mathematics

Article

Ekeland Variational Principle in the Variable Exponent Sequence Spaces $\ell_{p(\cdot)}$

Monther R. Alfuraidan [1] and Mohamed A. Khamsi [2,*]

[1] Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia; monther@kfupm.edu.sa
[2] Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, TX 79968, USA
* Correspondence: mohamed@utep.edu

Received: 12 February 2020; Accepted: 2 March 2020; Published: 7 March 2020

Abstract: In this work , we investigate the modular version of the Ekeland variational principle (EVP) in the context of variable exponent sequence spaces $\ell_{p(\cdot)}$. The core obstacle in the development of a modular version of the EVP is the failure of the triangle inequality for the module. It is the lack of this inequality, which is indispensable in the establishment of the classical EVP, that has hitherto prevented a successful treatment of the modular case. As an application, we establish a modular version of Caristi's fixed point theorem in $\ell_{p(\cdot)}$.

Keywords: Caristi; Ekeland Variational Principle; Electrorheological fluids; fixed point; modular vector spaces; Nakano; variable exponent sequence spaces

MSC: primary 47H09; 47H10

1. Introduction

The variable exponent sequence spaces can be traced back to the seminal work by W. Orlicz [1] where he introduced the vector space

$$\ell_{p(\cdot)} = \Big\{ \{x_n\} \subset \mathbb{R}^{\mathbb{N}};\ \sum_{n=0}^{\infty} |\lambda\, x_n|^{p(n)} < \infty \ \text{ for some } \lambda > 0 \Big\},$$

where $\{p(n)\} \subset [1, \infty)$. The variable exponent sequence spaces were thoroughly examined by many, among others: [2–6]. Their generalization, the function spaces $L^{p(\cdot)}$, is currently an active field of research extending into very diverse mathematical and applied areas [7]. In particular, variable exponent Lebesgue spaces $L^{p(\cdot)}$ are the natural spaces for the mathematical description of non-Newtonian fluids [8,9]. Non-Newtonian fluids (also known as smart fluids or electro-rheological fluids) have a wide range of applications, including military science, civil engineering, and medicine.

This work is devoted to the investigation of the modular version of the Ekeland variational principle (EVP) in the spaces $\ell_{p(\cdot)}$. This line of research has never been undertaken due to the lack of the triangle inequality for the modular version. In the absence of the Δ_2-condition, it is unclear how to approach this problem even if one wants to use the Luxemburg distance. As a byproduct of our result, we present a modular version of the Caristi fixed point theorem. The vastness of the subject known as metric fixed point theory prevents us from including the necessary background in this work. The reader is referred to [10,11] for background material.

2. Preliminaries

We open the discussion by presenting some definitions and basic facts about the space $\ell_{p(\cdot)}$.

Definition 1 ([1]). *Consider the vector space*

$$\ell_{p(\cdot)} = \left\{ \{x_n\} \subset \mathbb{R}^{\mathbb{N}};\ \sum_{n=0}^{\infty} \frac{1}{p(n)} \, |\lambda \, x_n|^{p(n)} < \infty \ \text{for some } \lambda > 0 \right\},$$

where $p : \mathbb{N} \to [1, \infty)$.

Though not under this name, these spaces were first considered by Orlicz [1]. It was at a later stage that the importance of these sequence spaces and their continuous counterpart, the Lebesgue spaces of variable exponent, became major objects of research. Inspired by the structure of these spaces, Nakano [4,12] introduced the notion of modular vector space.

Proposition 1 ([3,5,12]). *Consider the vector space* $\ell_{p(\cdot)}$. *The function* $v : \ell_{p(\cdot)} \to [0, \infty]$, *defined by*

$$v(x) = v((x_n)) = \sum_{n=0}^{\infty} \frac{1}{p(n)} \, |x_n|^{p(n)},$$

has the following properties:

(i) $v(x) = 0$ *if, and only if,* $x = 0$;
(ii) $v(\gamma x) = v(x)$, *if* $|\gamma| = 1$;
(iii) For arbitrary $x, y \in \ell_{p(\cdot)}$ *and any* $t : 0 \leq t \leq 1$, *the inequality*

$$v(t\, x + (1-t)\, y) \leq t\, v(x) + (1-t)\, v(y)$$

holds.

A function satisfying the preceding set of properties is said to be convex modular.

We stress the left continuity of v, i.e., the fact that $\lim_{\alpha \to 1-} v(\alpha x) = v(x)$, for any $x \in \ell_{p(\cdot)}$. Next, we introduce the modular version of some properties known in the metric setting.

Definition 2 ([11]).

(a) *A sequence* $\{x_n\} \subset \ell_{p(\cdot)}$ *is* v*-convergent to* $x \in \ell_{p(\cdot)}$ *if, and only if,* $v(x_n - x) \to 0$. *Note that the* v*-limit is unique if it exists.*
(b) *A sequence* $\{x_n\} \subset \ell_{p(\cdot)}$ *is* v*-Cauchy if* $v(x_n - x_m) \to 0$ *as* $n, m \to \infty$.
(c) *A subset* $C \subset \ell_{p(\cdot)}$ *is* v*-closed if for any sequence* $\{x_n\} \subset C$ *that* v*-converges to* x, *it holds* $x \in C$.

We emphasize the fact that v satisfies the Fatou's property, namely, for any sequence $\{y_n\} \subseteq \ell_{p(\cdot)}$ which v-converges to y and any $x \in \ell_{p(\cdot)}$, it holds that

$$v(x - y) \leq \liminf_{n \to \infty} v(x - y_n).$$

The next property, called the Δ_2-condition, plays a crucial role in the study of modular vector spaces.

Definition 3. v *is said to fulfill the* Δ_2*-condition if, for some* $K \geq 0$, *it holds that*

$$v(2x) \leq K\, v(x),$$

for any $x \in \ell_{p(\cdot)}$.

It is a matter of routine to verify that v satisfies the Δ_2-condition if, and only if, $p^+ = \sup_{n\in\mathbb{N}} p(n) < \infty$ [3,5,12]. The validity of this condition has far reaching implications in the study of modular vector spaces [11,13,14].

3. Main Results

The modular version of EVP was difficult to establish because the modular fails the triangle inequality, which is indispensable in the establishment of EVP in metric spaces. In the spirit of the work by Farkas [15], we present the following result:

Theorem 1. *Let C be a nonempty, v-closed subset of $\ell_{p(\cdot)}$ and $f : C \to \mathbb{R} \cup \{+\infty\}$ be a proper, v-lower semi-continuous function bounded from below, i.e., $\inf_{x\in C} f(x) > -\infty$. Fix $\{\delta_n\} \subset (0, +\infty)$ and $\varepsilon > 0$. Let $x_0 \in C$ be such that*

$$f(x_0) \leq \inf_{x\in C} f(x) + \varepsilon.$$

Then, there exists $\{x_n\}$ in C which v-converges to some x_ε, such that

(i) $v(x_\varepsilon - x_n) \leq \varepsilon/(2^n\delta_0)$, *for any* $n \in \mathbb{N}$;

(ii) $f(x_\varepsilon) + \sum_{n=0}^{+\infty} \delta_n \, v(x_\varepsilon - x_n) \leq f(x_0)$;

(iii) and for any $x \neq x_\varepsilon$, we