = 1, 2, \ldots, m_n$, the product is taken to be 1. The second and third indicators, nonnegative integers $\sigma(\lambda)$ and $\gamma(\lambda)$, are defined as follows. We say that $\gamma(\lambda) := \nu$ if

$$\lim_{n\to\infty} |\lambda_{n,\nu} - \lambda| = 0 \qquad \text{and} \qquad \limsup_{n\to\infty} |\lambda_{n,\nu+1} - \lambda| > 0$$

(for $\nu > m_n$ by convention $|\lambda_{n,\nu} - \lambda| := 1$, and when $\limsup_{n\to\infty} |\lambda_{n,1} - \lambda| > 0$, we take $\gamma(\lambda) = 0$). Similarly, $\sigma(\lambda) := \nu$ if

$$\limsup_{n\to\infty} |\lambda_{n,\nu} - \lambda|^{1/n} < 1 \qquad \text{and} \qquad \limsup_{n\to\infty} |\lambda_{n,\nu+1} - \lambda|^{1/n} \geq 1.$$

Moreover, we define

$$\delta_j(\lambda) := \limsup_{n\to\infty} |\lambda_{n,j} - \lambda|_1^{1/n}.$$

Clearly, $\gamma(\lambda) \geq \sigma(\lambda)$, and the statements $\Delta(\lambda) < 1$ and $\sigma(\lambda) \geq 1$ are equivalent.

The following theorem (see [31], Theorem 1) asserts that a pole of F of order ν in $\mathbb{B}_{R_m(F)}$ attracts with geometric rate exactly ν zeros of the polynomials $Q_{n,m}$.

Theorem 2 (Gonchar's theorem)**.** *Let F be defined as in Equation (1), $m \in \mathbb{N}$, and let $0 \neq \lambda \in \mathbb{C}$. The following two assertions are equivalent:*

(a) $\lambda \in \mathbb{B}_{R_m(F)}$ *and F has a pole at λ.*

(b) $\Delta(\lambda) < 1$.

If either (a) or (b) takes place, then

$$\Delta(\lambda) = \frac{|\lambda|}{R_m(F)} \qquad \text{and} \qquad \sigma(\lambda) = \nu,$$

where ν is the order of the pole at λ. Moreover, if we assume further that $\liminf_{n\to\infty} |\lambda - \lambda_{n,\nu+1}| > 0$, *then*

$$\delta_1(\lambda) = \delta_2(\lambda) = \ldots = \delta_\nu(\lambda) = \left(\frac{|\lambda|}{R_m(F)}\right)^{1/\nu}.$$

Naively, Gonchar asked what we can say about λ if λ attracts a certain numbers of zeros of the polynomials $Q_{n,m}$ without such geometric rate in the above theorem. In [31], Gonchar also proposed the following conjecture.

Theorem 3 (Gonchar's conjecture)**.** *Fix $m \in \mathbb{N}$. Let F be defined as in Equation (1). Assume that $\gamma(\lambda) \geq 1$ and $\lambda \neq 0$. Then, this series defines a function which is holomorphic at $z = 0$, $R_{m-1}(F) \geq |\lambda|$, and λ is a singularity of F.*

Gonchar's conjecture remains open. Special cases of the conjecture were proved by Vavilov, López, Prokhorov, and Suetin (see [32–35]). In the final progress, the following weaker version of Gonchar's conjecture was proved by Suetin [35].

Theorem 4 (Suetin's theorem)**.** *Assume that the formal power series $F(z) = \sum_{k=0}^{\infty} f_k z^k$ has coefficients such that for fixed $m \in \mathbb{N}$ and sufficiently large $n \in \mathbb{N}$ the approximants $R_{n,m}$ have precisely m finite poles $\lambda_{n,1}, \ldots, \lambda_{n,m}$, which are convergent:*

$$\lim_{n\to\infty} \lambda_{n,j} = \lambda_j \neq 0, \qquad j = 1, \ldots, m.$$

Then,

(i) The power series defines a holomorphic function F in the disk $\mathbb{B}_{R_{\min}}$, where $R_{\min} := \min_{1\leq j\leq m} |\lambda_j|$.
(ii) $R_{m-1}(F) = \max_{1\leq j\leq m} |\lambda_j|$.
(iii) All points $\lambda_1, \ldots, \lambda_m$ are singularities of F, the ones lying in the disk $\mathbb{B}_{R_{m-1}(F)}$ are poles, and F has no other poles in this disk.

It is easy to check that if the Taylor coefficients of $F(z) = \sum_{k=0}^{\infty} f_k z^k$ satisfy $f_k \neq 0$ and $f_{k+1} \neq 0$, thenf_k/f_{k+1} is a zero of $Q_{n,1}$. Therefore, when $m = 1$, Gonchar's conjecture and Suetin's theorem reduce to the following Fabry ratio theorem (see [36]).

Theorem 5 (Fabry ratio theorem)**.** *Suppose that the coefficients of a power series $F(z) = \sum_{k=0}^{\infty} f_k z^k$ are such that the limit*

$$\lim_{k\to\infty} \frac{f_k}{f_{k+1}} = \lambda \neq 0$$

exists. Then, the series converges uniformly on each compact subset of the disk $\mathbb{B}_{|\lambda|}$ and λ is a singular point of the function F.

Recently, extensions of Montessus de Ballore–Gonchar's theorem to Type II Hermite–Padé approximation and its generalizations were proved. As time progresses, there were many results on Type II Hermite–Padé approximation in the direction of Gonchar's conjecture. The purpose of this survey paper is to review all those results and collect open problems in this respect.

2. Hermite–Padé Approximation

2.1. Definition and Notation

Type II Hermite–Padé approximation involves the approximation of a vector of functions by a vector of rational functions with the same denominator. Let $\mathbf{F} = (F_1, F_2, \ldots, F_d)$ be a system of d formal Taylor expansions at the origin; that is, for each $\ell = 1, 2, \ldots, d$, we have

$$F_\ell(z) = \sum_{k=0}^{\infty} f_{k,\ell} z^k, \quad f_{k,\ell} \in \mathbb{C}. \tag{5}$$

In what follows, $\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{P}_n$ is the set of all polynomials of degree at most n, and for a given multi-index $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$, we define

$$|\mathbf{m}| := m_1 + m_2 + \cdots + m_d.$$

Definition 1. *Let $\mathbf{F} = (F_1, F_2, \ldots, F_d)$ be a system of d formal Taylor expansions as in Equation (5) and $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$ be a fixed multi-index. Then, for each $n \geq \max\{m_1, m_2, \ldots, m_d\}$, there exist $Q \in \mathbb{P}_{|\mathbf{m}|}$ and $P_\ell \in \mathbb{P}_{n-m_\ell}$ for all $\ell = 1, 2, \ldots, d$ such that $Q \not\equiv 0$ and*

$$Q(z)F_\ell(z) - P_\ell(z) = \mathcal{O}(z^{n+1}), \qquad \text{as } z \to 0. \tag{6}$$

The vector of rational functions $\mathbf{R}_{n,\mathbf{m}} = (P_1/Q, P_2/Q, \ldots, P_d/Q)$ is called an $(n, \mathbf{m})$ Type II Hermite–Padé (HP) approximant of $\mathbf{F}$.

Traditionally, the numbers of interpolation conditions of F_ℓ, for $\ell = 1, 2, \ldots, d$, in Equation (6) at 0 are selected to be the same, which is $n + 1$. Since $m_\ell = 0$ does not provide any advantage in locating singularities of F_ℓ, we can restrict our multi-indices $\mathbf{m} \in \mathbb{N}^d$ as stated in Definition 1. For any fixed $(n, \mathbf{m}) \in \mathbb{N} \times \mathbb{N}^d$, in general, $\mathbf{R}_{n,\mathbf{m}}$ may not be unique and we assume that, given $(n, \mathbf{m})$, one particular solution is assigned. We write

$$\mathbf{R}_{n,\mathbf{m}} = (R_{n,\mathbf{m},1}, R_{n,\mathbf{m},2}, \ldots, R_{n,\mathbf{m},d}) = (P_{n,\mathbf{m},1}, P_{n,\mathbf{m},2}, \ldots, P_{n,\mathbf{m},d})/Q_{n,\mathbf{m}}, \tag{7}$$

where $P_{n,\mathbf{m},\ell}$, $\ell = 1, 2, \ldots, d$, and $Q_{n,\mathbf{m}}$ are chosen so that $Q_{n,\mathbf{m}}$ is a monic polynomial that has no common zero with all the $P_{n,\mathbf{m},\ell}$. When $\mathbf{m}$ remains fixed, we call the sequences $\{\mathbf{R}_{n,\mathbf{m}}\}_{n \geq |\mathbf{m}|}$ $\mathbf{m}$ *row sequences*. When n and $\mathbf{m}$ have the relation,

$$m_1 = \cdots = m_d = m, \quad n = (d+1)m, \quad m \in \mathbb{N}$$

(or nearby configurations of multi-indices), the sequences $\{\mathbf{R}_{n,\mathbf{m}}\}$ are called *diagonal sequences*. Another construction called Type I HP approximants has very close algebraic relation to Type II HP approximants. For Type I HP approximation, one wants to approximate polynomial combination of the vector $(F_1, F_2, \ldots, F_d)$ by a polynomial. However, in Type II HP approximation, one wants to approximate F_ℓ separately by rational functions with the same denominator. For interested readers, we refer to Chapter 4 of [37] for more details about Type I HP approximation. Since we only provide the survey of the studies of Type II HP approximants, we omit the word "Type II" when we refer to Type II HP approximants. Moreover, we would like to emphasize that, for $d = 1$, the HP approximant $\mathbf{R}_{n,\mathbf{m}}$ reduces to the $(n - m, m)$ classical Padé approximant. However, because we are interested in the cases when m is fixed and $n \to \infty$, all theorems in Section 1 hold for the scalar HP approximation.

2.2. Results and Conjectures

In the direction of row sequences, the paper [38] by Graves-Morris and Saff was a pioneering result in the sense that it initiated a convergence theory for HP approximants. In [38], they proved a Montessus de Ballore type theorem for HP approximants under the concept of polewise independence. Later, Cacoq, de la Calle, and López [39] improved the results [38] in several ways; namely, improving the estimates on the rates of convergences of $\{\mathbf{R}_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$ and $\{Q_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$ and weakening the assumption of polewise independence. Note that in the Montessus de Ballore type theorem for HP approximants in [39], they found the exact rate of convergence of $\{\mathbf{R}_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$ but not of $\{Q_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$. A significant contribution in the direction of row sequences is due to Cacoq, de la Calle, and López [40] where they gave necessary and sufficient conditions for the convergence with geometric rate of $\{Q_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$ and calculated the exact rate of convergence of $\{Q_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$. To explain the results in [40], we need to state some definitions.

Definition 2. *Let $\mathbf{\Omega} := (\Omega_1, \Omega_2, \ldots, \Omega_d)$ be a system of domains such that, for each $\ell = 1, 2, \ldots, d$, F_ℓ is meromorphic in Ω_ℓ. We say that $\lambda \in \mathbb{C}$* is a pole of $\mathbf{F}$ in $\mathbf{\Omega}$ of order τ *if there exists an index $\ell \in \{1, 2, \ldots, d\}$ such that $\lambda \in \Omega_\ell$ is a pole of F_ℓ of order τ, and for $\beta \neq \ell$ either λ is a pole of F_β of order less than or equal to τ or $\lambda \notin \Omega_\beta$. When $\mathbf{\Omega} = (\Omega, \Omega, \ldots, \Omega)$, we say that λ* is a pole of $\mathbf{F}$ in $\mathbf{\Omega}$.

Let $R_0(\mathbf{F})$ be the radius of the largest disk $\mathbb{B}_{R_0(\mathbf{F})}$ to which all the expansions F_ℓ, $\ell = 1, 2, \ldots, d$ can be extended analytically. If $R_0(\mathbf{F}) = 0$, we take $\mathbb{B}_{R_m(\mathbf{F})} = \emptyset$, $m \geq 0$; otherwise, $R_m(\mathbf{F})$ is the radius of the largest disk $\mathbb{B}_{R_m(\mathbf{F})}$ centered at the origin to which all the analytic elements $(F_\ell, \mathbb{B}_{R_0(F_\ell)})$ can be extended so that $\mathbf{F}$ has at most m poles counting multiplicities.

To prove an analog of Montessus de Ballore–Gonchar's theorem for HP approximants, we have to decide what actually is the limit of $\{Q_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$. We found that the convergence of $\{Q_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$ for the vector case is more complicated than the one for the scalar case, as the following example shows.

Example 1. *Let $\mathbf{F} = (F_1, F_2)$, where*

$$F_1(z) = \frac{1}{z-2} + \sum_{k=0}^{\infty} \left(\frac{z}{3}\right)^{2^k} + \frac{1}{z-4}, \qquad F_2(z) = \frac{1}{z-2} + \sum_{k=0}^{\infty} \left(\frac{z}{3}\right)^{2^k}$$

and fix $\mathbf{m} = (1, 1)$. Define $\hat{\mathbf{F}} := (\hat{F}_1, \hat{F}_2)$, where

$$\hat{F}_1(z) = F_1(z) - F_2(z) = \frac{1}{z-4}, \qquad \hat{F}_2(z) = F_2(z) = \frac{1}{z-2} + \sum_{k=0}^{\infty} \left(\frac{z}{3}\right)^{2^k}$$

It is not difficult to check that HP approximants of $(\mathbf{F}, \mathbf{m})$ and $(\hat{\mathbf{F}}, \mathbf{m})$ have the same $Q_{n,\mathbf{m}}$. Applying Theorem 4.4 in [39] for $\hat{\mathbf{F}}$, 2 and 4 which are poles of $\mathbf{F}$ attract zeros of $Q_{n,\mathbf{m}}$ with geometric rate as $n \to \infty$. To be precise, we obtain

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}} - (z-2)(z-4)\|^{1/n} < 1.$$

However, since both F_1 and F_2 are meromorphic up to $\mathbb{B}_3$, 4 is not a pole of $\mathbf{F}$.

Based on the above example, the idea of pole in Definition 2 is not suitable when we study a system of functions. The authors of [40] proposed a new definition of pole (called "system pole") below.

Definition 3. *Given* $\mathbf{F} = (F_1, F_2, \ldots, F_d)$ *in as Equation (5) and* $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$, *we say that* $\lambda \in \mathbb{C} \setminus \{0\}$ *is a* system pole of order τ of $\mathbf{F}$ with respect to $\mathbf{m}$ *if* τ *is the largest positive integer such that for each* $t = 1, 2, \ldots, \tau$, *there exists at least one polynomial combination of the form*

$$\sum_{\ell=1}^{d} v_\ell F_\ell, \qquad \deg v_\ell < m_\ell, \quad \ell = 1, 2, \ldots, d, \tag{8}$$

which is holomorphic on a neighborhood of $\overline{\mathbb{B}_{|\lambda|}}$ *except for a pole at* $z = \lambda$ *of exact order* t.

Notice that in Definition 3, instead of examining poles of each function F_ℓ separately, we examine poles of the polynomial combinations of the vector $(F_1, F_2, \ldots, F_d)$. When $d = 1$, the statement $\mathbf{F}$ has a system pole of order τ with respect to $\mathbf{m} = m$ simply means that F has a pole of order τ in $\mathbb{B}_{R_m(F)}$. From Example 1, 2 and 4 are system poles of order 1 of $\mathbf{F}$ with respect to $\mathbf{m}$ but 4 is not a pole of $\mathbf{F}$. Conversely, the following example shows that a pole of $\mathbf{F}$ may not be a system pole.

Example 2. *Let* $\mathbf{F} = (F_1, F_2)$, *where*

$$F_1(z) = \frac{1}{z-1} + \frac{1}{z-2}, \qquad F_2(z) = \frac{1}{z-3}$$

and fix $\mathbf{m} = (1, 1)$. *Clearly,* 2 *is a pole of* $\mathbf{F}$. *Since* 1 *and* 3 *are only system poles of* $\mathbf{F}$, 2 *is not a system pole of* $\mathbf{F}$.

In conclusion, a system pole may not be a pole of $\mathbf{F}$ or vice versa. Note that a system $\mathbf{F}$ cannot have more than $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ counting their order (see Lemma 3.5 of [40]).

To state the main result (see Theorem 6 below) in [40], we need a generalization of the notion $R_m(F)$ for a system of functions $\mathbf{F}$. For each system pole λ of $\mathbf{F}$ with respect to $\mathbf{m}$, we want to define a corresponding characteristic number $R_\lambda(\mathbf{F}, \mathbf{m})$ as follows. Let τ be the order of λ as a system pole of $\mathbf{F}$. For each $t = 1, \ldots, \tau$, denote by $r_{\lambda,t}(\mathbf{F}, \mathbf{m})$ the largest of all the numbers $R_t(g)$ (the radius of the largest disk containing at most t poles of g), where g is a polynomial combination of type Equation (8) that is analytic on a neighborhood of $\overline{\mathbb{B}_{|\lambda|}}$ except for a pole at $z = \lambda$ of order t. Then, we define

$$R_\lambda(\mathbf{F}, \mathbf{m}) := \min_{k=1,2,\ldots,\tau} r_{\lambda,k}(\mathbf{F}, \mathbf{m}).$$

As in the scalar case, we denote by $Q_{\mathbf{m}}(\mathbf{F})$ the monic polynomial whose zeros are the system poles of $\mathbf{F}$ with respect to $\mathbf{m}$ taking account of their order. The set of distinct zeros of $Q_{\mathbf{m}}(\mathbf{F})$ is denoted by $\mathcal{P}(\mathbf{F}, \mathbf{m})$.

An analog of Montessus de Ballore–Gonchar's theorem for HP approximation (see Theorem 1.4 of [40]) is stated below.

Theorem 6. *Let* $\mathbf{F}$ *be a vector of formal Taylor expansions at the origin and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$. *Then, the following two assertions are equivalent:*

(a) $R_0(\mathbf{F}) > 0$ *and* $\mathbf{F}$ *has exactly* $|\mathbf{m}|$ *system poles with respect to* $\mathbf{m}$ *counting multiplicities.*

(b) *The denominators* $Q_{n,\mathbf{m}}$, $n \geq |\mathbf{m}|$, *of the HP approximants of* $\mathbf{F}$ *are uniquely determined for all sufficiently large* n, *and there exists a polynomial* $Q_{|\mathbf{m}|}$ *of degree* $|\mathbf{m}|$, $Q_{|\mathbf{m}|}(0) \neq 0$, *such that*

$$\limsup_{n\to\infty} \|Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}}\|^{1/n} = \theta < 1, \tag{9}$$

where $\|\cdot\|$ *denotes the coefficient norm in the space of polynomials.*

Moreover, if either (a) or (b) takes place, then $Q_{|\mathbf{m}|} \equiv Q_{\mathbf{m}}(\mathbf{F})$ *and*

$$\theta = \max\left\{ \frac{|\lambda|}{R_\lambda(\mathbf{F},\mathbf{m})} : \mathcal{P}(\mathbf{F},\mathbf{m}) \right\}.$$

An exact expression for the rate and region of convergence of $\{\mathbf{R}_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$ to $\mathbf{F}$ is given in Theorem 3.7 of [40]. Theorem 6 is considered to be a cornerstone in the study of HP approximation on row sequences in the sense that it contains the first inverse type result.

In the spirit of Gonchar's theorem and Gonchar's conjecture, López and Zaldivar [41] proposed the two conjectures below (Conjectures 1 and 2). Note that the notations Δ, σ, γ, and δ_j in Conjectures 1–3 and Theorem 7 are defined as in Section 1 taking

$$\mathcal{P}_{n,\mathbf{m}} := \{\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,u_n}\}, \qquad u_n \leq |\mathbf{m}|, \qquad n \in \mathbb{N},$$

to be the collection of zeros of $Q_{n,\mathbf{m}}$.

Conjecture 1. *Let* $\mathbf{F}$ *be a vector of formal Taylor expansions at the origin and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$. *If the denominators* $Q_{n,\mathbf{m}}$ *are uniquely determined for all sufficiently large* n *and* $\sigma(\lambda) \geq 1$, *then* λ *is a system pole of* $\mathbf{F}$ *with respect to* $\mathbf{m}$ *of order* $\tau = \sigma(\lambda)$.

The generalization of "singularity" when we consider a system of functions is stated below.

Definition 4. Given $\mathbf{F} = (F_1, F_2, \ldots, F_d)$ as in Equation (5) and $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$, we say that $\lambda \in \mathbb{C} \setminus \{0\}$ *is a system singularity of* $\mathbf{F}$ *with respect to* $\mathbf{m}$ if there exists at least one polynomial combination G of the form in Equation (8) analytic on $\mathbb{B}_{|\lambda|}$ and λ is a singular point of G.

Conjecture 2. *Let* $\mathbf{F}$ *be a vector of formal Taylor expansions at the origin and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$. *If the denominators* $Q_{n,\mathbf{m}}$ *are uniquely determined for all sufficiently large* n *and* $\gamma(\lambda) \geq 1$, *then* λ *is a system singularity of* $\mathbf{F}$ *with respect to* $\mathbf{m}$.

In ([41], p. 155), López and Zaldivar gave an example to support their conjectures. That example shows that a point in $\mathbb{C} \setminus \{0\}$ can be a system pole or a system singularity depending how it attracts zeros of $Q_{n,\mathbf{m}}$. For the self-contained purpose, we show their example here.

Example 3. *Consider* $F = (F_1, F_2)$, *where*

$$F_1(z) := \frac{1}{z-1} + e^z \qquad \text{and} \qquad F_2(z) := \log(z-1)$$

and $\mathbf{m} := (1,1)$. *Clear,* 1 *is a system pole of order* 1 *and a system singularity of* $\mathbf{F}$ *with respect to* $\mathbf{m}$. *Their experiment states that* $\deg(Q_{n,\mathbf{m}}) = 2$ *for* n *sufficiently large and*

$$\limsup_{n\to\infty} |\lambda_{n,1} - 1|^{1/n} = 0 \qquad \text{and} \qquad |\lambda_{n,2} - 1| \sim \frac{1}{n}, \qquad n \to \infty,$$

where $Q_{n,\mathbf{m}}(z) = (z - \lambda_{n,1})(z - \lambda_{n,2})$ *and* $|\lambda_{n,1} - 1| \leq |\lambda_{n,2} - 1|$.

Conjecture 1 for the scalar case ($d = 1$) is the part "(b)⇒(a)" in Gonchar's theorem. However, Conjecture 2 for the scalar case is Gonchar's conjecture which remains open. The converse statement of Conjecture 1 proved in Theorem 2.1 of [41] is stated as follows.

Theorem 7. *Let* $\mathbf{F}$ *be a vector of formal Taylor expansions at the origin and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$. *Assume that* λ *is a system pole of order* τ *of* $\mathbf{F}$ *with respect to* $\mathbf{m}$. *Then,*

$$\Delta(\lambda) \leq \frac{|\lambda|}{R_\lambda(\mathbf{F}, \mathbf{m})} \qquad \text{and} \qquad \sigma(\lambda) \geq \tau. \tag{10}$$

According to Gonchar's theorem, a natural conjecture is

Conjecture 3. *Let* $\mathbf{F}$ *be a vector of formal Taylor expansions at the origin and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$. *Assume that* λ *is a system pole of order* τ *of* $\mathbf{F}$ *with respect to* $\mathbf{m}$. *Then, the inequalities in Equation (10) are equalities. Moreover, if we assume further that* $\liminf_{n\to\infty} |\lambda - \lambda_{n,\tau+1}| > 0$, *then*

$$\delta_1(\lambda) = \delta_2(\lambda) = \ldots = \delta_\tau(\lambda) = \left(\frac{|\lambda|}{R_\lambda(\mathbf{F}, \mathbf{m})}\right)^{1/\tau}.$$

In the past few years, some progress (see [41–43]) on Conjecture 2 was made. In those papers, the authors studied Conjecture 2 with an additional assumption, namely,

$$\lim_{n\to\infty} Q_{n,\mathbf{m}} = Q_{|\mathbf{m}|}, \qquad \deg(Q_{|\mathbf{m}|}) = |\mathbf{m}|, \qquad Q_{|\mathbf{m}|}(0) \neq 0. \tag{11}$$

Without loss of generality, let $\lambda_1, \lambda_2, \ldots, \lambda_{|\mathbf{m}|}$ be the zeros of $Q_{|\mathbf{m}|}$ in such a way that

$$0 < |\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_{|\mathbf{m}|}| \tag{12}$$

and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,|\mathbf{m}|}$ be the zeros of $Q_{n,\mathbf{m}}$ indexed so that

$$\lim_{n\to\infty} \lambda_{n,k} = \lambda_k, \qquad k = 1, 2, \ldots, |\mathbf{m}|.$$

The latest progress on Conjecture 2 with the condition in Equation (11) is the following theorem (see Theorem 4.4 of [42]).

Theorem 8. *Let* $\mathbf{F}$ *be a vector of formal Taylor expansions at the origin and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$. *Assume that Equations (11) and (12) hold. Assume further that, if* $\lambda_j, \lambda_{j+1}, \ldots, \lambda_{j+N-1}$, *where* $N > 1$, *are the zeros of* $Q_{|\mathbf{m}|}$ *lying on the same circle (namely, they have the same modulus), then*

$$\limsup_{n\to\infty} |\lambda_{n,k} - \lambda_k|^{1/n} < 1, \qquad k = j, j+1, \ldots, j+N-1. \tag{13}$$

Then, each $\lambda_k, k = 1, 2, \ldots, |\mathbf{m}|$, *is a system singularity of* $\mathbf{F}$ *with respect to* $\mathbf{m}$. *Moreover, if* λ_k *is a zero of multiplicity* τ_k *of* $Q_{|\mathbf{m}|}$ *which verifies Equation (13), then it is a system pole of* $\mathbf{F}$ *with respect to* $\mathbf{m}$ *of order* τ_k.

Considering the coefficients of $Q_{n,\mathbf{m}}$ as the coefficients of a certain recurrence relation, López and Gerpe [42] employed Buslaev's results (extensions of Poincaré's theorem on recursion relations) in Theorems 1 and 2 of [44] to prove the above theorem.

Note that the following special case of Conjecture 2 (an analog of Suetin's theorem) remains unsolved.

Conjecture 4. *Let* $\mathbf{F}$ *be a vector of formal Taylor expansions at the origin and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$. *Assume that* $Q_{n,\mathbf{m}}$ *is unique for all sufficiently large* n, *Equation* (11) *takes place, and* $Q_{|\mathbf{m}|}(\lambda) = 0$. *Then,* λ *is a system singularity of* $\mathbf{F}$ *with respect to* $\mathbf{m}$.

2.3. Some Remarks

- The statement (b)⇒(a) in Theorem 6 was the first inverse-type result on the study of HP approximation on row sequences. Its proof is very constructive. It suggests how to find a polynomial combination in Equation (8) verifying that all zeros of $Q_{|\mathbf{m}|}$ in Equation (9) are the system poles of $\mathbf{F}$ with respect to $\mathbf{m}$.
- The study of zeros of $P_{n,\mathbf{m},\ell}, \ell = 1, 2, \ldots, d$, in Equation (7) is irrelevant to our interest in this paper. However, it is worth mentioning the paper [45] where la Calle Ysern and Mínguez Ceniceros studied the distribution of zeros of $P_{n,\mathbf{m},\ell}, \ell = 1, 2, \ldots, d$, as $\mathbf{m}$ is fixed and $n \to \infty$.

3. Generalized Hermite–Padé Approximations

3.1. Definitions and Notation

After an analog of Montessus de Ballore–Gonchar's theorem for HP approximation (Theorem 6) was proved, Theorem 6 was extended for various generalizations of HP approximation. These generalizations are formulated to approximate a vector of functions holomorphic on a neighborhood of the following sets E.

Let E be an infinite compact subset of the complex plane $\mathbb{C}$ such that $\overline{\mathbb{C}} \setminus E$ is simply connected. Whenever we consider set E, E is described as above, unless we specifically say otherwise. Denote by $\mathcal{H}(E)$ the space of all functions holomorphic in some neighborhood of E. We define

$$\mathcal{H}(E)^d := \{(F_1, F_2, \ldots, F_d) : F_\ell \in \mathcal{H}(E), \ell = 1, 2, \ldots, d\}.$$

In Section 3, we are mainly interested in approximating a vector in the space $\mathcal{H}(E)^d$.

3.1.1. Orthogonal Hermite–Padé Approximations

The first two approximations are constructed from orthogonal polynomials on E. Let μ be a finite positive Borel measure with infinite support $\mathrm{supp}(\mu)$ contained in E. We write $\mu \in \mathcal{M}(E)$ and define the associated inner product,

$$\langle g, h \rangle_\mu := \int g(\zeta)\overline{h(\zeta)}d\mu(\zeta), \quad g, h \in L_2(\mu). \tag{14}$$

Using the Gram–Schmidt process, we can generate the sequence of the orthonormal polynomials with positive leading coefficients

$$p_n(z) := \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, 2, \ldots,$$

corresponding to the inner product in Equation (14). It is well-known that, for each $n \geq 0$, such orthonormal polynomial p_n is unique. Combining the concepts of HP approximation and orthogonal polynomials, we define two types of orthogonal Hermite–Padé approximations. The first one is a natural extension of HP approximation.

Definition 5. *Let* $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$ *and* $\mu \in \mathcal{M}(E)$. *Fix a multi-index* $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$. *Then, for each* $n \geq \max\{m_1, m_2, \ldots, m_d\}$, *there exists* $Q^{\mu,S}_{n,\mathbf{m}} \in \mathbb{P}_{|\mathbf{m}|}$ *such that* $Q^{\mu,S}_{n,\mathbf{m}} \not\equiv 0$ *and*

$$\langle Q^{\mu,S}_{n,\mathbf{m}} F_\ell,\, p_k \rangle_\mu = 0 \qquad \text{for all } k = n - m_\ell + 1, n - m_\ell + 2, \ldots, n \text{ and } \ell = 1, 2, \ldots, d. \tag{15}$$

The corresponding vector of rational functions

$$\mathbf{R}^{\mu,S}_{n,\mathbf{m}} := (R^{\mu,S}_{n,\mathbf{m},1}, R^{\mu,S}_{n,\mathbf{m},2}, \ldots, R^{\mu,S}_{n,\mathbf{m},d})$$

$$= \left(\frac{\sum_{j=0}^{n-m_1} \langle Q^{\mu,S}_{n,\mathbf{m}} F_1,\, p_j \rangle_\mu p_j}{Q^{\mu,S}_{n,\mathbf{m}}}, \frac{\sum_{j=0}^{n-m_2} \langle Q^{\mu,S}_{n,\mathbf{m}} F_2,\, p_j \rangle_\mu p_j}{Q^{\mu,S}_{n,\mathbf{m}}}, \ldots, \frac{\sum_{j=0}^{n-m_d} \langle Q^{\mu,S}_{n,\mathbf{m}} F_d,\, p_j \rangle_\mu p_j}{Q^{\mu,S}_{n,\mathbf{m}}} \right)$$

are called an $(n, \mathbf{m})$ *standard orthogonal Hermite–Padé (SOHP) approximant of* $\mathbf{F}$ *with respect to* μ.

When $E = \{z \in \mathbb{C} : |z| \leq 1\}$ and μ is the normalized arc length on the unit circle, $p_n(z) = z^n$ for all $n \geq 0$ and the system of linear equations in Equation (15) reduces to

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{Q^{\mu,S}_{n,\mathbf{m}}(z) F(z)}{z^{k+1}} dz = 0 \qquad \text{for all} k = n - m_\ell + 1, n - m_\ell + 2, \ldots, n \text{and} \ell = 1, 2, \ldots, d.$$

Then, the above polynomial $Q^{\mu,S}_{n,\mathbf{m}}$ coincides with the polynomial Q in Equation (6), which further implies that, for each $\ell = 1, 2, \ldots, d$, the corresponding polynomial $\sum_{j=0}^{n-m_\ell} \langle Q^{\mu,S}_{n,\mathbf{m}} F_\ell,\, p_j \rangle_\mu p_j$ coincides with the polynomial P_ℓ in Equation (6). Moreover, when $d = 1$, the rational function $\mathbf{R}^{\mu,S}_{n,\mathbf{m}}$ is the usual $(n - m, m)$ orthogonal Padé approximant defined in [46]. Therefore, the approximation in Definition 5 is a natural generalization of orthogonal Padé approximation to the vector case.

The definition of the other orthogonal Hermite–Padé approximation was recently introduced by Bosuwan and López [47] in the problem concerning localization of system poles of $\mathbf{F} \in \mathcal{H}(E)^d$ around the set E.

Definition 6. *Let* $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$ *and* $\mu \in \mathcal{M}(E)$. *Fix a multi-index* $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$ *and* $n \in \mathbb{N}$. *Then, there exists* $Q^{\mu,M}_{n,\mathbf{m}} \in \mathbb{P}_{|\mathbf{m}|}$ *such that* $Q^{\mu,M}_{n,\mathbf{m}} \not\equiv 0$ *and*

$$\langle z^k Q^{\mu,M}_{n,\mathbf{m}} F_\ell,\, p_n \rangle_\mu = 0 \qquad \text{for all } k = 0, 1, \ldots, m_\ell - 1 \text{ and } \ell = 1, 2, \ldots, d. \tag{16}$$

The corresponding vector of rational functions

$$\mathbf{R}^{\mu,M}_{n,\mathbf{m}} := (R^{\mu,M}_{n,\mathbf{m},1}, R^{\mu,M}_{n,\mathbf{m},2}, \ldots, R^{\mu,M}_{n,\mathbf{m},d})$$

$$= \left(\frac{\sum_{j=0}^{n-1} \langle Q^{\mu,M}_{n,\mathbf{m}} F_1,\, p_j \rangle_\mu p_j}{Q^{\mu,M}_{n,\mathbf{m}}}, \frac{\sum_{j=0}^{n-1} \langle Q^{\mu,M}_{n,\mathbf{m}} F_2,\, p_j \rangle_\mu p_j}{Q^{\mu,M}_{n,\mathbf{m}}}, \ldots, \frac{\sum_{j=0}^{n-1} \langle Q^{\mu,M}_{n,\mathbf{m}} F_d,\, p_j \rangle_\mu p_j}{Q^{\mu,M}_{n,\mathbf{m}}} \right)$$

are called an $(n, \mathbf{m})$ *modified orthogonal Hermite–Padé (MOHP) approximant of* $\mathbf{F}$ *with respect to* μ.

The motivation of the above definition is in the paragraph that contains Equation (39).

3.1.2. Faber–Hermite–Padé Approximations

Now, we want to combine the ideas of HP approximation and Faber polynomials on E. Let Φ be the unique Riemann mapping function from $\overline{\mathbb{C}} \setminus E$ to the exterior of the closed unit disk verifying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. For each $\rho > 1$, the *level curve of index* ρ and the *canonical domain of index* ρ are defined by

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\} \qquad \text{and} \qquad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\},$$

respectively. Let $\mathbf{F} \in \mathcal{H}(E)^d$. Denote by $\rho_m(\mathbf{F})$ the index $\rho > 1$ of the largest canonical domain D_ρ to which $\mathbf{F}$ has at most m poles counting multiplicities.

The *Faber polynomial* of E of degree n is

$$\Phi_n(z) := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\Phi^n(t)}{t - z} dt, \qquad z \in D_\rho, \qquad n = 0, 1, 2, \ldots.$$

One can also define Φ_n as the polynomial part of the Laurent expansion of Φ^n at infinity. The nth *Faber coefficient* of $F \in \mathcal{H}(E)$ with respect to Φ_n is defined by the formula

$$[F]_n := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{F(t)\Phi'(t)}{\Phi^{n+1}(t)} dt,$$

where $\rho \in (1, \rho_0(F))$.

As with the SOHP and MOHP approximations, we have two ways to define Faber–Hermite–Padé approximations.

Definition 7. *Let* $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$. *Fix* $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$. *Then, for each* $n \geq \max\{m_1, m_2, \ldots, m_d\}$, *there exists* $Q^{E,S}_{n,\mathbf{m}} \in \mathbb{P}_{|\mathbf{m}|}$ *such that* $Q^{E,S}_{n,\mathbf{m}} \not\equiv 0$ *and*

$$[Q^{E,S}_{n,\mathbf{m}} F_\ell]_k = 0 \qquad \text{for all } k = n - m_\ell + 1, n - m_\ell + 2, \ldots, n \text{ and } \ell = 1, 2, \ldots, d. \tag{17}$$

The corresponding vector of rational functions

$$\mathbf{R}^{E,S}_{n,\mathbf{m}} := (R^{E,S}_{n,\mathbf{m},1}, R^{E,S}_{n,\mathbf{m},2}, \ldots, R^{E,S}_{n,\mathbf{m},d})$$
$$= \left(\frac{\sum_{j=0}^{n-m_1} [Q^{E,S}_{n,\mathbf{m}} F_1]_j \Phi_j}{Q^{E,S}_{n,\mathbf{m}}}, \frac{\sum_{j=0}^{n-m_2} [Q^{E,S}_{n,\mathbf{m}} F_2]_j \Phi_j}{Q^{E,S}_{n,\mathbf{m}}}, \ldots, \frac{\sum_{j=0}^{n-m_d} [Q^{E,S}_{n,\mathbf{m}} F_d]_j \Phi_j}{Q^{E,S}_{n,\mathbf{m}}} \right)$$

are called an $(n, \mathbf{m})$ *standard Faber–Hermite–Padé (SFHP) approximant of* $\mathbf{F}$ *with respect to* E.

One can easily check that, when $E = \{z \in \mathbb{C} : |z| \leq 1\}$, the SFHP approximant $\mathbf{R}^{E,S}_{n,\mathbf{m}}$ is the same as a HP approximant $\mathbf{R}_{n,\mathbf{m}}$. Furthermore, when $d = 1$, the approximant $\mathbf{R}^{E,S}_{n,\mathbf{m}}$ reduces to the usual $(n - m, m)$ Padé-Faber approximant (see its definition in Section 3 of [48]).

Definition 8. *Let* $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$. *Fix* $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$ *and* $n \in \mathbb{N}$. *Then, there exists* $Q^{E,M}_{n,\mathbf{m}} \in \mathbb{P}_{|\mathbf{m}|}$ *such that* $Q^{E,M}_{n,\mathbf{m}} \not\equiv 0$ *and*

$$[z^k Q^{E,M}_{n,\mathbf{m}} F_\ell]_n = 0 \qquad \text{for all } k = 0, 1, \ldots, m_\ell - 1 \text{ and } \ell = 1, 2, \ldots, d. \tag{18}$$

The corresponding vector of rational functions

$$\mathbf{R}^{E,M}_{n,\mathbf{m}} := (R^{E,M}_{n,\mathbf{m},1}, R^{E,M}_{n,\mathbf{m},2}, \ldots, R^{E,M}_{n,\mathbf{m},d})$$
$$= \left(\frac{\sum_{j=0}^{n-1} [Q^{E,M}_{n,\mathbf{m}} F_1]_j \Phi_j}{Q^{E,M}_{n,\mathbf{m}}}, \frac{\sum_{j=0}^{n-1} [Q^{E,M}_{n,\mathbf{m}} F_2]_j \Phi_j}{Q^{E,M}_{n,\mathbf{m}}}, \ldots, \frac{\sum_{j=0}^{n-1} [Q^{E,M}_{n,\mathbf{m}} F_d]_j \Phi_j}{Q^{E,M}_{n,\mathbf{m}}} \right)$$

are called an $(n, \mathbf{m})$ modified Faber–Hermite–Padé (MFHP) approximant of $\mathbf{F}$ with respect to E.

The above MFHP approximation was recently introduced by Bosuwan and López [49]. In [49], MFHP approximation was proved to be an effective tool to locate system poles of $\mathbf{F} \in \mathcal{H}(E)^d$ around the set E.

3.1.3. Multipiont Hermite–Padé Approximations

Let $\alpha \subset E$ be a table of points; more precisely, $\alpha = \{\alpha_{n,k}\}$, $k = 1, \ldots, n$, $n = 1, 2, \ldots$. The definition of multipoint Hermite–Padé approximation is

Definition 9. *Let $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$. Set*

$$a_n(z) := \prod_{k=1}^{n} (z - \alpha_{n,k})$$

and fix a multi-index $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$. Then, for each $n \geq \max\{m_1, m_2, \ldots, m_d\}$, there exist $Q^{\alpha}_{n,\mathbf{m}} \in \mathbb{P}_{|\mathbf{m}|}$ and $P^{\alpha}_{n,\mathbf{m},\ell} \in \mathbb{P}_{n-m_\ell}$ for all $\ell = 1, 2, \ldots, d$ such that $Q^{\alpha}_{n,\mathbf{m}} \not\equiv 0$ and

$$(Q^{\alpha}_{n,\mathbf{m}} F_\ell - P^{\alpha}_{n,\mathbf{m},\ell}) / a_{n+1} \in \mathcal{H}(E), \qquad \text{for all } \ell = 1, 2, \ldots, d. \tag{19}$$

The corresponding vector of rational functions

$$\mathbf{R}^{\alpha}_{n,\mathbf{m}} = \left(R^{\alpha}_{n,\mathbf{m},1}, R^{\alpha}_{n,\mathbf{m},2}, \ldots, R^{\alpha}_{n,\mathbf{m},d} \right) = \left(\frac{P^{\alpha}_{n,\mathbf{m},1}}{Q^{\alpha}_{n,\mathbf{m}}}, \frac{P^{\alpha}_{n,\mathbf{m},2}}{Q^{\alpha}_{n,\mathbf{m}}}, \ldots, \frac{P^{\alpha}_{n,\mathbf{m},d}}{Q^{\alpha}_{n,\mathbf{m}}} \right) \tag{20}$$

is called an $(n, \mathbf{m})$ multipoint Hermite–Padé (MHP) approximant of $\mathbf{F}$ with respect α.

This approximation was introduced in [2] by Mahler long before other generalizations. Note that, if all the interpolation points are 0, then the MHP approximants reduce to HP approximants.

3.1.4. Some Remarks

- Finding $Q^{\mu,S}_{n,\mathbf{m}}, Q^{\mu,M}_{n,\mathbf{m}}, Q^{E,S}_{n,\mathbf{m}}, or Q^{E,M}_{n,\mathbf{m}}$ is equivalent to solving $|\mathbf{m}| + 1$ unknowns from linear system of $|\mathbf{m}|$ equations in Equations (15)–(18), respectively. Moreover, finding $P^{\alpha}_{n,\mathbf{m},\ell}$, $\ell = 1, 2, \ldots, d$ and $Q^{\alpha}_{n,\mathbf{m}}$ is equivalent to solving $(n+1)d + 1$ unknowns from linear system of $(n+1)d$ equations in Equation (19). Therefore, these polynomials, $Q^{\mu,S}_{n,\mathbf{m}}, Q^{\mu,M}_{n,\mathbf{m}}, Q^{E,S}_{n,\mathbf{m}}, Q^{E,M}_{n,\mathbf{m}}$, and $Q^{\alpha}_{n,\mathbf{m}}$, always exist but may not be unique. Since such polynomials are not the zero function, we normalize them to be "monic" polynomials. Moreover, $Q^{\alpha}_{n,\mathbf{m}}$ in Equation (20) is chosen so that it does not have a common zero with all the $P^{\alpha}_{n,\mathbf{m},\ell}$. We would like to emphasize that for any $(n, \mathbf{m}) \in \mathbb{N} \times \mathbb{N}^d$, $\mathbf{R}^{\mu,S}_{n,\mathbf{m}}, \mathbf{R}^{\mu,M}_{n,\mathbf{m}}, \mathbf{R}^{E,S}_{n,\mathbf{m}}, \mathbf{R}^{E,M}_{n,\mathbf{m}}$, and $\mathbf{R}^{\alpha}_{n,\mathbf{m}}$ may not be unique.

- Extensions of generalized HP approximations in Definitions 5–9 to a compact set E whose complement is connected are possible. However, the results in this survey paper are restricted to the case when E is a compact subset of the complex plane with simply connected complement in the extended complex plane. This is because, for the sets E which are disconnected, the zeros of the corresponding orthonormal polynomials p_n, second type functions s_n (defined in Equation (21)), or Faber polynomials Φ_n may lie in $\mathbb{C} \setminus E$ which may be the locations of system poles.

3.1.5. Classes of Measures in $\mathcal{M}(E)$

For the studies of SOHP and MOHP approximations, we need to impose asymptotic properties of the sequences of the orthonormal polynomials $\{p_n\}_{n\geq 0}$ and the corresponding second type functions $\{s_n\}_{n\geq 0}$ defined below

$$s_n(z) := \int \frac{\overline{p_n(\zeta)}}{z-\zeta} d\mu(\zeta), \qquad z \in \overline{\mathbb{C}} \setminus \operatorname{supp}(\mu). \tag{21}$$

We keep classes of measures in $\mathcal{M}(E)$ here so that they do not disturb the flow of our paper. The readers who are not interested in SOHP or MOHP approximations may skip Section 3.1.5.

Definition 10. *Let $\mu \in \mathcal{M}(E)$.*

(a) $\mu \in \mathbf{Reg}_1(E)$ *if and only if*

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)|. \tag{22}$$

(b) $\mu \in \mathbf{Reg}_2(E)$ *if and only if*

$$\lim_{n\to\infty} |s_n(z)|^{1/n} = |\Phi(z)|^{-1}. \tag{23}$$

(c) $\mu \in \mathbf{Reg}_{1,2}(E)$ *if and only if* $\mu \in \mathbf{Reg}_1(E) \cap \mathbf{Reg}_2(E)$.

(d) $\mu \in \mathbf{Reg}^m_{1,2}(E)$ *if and only if* $\mu \in \mathbf{Reg}_{1,2}(E)$ *and there exists a positive constant c such that*

$$\frac{\kappa_{n-m}}{\kappa_n} \geq c, \qquad n \geq n_0.$$

(e) $\mu \in \mathbf{Rat}_1(E)$ *if and only if*

$$\lim_{n\to\infty} \frac{p_n(z)}{p_{n+1}(z)} = \frac{1}{\Phi(z)}. \tag{24}$$

(f) $\mu \in \mathbf{Rat}_2(E)$ *if and only if*

$$\lim_{n\to\infty} \frac{s_{n+1}(z)}{s_n(z)} = \frac{1}{\Phi(z)}. \tag{25}$$

(g) $\mu \in \mathbf{Rat}_{1,2}(E)$ *if and only if* $\mu \in \mathbf{Rat}_1(E) \cap \mathbf{Rat}_2(E)$.

(h) $\mu \in \mathcal{S}(E)$ *if and only if* $\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = 1$ *and*

$$\lim_{n\to\infty} \frac{p_n(z)}{c_n \Phi^n(z)} = S(z), \tag{26}$$

where the c_ns are positive constants and S is a non-vanishing holomorphic function on $\overline{\mathbb{C}} \setminus E$.

The limits in Equations (22) and (23) are assumed to hold uniformly on compact subsets of $\mathbb{C} \setminus E$ and the ones in Equations (24)–(26) are assumed to hold uniformly on compact subsets of $\overline{\mathbb{C}} \setminus E$.

The classes $\mathbf{Reg}_1(E)$ and $\mathbf{Reg}_2(E)$ are more or less the same in some cases. In particular, if E is convex, then $\mathbf{Reg}_1(E) = \mathbf{Reg}_2(E)$ and these two classes coincide with the regular class in the usual sense (see Definition 3.1.2 of [50] for the definition of the regular class in the usual sense). Clearly, $\mathcal{S}(E) \subset \mathbf{Rat}_1(E) \subset \mathbf{Reg}_1(E)$ and $\mathbf{Rat}_2(E) \subset \mathbf{Reg}_2(E)$.

Denote by $\mathcal{K}_1$ the collection of all compact sets E (stated at the beginning of Section 3) satisfying the condition that the inverse of the exterior conformal function Φ^{-1} can be extended continuously to $\overline{\mathbb{C}} \setminus \{w \in \mathbb{C} : |w| < 1\}$. If we assume that $E \in \mathcal{K}_1$ and $\mu \in \mathbf{Rat}_1(E)$, then

$$\lim_{n\to\infty} p_n(z)s_n(z) = \frac{\Phi'(z)}{\Phi(z)},$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus E$ (see Lemma 3.1 of [46]). This statement implies that, if $E \in \mathcal{K}_1$, then $\mathcal{S}(E) \subset \mathbf{Rat}_1(E) \subset \mathbf{Rat}_2(E)$. Moreover, it is well-known that, if $\mu \in \mathbf{Rat}_1(E)$, then

$$\lim_{n\to\infty} \frac{\kappa_n}{\kappa_{n+1}} = \mathrm{cap}(E)$$

(see, e.g., Lemma 2 of [51]). Therefore, for each $m \in \mathbb{N}$, $\mathbf{Rat}_1(E) \subset \mathbf{Reg}_1^m(E)$ and $\mathbf{Rat}_{1,2}(E) \subset \mathbf{Reg}_{1,2}^m(E)$. For a general compact set E, the class of measures $\mathbf{Reg}_1(E)$ has been well studied and characterized in terms of the analytic properties of the measure or of the corresponding sequence of leading coefficients (see, e.g., [50], Theorem 3.1.1). The situation is not quite the same for other classes of measures. To discuss them would take us too far from our main direction; rather we refer the reader to Pages 532–533 in [52] for more details and references.

3.1.6. Classes of Tables of Interpolation Points

When we state results on MHP approximation, we refer to two classes of tables of interpolation points.

Definition 11. *Let $\alpha \subset E$ be a table of interpolation points, namely*

$$\alpha = \{\alpha_{n,k}\}, \qquad k = 1, \ldots, n, \qquad n = 1, 2, \ldots.$$

Set

$$a_n(z) := \prod_{k=1}^{n} (z - \alpha_{n,k}).$$

(a) $\alpha \in \mathbf{Strong}(E)$ *if and only if the corresponding polynomials a_n satisfy the following strong asymptotics:*

$$\lim_{n\to\infty} \frac{a_n(z)}{(\mathrm{cap}(E))^n \Phi^n(z)} = G(z) \neq 0,$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus E$.

(b) $\alpha \in \mathbf{Root}(E)$ *if and only if the corresponding polynomials a_n satisfy the following nth root asymptotics:*

$$\lim_{n\to\infty} |a_n(z)|^{1/n} = \mathrm{cap}(E)|\Phi(z)|,$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus E$,

Clearly, $\mathbf{Strong}(E) \subset \mathbf{Root}(E)$. Moreover, it is well known that $\mathbf{Strong}(E) \neq \emptyset$ and $\mathbf{Root}(E) \neq \emptyset$ (see Chapters 8 and 9 of [53] for more details about both classes).

3.2. Results and Conjectures

3.2.1. The Scalar Case

We add a section dedicated to the scalar case of all generalizations (defined in Section 3.1) here. The purpose is not only to help the readers consolidate their understanding of the vector case but also to discuss some interesting theorems and conjectures corresponding to such scalar case. When $d = 1$, we have $\mathbf{m} = m$ and write

$$\mathbf{R}_{n,\mathbf{m}}^{\mu,S} = R_{n,m}^{\mu,S}, \qquad \mathbf{R}_{n,\mathbf{m}}^{\mu,M} = R_{n,m}^{\mu,M}, \qquad \mathbf{R}_{n,\mathbf{m}}^{E,S} = R_{n,m}^{E,S}, \qquad \mathbf{R}_{n,\mathbf{m}}^{E,M} = R_{n,m}^{E,M}, \qquad \mathbf{R}_{n,\mathbf{m}}^{\alpha} = R_{n,m}^{\alpha}.$$

The first theorem simply says that one can use the poles of the mth row sequences $\{R_{n,m}^{\mu,S}\}_{n\geq m}, \{R_{n,m}^{\mu,M}\}_{n\geq m}, \{R_{n,m}^{E,S}\}_{n\geq m}, \{R_{n,m}^{E,M}\}_{n\geq m}$, or $\{R_{n,m}^{\alpha}\}_{n\geq m}$ to detect the m poles of $F \in \mathcal{H}(E)$ nearest E (with respect to the level curves).

Theorem 9. *Let $F \in \mathcal{H}(E)$ and $m \in \mathbb{N}$ be fixed. Then, the following statements are equivalent.*

(a) *F has exactly m poles counting multiplicities in $D_{\rho_m(F)}$.*

(b) *For all n sufficiently large, $Q_{n,m}^{\mu,S}$ corresponding to $\mu \in \mathbf{Rat}_{1,2}(E)$ has degree m and there exists a polynomial $\hat{Q}_m$ of degree m such that*

$$\limsup_{n\to\infty} \|Q_{n,m}^{\mu,S} - \hat{Q}_m\|^{1/n} = \hat{\theta} < 1.$$

(For $\mu \notin \mathbf{Rat}_{1,2}(E)$, the statement of the theorem does not include this assertion.)

(c) *For all n sufficiently large, $Q_{n,m}^{\mu,M}$ corresponding to $\mu \in \mathbf{Reg}_{1,2}^{m}(E)$ has degree m and there exists a polynomial $\check{Q}_m$ of degree m such that*

$$\limsup_{n\to\infty} \|Q_{n,m}^{\mu,M} - \check{Q}_m\|^{1/n} = \check{\theta} < 1.$$

(For $\mu \notin \mathbf{Reg}_{1,2}^{m}(E)$, the statement of the theorem does not include this assertion.)

(d) *For all n sufficiently large, $Q_{n,m}^{E,S}$ has degree m and there exists a polynomial $\ddot{Q}_m$ of degree m such that*

$$\limsup_{n\to\infty} \|Q_{n,m}^{E,S} - \ddot{Q}_m\|^{1/n} = \ddot{\theta} < 1.$$

(e) *For all n sufficiently large, $Q_{n,m}^{E,M}$ has degree m and there exists a polynomial $\tilde{Q}_m$ of degree m such that*

$$\limsup_{n\to\infty} \|Q_{n,m}^{E,M} - \tilde{Q}_m\|^{1/n} = \tilde{\theta} < 1.$$

(f) *For all n sufficiently large, $Q_{n,m}^{\alpha}$ corresponding to $\alpha \in \mathbf{Root}(E)$ has degree m and there exists a polynomial $\breve{Q}_m$ of degree m such that*

$$\limsup_{n\to\infty} \|Q_{n,m}^{\alpha} - \breve{Q}_m\|^{1/n} = \breve{\theta} < 1.$$

(For $\alpha \notin \mathbf{Root}(E)$, the statement of the theorem does not include this assertion.)

Moreover, if one of Assertions (a)–(f) takes place, then

(i)

$$\hat{Q}_m = \check{Q}_m = \ddot{Q}_m = \tilde{Q}_m = \breve{Q}_m = Q_m(F),$$

where $Q_m(F)$ is the monic polynomial whose zeros are the poles of F in $D_{\rho_m(F)}$ taking account of their order.

(ii)

$$\hat{\theta} = \check{\theta} = \ddot{\theta} = \tilde{\theta} = \breve{\theta} = \frac{\max\{|\Phi(\lambda)| : \lambda \in \mathcal{P}_m(F)\}}{\rho_m(F)},$$

where $\mathcal{P}_m(F)$ is the set of the distinct zeros of $Q_m(F)$.
(iii) For any compact subset K of $D_{\rho_m(F)} \setminus \mathcal{P}_m(F)$,

$$\limsup_{n\to\infty} \|F - R^{\mu,S}_{n,m}\|_K^{1/n} \le \frac{\|\Phi\|_K}{\rho_m(F)}, \tag{27}$$

$$\limsup_{n\to\infty} \|F - R^{\mu,M}_{n,m}\|_K^{1/n} \le \frac{\|\Phi\|_K}{\rho_m(F)}, \tag{28}$$

$$\limsup_{n\to\infty} \|F - R^{E,S}_{n,m}\|_K^{1/n} \le \frac{\|\Phi\|_K}{\rho_m(F)}, \tag{29}$$

$$\limsup_{n\to\infty} \|F - R^{E,M}_{n,m}\|_K^{1/n} \le \frac{\|\Phi\|_K}{\rho_m(F)}, \tag{30}$$

$$\limsup_{n\to\infty} \|F - R^{\alpha}_{n,m}\|_K^{1/n} \le \frac{\|\Phi\|_K}{\rho_m(F)}, \tag{31}$$

under the convention that if $K \subset E$, then $\|\Phi\|_K$ is replaced by 1.

Remark 1. *The inequality in Equation (31) is equality when α is a Newton type.*

The proofs concerning (a)⇔(b), (a)⇔(c), (a)⇔(d), (a)⇔(e), and (a)⇔(f) are in Corollary 1 of [52], Theorem 1.2 of [47], Theorem 2 of [54], Corollary 1.6 of [49], and Theorem 1.3 of [55], respectively.

We strongly believe that for this scalar case, the exact rates of convergences of $\{R^{\mu,S}_{n,m}\}_{n\ge m}$, $\{R^{\mu,M}_{n,m}\}_{n\ge m}$, $\{R^{E,S}_{n,m}\}_{n\ge m}$, $\{R^{E,M}_{n,m}\}_{n\ge m}$, and $\{R^{\alpha}_{n,m}\}_{n\ge m}$, should be as indicated in the following conjecture.

Conjecture 5. *Let $F \in \mathcal{H}(E)$ and $m \in \mathbb{N}$. If F has exactly m poles in $D_{\rho_m(F)}$, then the inequalities in Equations (27)–(31) are equalities.*

Clearly, if $E = \{z \in \mathbb{C} : |z| \le 1\}$ and $d\mu = d\theta/2\pi$ on the boundary of E, then $R^{\mu,S}_{n,m}, R^{\mu,M}_{n,m}, R^{E,S}_{n,m}$ and $R^{E,M}_{n,m}$ are the $(n-m,m)$ classical Padé approximant and Conjecture 5 holds true because of Equation (4) in Montessus de Ballore–Gonchar's theorem. Another supporting evidence for the validity of Conjecture 5 is provided for the case when $K = E$, namely if F has exactly m poles in $D_{\rho_m(F)}$, then

$$\limsup_{n\to\infty} \|F - R^{\mu,S}_{n,m}\|_E^{1/n} = \limsup_{n\to\infty} \|F - R^{\mu,M}_{n,m}\|_E^{1/n} = \limsup_{n\to\infty} \|F - R^{E,S}_{n,m}\|_E^{1/n} = \limsup_{n\to\infty} \|F - R^{E,M}_{n,m}\|_E^{1/n}$$

$$= \limsup_{n\to\infty} \|F - R^{\alpha}_{n,m}\|_E^{1/n} = \limsup_{n\to\infty} \sigma_{n,m}^{1/n} = \frac{1}{\rho_m(F)}, \tag{32}$$

where

$$\sigma_{n,m} := \inf_r \|F - r\|_E,$$

and the infimum is taken over the class of all rational functions of type (n,m)

$$r(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0}{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0}.$$

We refer the reader to [56] for the proof of the fifth equality in Equation (32).

For Gonchar's conjecture, we made good progress in the case when the denominators $Q_{n,m}^{\mu,S}, Q_{n,m}^{\mu,M}, Q_{n,m}^{E,S}, Q_{n,m}^{E,M}$, and $Q_{n,m}^{\alpha}$ converge to a polynomial of degree m as $n \to \infty$. An analog of Suetin's theorem for the scalar case of all generalized HP approximations (see Theorem 4 of [44], Theorem 2 of [57], Theorem 2.7 of [46], and Theorems 2.5 and 2.6 of [58]) is the following.

Theorem 10. *Let $F \in \mathcal{H}(E)$ and $m \in \mathbb{N}$ be fixed. If one of the following holds:*

(a) $E \in \mathcal{K}_1$, $\mu \in \mathcal{S}(E)$, *for all n sufficiently large, $Q_{n,m}^{\mu,S}$ has degree m, and* $\lim_{n\to\infty} Q_{n,m}^{\mu,S} = Q_m$;

(b) $E \in \mathcal{K}_1$, $\mu \in \mathcal{S}(E)$, *for all n sufficiently large, $Q_{n,m}^{\mu,M}$ has degree m, and* $\lim_{n\to\infty} Q_{n,m}^{\mu,M} = Q_m$;

(c) *for all n sufficiently large, $Q_{n,m}^{E,S}$ has degree m and* $\lim_{n\to\infty} Q_{n,m}^{E,S} = Q_m$;

(d) *for all n sufficiently large, $Q_{n,m}^{E,M}$ hhas degree m and* $\lim_{n\to\infty} Q_{n,m}^{E,M} = Q_m$; *or*

(e) $\alpha \in \mathbf{Strong}(E)$, *for all n sufficiently large, $Q_{n,m}^{\alpha}$ has degree m, and* $\lim_{n\to\infty} Q_{n,m}^{\alpha} = Q_m$,

then all of the following hold:

(i)

$$\rho_0(F) = \min\{|\Phi(\lambda)| : Q_m(\lambda) = 0\};$$

(ii)

$$\rho_{m-1}(F) = \max\{|\Phi(\lambda)| : Q_m(\lambda) = 0\};$$

and

(iii) all zeros of Q_m are singularities of F; those lying in $D_{\rho_{m-1}(F)}$ are poles (counting multiplicities), and F has no other poles in $D_{\rho_{m-1}(F)}$.

The proof of the above theorem relies on deep results on refinements of Poincaré's theorem on recurrence relations developed by Buslaev in Theorems 5 and 6 of [57]. These Buslaev's results connect Suetin's theorem (for classical Padé approximants) and analogous ones for generalized Padé approximants (see Theorem 3 of [57] and Theorems 2.7 and 2.8 of [58]).

An analog of Gonchar's theorem for $Q_{n.m}^{\alpha}$ when α is a Newton type was proved by himself in Theorem 2 of [31]. For other generalizations, we have proofs only for an analog of (a)⇒(b) in Gonchar's theorem (see Corollary 5.6 of [51], Corollary 1 of [59], Theorems 2.1 and 2.3 of [58]).

A direct analog of the Fabry ratio theorem for orthogonal and Faber polynomial expansions is

Theorem 11. *Let $F \in \mathcal{H}(E)$. If one of the following holds:*

(a)

$$\lim_{n\to\infty} \frac{[F]_n}{[F]_{n+1}} = \xi;$$

or

(b) $E \in \mathcal{K}_1$, $\mu \in \mathcal{S}(E)$, *and*

$$\lim_{n\to\infty} \frac{\langle F, p_n \rangle_\mu}{\langle F, p_{n+1} \rangle_\mu} = \xi,$$

then $\Phi^{-1}(\xi)$ is a singularity of F and $\rho_0(F) = |\xi|$.

The part of the above theorem concerning the limit of the ratio of Faber coefficients follows from a straightforward change of variables:

$$[F]_n = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{F(t)\Phi'(t)}{\Phi^{n+1}(t)}dt = \frac{1}{2\pi i}\int_{|z|=\rho} \frac{F(\Phi^{-1}(z))}{z^{n+1}}dz, \qquad \text{where} t = \Phi^{-1}(z)$$

and the use of the Fabry ratio theorem for the analytic part of $F \circ \Phi^{-1}$. The part of the above theorem concerning the limit of the ratio of Fourier coefficients is in Theorem 2.3 of [46].

We end this section by stating an application of Theorem 11 to orthogonal polynomial theory on the unit circle. Firstly, we recall some facts. If $E = \overline{\mathbb{B}_1}$ and the measure μ supported on the unit circle satisfies the Szegő condition,

$$\int_0^{2\pi} \log w(\theta) d\theta > -\infty, \tag{33}$$

where $d\mu(\theta) = w(\theta)d\theta/2\pi + d\mu_s(\theta)$ is the Radon–Nikodym decomposition of μ, then it is well known that $\mu \in \mathcal{S}(E)$, the leading coefficients of the orthonormal polynomials p_n satisfy

$$\lim_{n\to\infty} \kappa_n = \kappa := \exp\left\{-\frac{1}{4\pi}\int_0^{2\pi} \log w(\theta)d\theta\right\},$$

and

$$\frac{1}{S_{\text{int}}(z)} = \frac{1}{\kappa}\sum_{k=0}^{\infty} \overline{p_k(0)}p_k(z), \quad \text{uniformly on compact subsets of } \mathbb{B},$$

where

$$S_{\text{int}}(z) := \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \log w(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta\right), \quad z \in \mathbb{B},$$

denotes the interior Szegő function (see ([60], pp. 19–20) for the proof). Therefore, Theorem 11 enables us to locate the singularity nearest the origin of the reciprocal of the interior Szegő function $1/S_{\text{int}}$ in terms of the Verblunsky coefficients

$$\alpha_n := -\overline{p_n(0)}/\kappa_n.$$

Theorem 12. *Let μ satisfy the Szegő condition in Equation (33) and assume that $1/S_{\text{int}} \in \mathcal{H}(\overline{\mathbb{B}})$. Suppose that the Verblunsky coefficients α_n corresponding to μ verify*

$$\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = \lambda.$$

Then, λ is a singularity of $1/S_{\text{int}}$ and $1/S_{\text{int}}$ is holomorphic on $\mathbb{B}_{|\lambda|}$.

The above result was stated in Corollary 2.4 of [46].

3.2.2. The Vector Case

Graves-Morris and Saff were the first to prove an extension of Montessus de Ballore's theorem to MHP approximation in Theorem 3 of [38]. Their theorem relies on the concept of polewise independence, which influenced the current author to define the following adapted polewise independence.

Definition 12. *Let $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$ be a vector of functions meromorphic in some canonical domain D_ρ and let $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$ be the multi-index. Then, the function $\mathbf{F}$ is said to be* polewise independent with respect to the multi-index $\mathbf{m}$ in D_ρ *if and only if there do not exist polynomials* $v_1, v_2, \ldots, v_d$, *at least one of which being non-null, satisfying:*

(i) $\deg v_\ell \leq m_\ell - 1$, $\ell = 1, 2, \ldots, d$.
(ii) $\sum_{\ell=1}^{d} (v_\ell \circ \Phi) \cdot F_\ell \in \mathcal{H}(D_\rho \setminus E)$.

Note that, if we replace Φ in (ii) of Definition 12 by the identity mapping, then the above definition reduces the definition of polewise independence in [38].

Putting together Theorem 2.3 in [51] and Theorem 1 in [61], we obtain the following analog of Montessus de Ballore's theorem for SOHP and SFHP approximations under the concept of polewise independence in Definition 12.

By $Q^{\mathbf{F}}_{|\mathbf{m}|}$, we denote the monic polynomial whose zeros are the poles of $\mathbf{F}$ in $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$ taking account of their order. The set of distinct zeros of $Q^{\mathbf{F}}_{|\mathbf{m}|}$ is denoted by $\mathcal{P}_{|\mathbf{m}|}(\mathbf{F})$.

Theorem 13. *Let* $\mathbf{m} \in \mathbb{N}^d$ *be a fixed multi-index,* $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$, *and* $\mu \in \mathbf{Rat}_{1,2}(E)$. *Suppose that* $\mathbf{F}$ *is polewise independent (in the sense of Definition 12) with respect to the multi-index* $\mathbf{m}$ *in* $D_{\rho_{|\mathbf{m}|}(\mathbf{F})}$. *Then,* $\mathbf{R}^{\mu,S}_{n,\mathbf{m}}$ *and* $\mathbf{R}^{E,S}_{n,\mathbf{m}}$ *are uniquely determined for all sufficiently large* n, *and for each* $\ell = 1, 2, \ldots, d$ *and for any compact set* $K \subset D_{\rho_{|\mathbf{m}|}(\mathbf{F})} \setminus \mathcal{P}_{|\mathbf{m}|}(\mathbf{F})$,

$$\limsup_{n\to\infty} \|F_\ell - R^{\mu,S}_{n,\mathbf{m},\ell}\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{|\mathbf{m}|}(\mathbf{F})} \qquad \text{and} \qquad \limsup_{n\to\infty} \|F_\ell - R^{E,S}_{n,\mathbf{m},\ell}\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{|\mathbf{m}|}(\mathbf{F})}. \tag{34}$$

Moreover,

$$\limsup_{n\to\infty} \|Q^{\mu,S}_{n,\mathbf{m}} - Q^{\mathbf{F}}_{|\mathbf{m}|}\|^{1/n} \leq \frac{\max_{\lambda\in\mathcal{P}_{|\mathbf{m}|}(\mathbf{F})} |\Phi(\lambda)|}{\rho_{|\mathbf{m}|}(\mathbf{F})},$$
$$\limsup_{n\to\infty} \|Q^{E,S}_{n,\mathbf{m}} - Q^{\mathbf{F}}_{|\mathbf{m}|}\|^{1/n} \leq \frac{\max_{\lambda\in\mathcal{P}_{|\mathbf{m}|}(\mathbf{F})} |\Phi(\lambda)|}{\rho_{|\mathbf{m}|}(\mathbf{F})}. \tag{35}$$

In fact, Cacoq and López [62] were the first who proved the above theorem but in the context of SOHP approximation for the case when E is the closed unit disk and μ is supported on the unit circle.

Making use of incomplete orthogonal Padé and incomplete Padé–Faber approximations (see Definition 5.1 of [51] and Definition 5 of [59]), Bosuwan proved another Montessus de Ballore type theorem for SOHP and SFHP approximations in Theorem 2.4 of [51] and Theorem 1 of [59]. The idea of incomplete approximants allows us to analyze each function F_ℓ individually. Let us define some more notation about the region of convergence.

Given a system $\mathbf{F} = (F_1, F_2, \ldots, F_d)$ and a multi-index $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d \setminus \{\mathbf{0}\}$, we define

$$\mathbf{D_m}(\mathbf{F}) := (D_{\rho_{m_1}(F_1)}, D_{\rho_{m_2}(F_2)}, \ldots, D_{\rho_{m_d}(F_d)}).$$

By $Q^{\mathbf{F}}_{\mathbf{m}}$, we denote the monic polynomial whose zeros are the poles of $\mathbf{F}$ in $\mathbf{D_m}(\mathbf{F})$ counting multiplicities. This set of poles is denoted by $\mathcal{P}_{\mathbf{m}}(\mathbf{F})$. For $\ell = 1, 2, \ldots, d$, set $\mathcal{P}_{\mathbf{m},\ell}(\mathbf{F}) := \mathcal{P}_{\mathbf{m}}(\mathbf{F}) \cap D_{\rho_{m_\ell}(F_\ell)}$.

To each pole λ of $\mathbf{F}$ in this system of domains

$$\mathbf{D_m}(\mathbf{F}) := (D_{\rho_{m_1}(F_1)}, D_{\rho_{m_2}(F_2)}, \ldots, D_{\rho_{m_d}(F_d)}),$$

we associate an index $\ell(\lambda) \in \{1, 2, \ldots, d\}$ as follows. The index $\ell(\lambda)$ verifies that $\lambda \in D_{\rho_{m_{\ell(\lambda)}}(F_{\ell(\lambda)})}$ and λ is a pole of $F_{\ell(\lambda)}$ of the same order as is a pole of $\mathbf{F}$ in $\mathbf{D}_{\mathbf{m}}(\mathbf{F})$. If there are several indices ℓ satisfying this condition, then we choose one among those with greatest $\rho_{m_\ell}(F_\ell)$.

Theorem 14. *Let $\mathbf{m} \in \mathbb{N}^d$ be a fixed multi-index, $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$, $\mathcal{P}_{\mathbf{m}}(\mathbf{F}) = \{\lambda_1, \lambda_2, \ldots, \lambda_q\}$, and $\mu \in \mathbf{Rat}_{1,2}(E)$. Suppose that $\mathbf{F} \in \mathcal{H}(E)^d$ has exactly $|\mathbf{m}|$ poles in $\mathbf{D}_{\mathbf{m}}(\mathbf{F})$. Then, $\mathbf{R}^{\mu,S}_{n,\mathbf{m}}$ and $\mathbf{R}^{E,S}_{n,\mathbf{m}}$ is uniquely determined for all sufficiently large n and for each $\ell = 1, 2, \ldots, d$, $R^{\mu,S}_{n,\mathbf{m},\ell}$ and $R^{E,S}_{n,\mathbf{m},\ell}$ converge uniformly to F_ℓ on compact subsets of $D_{\rho_{m_\ell}(F_\ell)} \setminus \mathcal{P}_{\mathbf{m},\ell}(\mathbf{F})$. Moreover, for all $\ell = 1, 2, \ldots, d$,*

$$\limsup_{n\to\infty} \|F_\ell - R^{\mu,S}_{n,\mathbf{m},\ell}\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{m_\ell}(F_\ell)} \qquad \text{and} \qquad \limsup_{n\to\infty} \|F_\ell - R^{E,S}_{n,\mathbf{m},\ell}\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_{m_\ell}(F_\ell)}, \tag{36}$$

where K is any compact subset of $D_{\rho_{m_\ell}(F_\ell)} \setminus \mathcal{P}_{\mathbf{m},\ell}(\mathbf{F})$ and if $K \subset E$, then $\|\Phi\|_K$ is replaced by 1. Additionally, we have

$$\limsup_{n\to\infty} \|Q^{\mathbf{F}}_{\mathbf{m}} - Q^{\mu,S}_{n,\mathbf{m}}\|^{1/n} \leq \max_{j=1,2,\ldots,q} \left\{ \frac{|\Phi(\lambda_j)|}{\rho_{m_{\ell(\lambda_j)}}(F_{\ell(\lambda_j)})} \right\},$$

$$\limsup_{n\to\infty} \|Q^{\mathbf{F}}_{\mathbf{m}} - Q^{E,S}_{n,\mathbf{m}}\|^{1/n} \leq \max_{j=1,2,\ldots,q} \left\{ \frac{|\Phi(\lambda_j)|}{\rho_{m_{\ell(\lambda_j)}}(F_{\ell(\lambda_j)})} \right\}, \tag{37}$$

Note that Theorems 13 and 14 have their own values. Section 3 in [51] and Section 2 in [59] gave examples showing when Theorem 13 is applicable but Theorem 14 is not applicable or vice versa.

During my visit to Universidad Carlos III de Madrid in the summer of 2016, López and I initiated a project emphasizing the study of analogs of the inverse statement of Montessus de Ballore–Gonchar's Theorem for SOHP and SFHP approximations. As guided by the principle of the definition of system pole in [40] (see Definition 3), we define

Definition 13. *Given $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$ and $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$, we say that $\lambda \in \mathbb{C}$ is a* system pole of order τ of $\mathbf{F}$ with respect to $\mathbf{m}$ *if τ is the largest positive integer such that, for each $t = 1, 2, \ldots, \tau$, there exists at least one polynomial combination of the form*

$$\sum_{\ell=1}^{d} v_\ell F_\ell, \qquad \deg(v_\ell) < m_\ell, \qquad \ell = 1, 2, \ldots, d, \tag{38}$$

which is holomorphic in a neighborhood of $\overline{D}_{|\Phi(\lambda)|}$ except for a pole at $z = \lambda$ of exact order t.

We had an application in my mind that we wanted to use the zeros of $\{Q^{\mu,S}_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$ and $\{Q^{E,S}_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$ to detect $|\mathbf{m}|$ system poles of $\mathbf{F} \in \mathcal{H}(E)^d$ nearest the set E. Although we are not able to solve our initial aim (Conjecture 6 below), we observed that the system poles of $\mathbf{F}$ with respect to $\mathbf{m}$ and their orders are the same as the system poles of

$$\overline{\mathbf{F}} := (F_1, \ldots, z^{m_1-1}F_1, F_2, \ldots, z^{m_d-1}F_d) \tag{39}$$

with respect to

$$\overline{\mathbf{m}} = (1, 1, \ldots, 1), \qquad \text{with} |\mathbf{m}| = |\overline{\mathbf{m}}|$$

and their orders. The creation of the vector in Equation (39) motivated us to define MOHP and MFHP approximations. Importantly, the zeros of $\{Q_{n,\mathbf{m}}^{\mu,M}\}_{n\geq|\mathbf{m}|}$ and $\{Q_{n,\mathbf{m}}^{E,M}\}_{n\geq|\mathbf{m}|}$ enable us to detect $|\mathbf{m}|$ system poles of $\mathbf{F} \in \mathcal{H}(E)^d$ nearest the set E (see [47,49]).

To state main results in [47,49,55], we need a generalization of $R_\lambda(\mathbf{F},\mathbf{m})$. For each system pole λ of $\mathbf{F}$ with respect to $\mathbf{m}$, we define a characteristic index $\overline{\rho}_\lambda(\mathbf{F},\mathbf{m})$ as follows. Let τ be the order of λ as a system pole of $\mathbf{F}$. For each $t = 1, 2, \ldots, \tau$, denote by $\rho_{\lambda,t}(\mathbf{F},\mathbf{m})$ the largest of all the numbers $\rho_t(G)$ (the index of the largest canonical domain containing at most t poles of G), where G is a polynomial combination of the type in Equation (38) that is holomorphic in a neighborhood of $\overline{D}_{|\Phi(\lambda)|}$ except for a pole at $z = \lambda$ of order t. There is only a finite number of such possible values so the maximum is indeed attained. Then, we define

$$\overline{\rho}_\lambda(\mathbf{F},\mathbf{m}) := \min_{t=1,2,\ldots,\tau} \rho_{\lambda,t}(\mathbf{F},\mathbf{m}).$$

Combining Theorem 1.2 in [47], Corollary 1.6 in [49], and Theorem 1.3 in [55], we arrive at the following theorem which contains analogs of Montessus de Ballore–Gonchar's Theorem for MOHP, MFHP, and MHP approximations.

Theorem 15. *Let $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$, $\mathbf{m} \in \mathbb{N}^d$ be a fixed multi-index, $\mu \in \mathbf{Reg}_{1,2}^{|\mathbf{m}|}(E)$, and $\alpha \in \mathbf{Strong}(E)$. Denote by $Q_\mathbf{m}(\mathbf{F})$ the monic polynomial whose zeros are the system poles of $\mathbf{F}$ with respect to $\mathbf{m}$ taking account of their order and by $\mathcal{P}(\mathbf{F},\mathbf{m})$ the set of all zeros of $Q_\mathbf{m}(\mathbf{F})$. Then, the following assertions are equivalent:*

(a) *$\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ counting multiplicities.*

(b) *The polynomials $Q_{n,\mathbf{m}}^{\mu,M}$ are uniquely determined for all sufficiently large n, and there exists a polynomial $\hat{Q}_{|\mathbf{m}|}$ of degree $|\mathbf{m}|$ such that*

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}}^{\mu,M} - \hat{Q}_{|\mathbf{m}|}\|^{1/n} = \hat{\theta} < 1. \tag{40}$$

(c) *The polynomials $Q_{n,\mathbf{m}}^{E,M}$ are uniquely determined for all sufficiently large n, and there exists a polynomial $\tilde{Q}_{|\mathbf{m}|}$ of degree $|\mathbf{m}|$ such that*

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}}^{E,M} - \tilde{Q}_{|\mathbf{m}|}\|^{1/n} = \tilde{\theta} < 1.$$

(d) *The polynomials $Q_{n,\mathbf{m}}^{\alpha}$ are uniquely determined for all sufficiently large n, and there exists a polynomial $\check{Q}_{|\mathbf{m}|}$ of degree $|\mathbf{m}|$ such that*

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}}^{\alpha} - \check{Q}_{|\mathbf{m}|}\|^{1/n} = \check{\theta} < 1. \tag{41}$$

Moreover, if one of Assertions (a)–(d) takes place, then $\hat{Q}_{|\mathbf{m}|} = \tilde{Q}_{|\mathbf{m}|} = \check{Q}_{|\mathbf{m}|} = Q_\mathbf{m}(\mathbf{F})$ and

$$\hat{\theta} = \tilde{\theta} = \check{\theta} = \max\left\{\frac{|\Phi(\lambda)|}{\overline{\rho}_\lambda(\mathbf{F},\mathbf{m})} : \lambda \in \mathcal{P}(\mathbf{F},\mathbf{m})\right\}. \tag{42}$$

Remark 2. *(i)* *To prove that (a) implies (b) with Equation (40) replaced by*

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}}^{\mu,M} - Q_\mathbf{m}(\mathbf{F})\|^{1/n} < 1,$$

we only need to impose that $\mu \in \mathbf{Reg}_2(E)$.

(ii) *To prove that (a) implies (d) with Equation (41) replaced by*

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}}^{\alpha} - Q_\mathbf{m}(\mathbf{F})\|^{1/n} < 1,$$

we only need to impose that $\alpha \in \mathbf{Root}(E)$.

(iii) For the estimates on the convergences of $\{\mathbf{R}^{\mu,M}_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$, $\{\mathbf{R}^{E,M}_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$, and $\{\mathbf{R}^{\alpha}_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$, we refer the readers to Theorem 1.2 of [47], Theorem 1.4 of [49], and Theorem 1.3 of [55] to avoid introducing a complicated collection of notations.

As Conjecture 5 (the scalar case) stands open, the exact rate of convergences $\{\mathbf{R}^{\mu,M}_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$, $\{\mathbf{R}^{E,M}_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$, and $\{\mathbf{R}^{\alpha}_{n,\mathbf{m}}\}_{n\geq|\mathbf{m}|}$ are still unknown.

Our expected result for the study SOHP and SFHP approximations is stated below.

Conjecture 6. *Let $\mathbf{F} \in \mathcal{H}(E)^d$, $\mathbf{m} \in \mathbb{N}^d$ be a fixed multi-index, and $\mu \in \mathbf{Rat}_{1,2}(E)$. Then, the following assertions are equivalent:*

(a) *$\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ counting multiplicities.*

(b) *The polynomials $Q^{\mu,S}_{n,\mathbf{m}}$ of $\mathbf{F}$ are uniquely determined for all sufficiently large n, and there exists a polynomial $\hat{Q}_{|\mathbf{m}|}$ of degree $|\mathbf{m}|$ such that*

$$\limsup_{n\to\infty} \|Q^{\mu,S}_{n,\mathbf{m}} - \hat{Q}_{|\mathbf{m}|}\|^{1/n} = \hat{\theta} < 1.$$

(c) *The polynomials $Q^{E,S}_{n,\mathbf{m}}$ of $\mathbf{F}$ are uniquely determined for all sufficiently large n, and there exists a polynomial $\tilde{Q}_{|\mathbf{m}|}$ of degree $|\mathbf{m}|$ such that*

$$\limsup_{n\to\infty} \|Q^{E,S}_{n,\mathbf{m}} - \tilde{Q}_{|\mathbf{m}|}\|^{1/n} = \tilde{\theta} < 1.$$

Moreover, if one of Assertions (a)–(c) takes place, then $\hat{Q}_{|\mathbf{m}|} = \tilde{Q}_{|\mathbf{m}|} = Q_{\mathbf{m}}(\mathbf{F})$ and

$$\hat{\theta} = \tilde{\theta} = \check{\theta} = \max\left\{\frac{|\Phi(\lambda)|}{\overline{\rho}_\lambda(\mathbf{F},\mathbf{m})} : \lambda \in \mathcal{P}(\mathbf{F},\mathbf{m})\right\}.$$

Concerning Conjectures 1–4 replacing $Q_{n,\mathbf{m}}$ by $Q^{\mu,S}_{n,\mathbf{m}}$, $Q^{\mu,M}_{n,\mathbf{m}}$, $Q^{E,S}_{n,\mathbf{m}}$, $Q^{E,M}_{n,\mathbf{m}}$, and $Q^{\alpha}_{n,\mathbf{m}}$, much less is known except the scalar case (see Theorem 10 above). Especially, the structures of $\mathbf{R}^{\mu,S}_{n,\mathbf{m}}$, $\mathbf{R}^{\mu,M}_{n,\mathbf{m}}$, $\mathbf{R}^{E,S}_{n,\mathbf{m}}$, and $\mathbf{R}^{E,M}_{n,\mathbf{m}}$ are much more complicated than HP approximants. There is no nice equality (similar to Equality (3.3) in [41]) which allows us to say something about the convergence and divergence of incomplete Padé approximation appear in SOHP, MOHP, SFHP, and MFHP approximations. Such equality is also a main ingredient for the proofs of Gonchar's theorem and Suetin's theorem for classical Padé approximation stated in Section 1 and an analog of Gonchar's theorem for the scalar MHP approximation when α is a Newton type in Theorem 2 of [31].

4. Conclusions

The study of classical Padé approximants on row sequences roots from the work of Montessus de Ballore [29] on the uniform convergence of row sequences of the approximants. The subject received renewed interest by Gonchar [30], who proved the converse statement of Montessus de Ballore's theorem. Soon after that, he studied an attraction of an individual pole of F in $\mathbb{B}_{R_m(F)}$ to the zeros of $Q_{n,m}$ when m is fixed as $n \to \infty$ in [31]. In the same paper, Gonchar proposed his conjecture (which is commonly called Gonchar's conjecture and remains unsolved). Since its introduction, several positive answers have supported the conjecture (see, e.g., [32–35]).

Graves-Morris and Saff [38] extended Montessus de Ballore's theorem to the vector case (which we call HP approximation). Recently, Cacoq, de la Calle Ysern, and López [40] defined the notion of system pole and proved the inverse result of Graves-Morris and Saff's result. In the spirit of Gonchar's conjecture, López and Gerpe proposed the following conjectures.

Conjecture 1. *Let* $\mathbf{F}$ *be a vector of formal Taylor expansions at the origin and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$*. If the denominators* $Q_{n,\mathbf{m}}$ *are uniquely determined for all sufficiently large* n *and* $\sigma(\lambda) \geq 1$*, then* λ *is a system pole of* $\mathbf{F}$ *with respect to* $\mathbf{m}$ *of order* $\tau = \sigma(\lambda)$*.*

Conjecture 2. *Let* $\mathbf{F}$ *be a vector of formal Taylor expansions at the origin and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$*. If the denominators* $Q_{n,\mathbf{m}}$ *are uniquely determined for all sufficiently large* n *and* $\gamma(\lambda) \geq 1$*, then* λ *is a system singularity of* $\mathbf{F}$ *with respect to* $\mathbf{m}$*.*

Some progress on Conjecture 2 was made in [41–43]. However, both conjectures are still open. Concurrently, generalizations of HP approximation were introduced and studied on row sequences [47,49,51,55,61]. Both conjectures can be asked for all generalizations of HP approximation. Although SOHP and SFHP approximations are natural, the proofs of analogs of Montessus de Ballore–Gonchar's theorem for such approximations are not available (see Conjecture 6 for our expectation).

Analogs of Montessus de Ballore–Gonchar's theorem and Suetin's theorem for the scalar case of all generalizations were completely proved. However, the exact rates of convergences of $\{R^{\mu,S}_{n,m}\}_{n\geq m}$, $\{R^{\mu,M}_{n,m}\}_{n\geq m}$, $\{R^{E,S}_{n,m}\}_{n\geq m}$, $\{R^{E,M}_{n,m}\}_{n\geq m}$, and $\{R^{\alpha}_{n,m}\}_{n\geq m}$ (when α is not a Newton type) are unknown (see Conjecture 5 for our expectation).

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Abbreviations

The following abbreviations are used in this manuscript:

HP approximation	Hermite–Padé approximation
SOHP approximation	standard orthogonal Hermite–Padé approximation
MOHP approximation	modified orthogonal Hermite–Padé approximation
SFHP approximation	standard Faber–Hermite–Padé approximation
MFHP approximation	modified Faber–Hermite–Padé approximation
MHP approximation	multipoint Hermite–Padé approximation

References

1. Hermite, C. Sur la fonction exponentielle. *C. R. Acad. Sci. Paris* **1873**, *77*, 18–24, 74–79, 226–233, 285–293.
2. Mahler, K. Perfect systems. *Compos. Math.* **1968**, *19*, 95–166.
3. Coates, J. On the algebraic approximation of functions. I, II, III. *Indag. Math.* **1966**, *28*, 421–461. [CrossRef]
4. Jager, H. A simultaneous generalization of the Padé table. I–VI. *Indag. Math.* **1964**, *26*, 193–249. [CrossRef]
5. Zudilin, W. Arithmetic of linear forms involving odd zeta values. *J. ThéOrie Nombres Bordx.* **2004**, *16*, 251–291. [CrossRef]
6. Ball, K.; Rivoal, T. Irrationalité d'une infinit'e de valeurs de la fonction zêta aux entiers impairs. *Invent. Math.* **2001**, *146*, 193–207. [CrossRef]

7. Lindemann, F. Über die Zahl π. *Math. Ann.* **1882**, *20*, 213–225. [CrossRef]
8. Apéry, R. Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque* **1979**, *61*, 11–13.
9. Van Assche, W. Analytic number theory and rational approximation. In *Coimbra Lecture Notes on Orthogonal Polynomials*; Branquinho, A., Foulquié, A., Eds.; Nova Science Pub.: New York, NY, USA, 2008; pp. 197–229.
10. Beckermann, B.; Labahn, G. A uniform approach for Hermite Padé and simultaneous Padé approximants and their matrix-type generalizations. *Numer. Algorithms* **1992**, *3*, 45–54. [CrossRef]
11. Beckermann, B.; Labahn, G. A uniform approach for the fast computation of matrix-type Padé approximants. *SIAM J. Matrix Anal. Appl.* **1994**, *15*, 804–823. [CrossRef]
12. Beckermann, B.; Labahn, G. Fraction-free computation of matrix rational interpolants and matrix GCDs. *SIAM J. Matrix Anal. Appl.* **2000**, *22*, 114–144. [CrossRef]
13. Borges, C.F. On a class of Gauss-like quadrature rules. *Numer. Math.* **1994**, *67*, 271–288. [CrossRef]
14. Cabay, S.; Jones, A.R.; Labahn, G. Computation of numerical Padé-Hermite and simultaneous Padé systems II: A weakly stable algorithm. *SIAM J. Matrix Anal. Appl.* **1996**, *17*, 268–297. [CrossRef]
15. Cabay, S.; Labahn, G. A superfast algorithm for multi-dimensional Padé systems. *Numer. Algorithms* **1992**, *2*, 201–224. [CrossRef]
16. Fidalgo Prieto, U.; Illán, J.; López Lagomasino, G. Hermite-Padé approximation and simultaneous quadrature formulas. *J. Approx. Theory* **2004**, *126*, 171–197. [CrossRef]
17. Lindman, E.L. Free-space boundary conditions for the time dependent wave equation. *J. Comput. Phys.* **1975**, *18*, 66–78. [CrossRef]
18. Coussement, J.; Van Assche, W. Gaussian quadrature for multiple orthogonal polynomials. *J. Comput. Appl. Math.* **2005**, *178*, 131–145. [CrossRef]
19. Kuijlaars, A.B.J. Multiple orthogonal polynomial ensembles. Recent trends in orthogonal polynomials and approximation theory. *Contemp. Math.* **2010**, *507*, 155–176.
20. Aptekarev, A.I. Asymptotics of simultaneously orthogonal polynomials in the Angelesco case. *Math. USSR Sb.* **1989**, *64*, 57–84. [CrossRef]
21. Aptekarev, A.I. Strong asymptotics of multiply orthogonal polynomials for Nikishin systems. *Sb. Math.* **1999**, *190*, 631–669. [CrossRef]
22. Martín P.; Baker, G.A., Jr. Two-point quasifractional approximant in physics. Truncation error. *J. Math. Phys.* **1991**, *32*, 1470–1477. [CrossRef]
23. Aptekarev, A.; Kaliaguine, V.; Iseghem J.V. The genetic sums' representation for the moments of a system of Stieltjes functions and its application. *Constr. Approx.* **2000**, *16*, 487–524. [CrossRef]
24. Daems, E.; Kuijlaars, A.B.J. Multiple orthogonal polynomials of mixed type and non-intersecting Brownian motions. *J. Approx. Theory* **2007**, *146*, 91–114. [CrossRef]
25. Bleher, P.M.; Kuijlaars, A.B.J. Random matrices with external source and multiple orthogonal polynomials. *Int. Math. Res. Not.* **2004**, *2004*, 109–129. [CrossRef]
26. Beckermann, B.; Kalyagin, V.; Matos, A.; Wielonsky, F. How well does the Hermite-Padé approximation smooth the Gibbs phenomenon? *Math. Comp.* **2011**, *80*, 931–958. [CrossRef]
27. Shang, Y. Analytical solution for an in-host viral infection model with time-inhomogeneous rates. *Acta Phys. Polon. B* **2015**, *46*, 1567–1577. [CrossRef]
28. Van Assche, W. Padé and Hermite-Padé approximation and orthogonality. *Surv. Approx. Theory* **2006** *2*, 61–91.
29. de Montessus de Ballore, R. Sur les fractions continues algébrique. *Bull. Soc. Math. Fr.* **1902**, *30*, 28–36. [CrossRef]
30. Gonchar, A.A. On convergence of Padé approximants for some classes of meromorphic functions. *Sb. Math.* **1975**, *26*, 555–575. [CrossRef]
31. Gonchar, A.A. Poles of rows of the Padé table and meromorphic continuation of functions. *Sb. Math.* **1981**, *43*, 527–546. [CrossRef]
32. Vavilov, V.V. On the singular points of a meromorphic function given by Its Taylor series. *Dokl. Akad. Nauk SSSR* **1976**, *231*, 1281–1284.
33. Vavilov, V.V.; López Lagomasino, G.; Prokhorov, V.A. On an Inverse Problem for the Rows of a Padé Table. *Mat. Sb.* **1979**, *110*, 117–129. [CrossRef]

34. Vavilov, V.V.; Prokhorov, V.A.; Suetin, S.P. The poles of the *m*th row of the Padé table and the singular points of a function. *Mat. Sb.* **1983**, *122*, 475–480.
35. Suetin, S.P. On an inverse problem for the *m*th row of the Padé table. *Sb. Math.* **1985**, *52*, 231–244. [CrossRef]
36. Fabry, E. Sur les points singuliers d'une fonction données par son développement de Taylor. *Ann. École Norm. Sup. Paris* **1896**, *13*, 367–399. [CrossRef]
37. Nikishin, E.M. *Rational Approximations and Orthogonality*; Amer. Math. Soc.: Providence, RI, USA, 1991.
38. Graves-Morris, P.R.; Saff, E.B. A de Montessus theorem for vector valued rational interpolants. In *Rational Approximation and Interpolation*; Springer: Berlin/Heidelberg, Germany, 1984; pp. 227–242.
39. Cacoq, J.; de la Calle Ysern, B.; López Lagomasino, G. Incomplete Padé approximation and convergence of row sequences of Hermite-Padé approximants. *J. Approx. Theory* **2013**, *170*, 59–77. [CrossRef]
40. Cacoq, J.; de la Calle Ysern, B.; López Lagomasino, G. Direct and inverse results on row sequences of Hermite-Padé approximation. *Constr. Approx.* **2013**, *38*, 133–160. [CrossRef]
41. Lagomasino, G.L.; Gerpe, Y.Z. Inverse results on row sequences of Hermite-Padé approximation. *Proc. Steklov Inst. Math.* **2017**, *298*, 152–169. [CrossRef]
42. López, G.L.; Gerpe, Y.Z. Higher order recurrences and row sequences of Hermite-Padé approximation. *J. Differ. Equ. Appl.* **2018**, *24*, 1830–1845. [CrossRef]
43. Lagomasino, G.L. On row sequences of Padé and Hermite-Padé approximation. In Proceedings of the Modern Trends in Constructive Function Theory: Conference in Honor of Ed Saff's 70th Birthday: Constructive Functions 2014, Vanderbilt University, Nashville, TN, USA, 26–30 May 2014; Volume 661.
44. Buslaev, V.I. Relations for the coefficients, and singular points of a function. *Math. USSR-Sb.* **1988**, *59*, 349–377. [CrossRef]
45. de la Calle Ysern, B.; Mínguez Ceniceros, J. Zero distribution of incomplete Padé and Hermite-Padé approximations. *J. Approx. Theory* **2016**, *201*, 13–29. [CrossRef]
46. Bosuwan, N.; López Lagomasino, G.; Saff, E.B. Determining singularities using row sequences of Padé-orthogonal approximants. *Jaen J. Approx.* **2013**, *5*, 179–208.
47. Bosuwan, N.; López Lagomasino, G. Determining system poles using row sequences of orthogonal Hermite-Padé approximants. *J. Approx. Theory* **2018**, *231*, 15–40. [CrossRef]
48. Suetin, S.P. On the convergence of rational approximations to polynomial expansions in domains of meromorphy of a given function. *Math. USSR Sb.* **1978**, *34*, 367–381. [CrossRef]
49. Bosuwan, N.; López Lagomasino, G. Direct and inverse results on row sequences of simultaneous Padé-Faber approximants. *Mediterr. J. Math.* **2019**, *16*, 36. [CrossRef]
50. Stahl, H.; Totik, V. *General Orthogonal Polynomials*; Cambridge University Press: Cambridge, UK, 1992; Volume 43.
51. Bosuwan, N. Convergence of row sequences of simultaneous Padé-orthogonal approximants. *Comput. Methods Funct. Theory* **2017**, *17*, 525–556. [CrossRef]
52. Bosuwan, N.; López Lagomasino, G. Inverse theorem on row sequences of linear Padé-orthogonal approximants. *Comput. Methods Funct. Theory* **2015**, *15*, 529–554. [CrossRef]
53. Walsh, J.L. *Interpolation and Approximation by Rational Functions in the Complex Domain*, 5th ed.; Colloquium Publications, American Mathematical Society: Providence, RI, USA, 1969.
54. Suetin, S.P. Inverse theorems on generalized Padé approximants. *Math. USSR Sb.* **1980**, *37*, 581–597. [CrossRef]
55. Bosuwan, N.; Lagomasino, G.L.; Gerpe, Y.Z. Direct and inverse results for multipoint Hermite-Padé approximants. *Anal. Math. Phys.* **2019**. [CrossRef]
56. Saff, E.B. Regions of meromorphy determined by the degree of best rational approximation. *Proc. Am. Math. Soc.* **1971**, *29*, 30–38. [CrossRef]
57. Buslaev, V.I. An analogue of Fabry's theorem for generalized Padé approximants. *Math. Sb.* **2009**, *200*, 39–106. [CrossRef]
58. Bosuwan, N. Direct and inverse results on row sequences of generalized Padé approximants to polynomial expansions. *Acta Math. Hung.* **2019**, *157*, 191–219. [CrossRef]
59. Bosuwan, N. On Montessus de Ballore's theorem for simultaneous Padé-Faber approximants. *Demonstr. Math.* **2018**, *51*, 45–61. [CrossRef]

60. Geronimus, L.Y. *Orthogonal Polynomials on a Circle and Interval*; Pergamon Press: Oxford, UK, 1960.
61. Bosuwan, N. Convergence of row sequences of simultaneous Padé-Faber approximants. *Math. Notes* **2018**, *103*, 643–656. [CrossRef]
62. Cacoq, J.; López Lagomasino, G. Convergence of row sequences of simultaneous Fourier-Padé approximation. *Jaen J. Approx.* **2012**, *4*, 101–120.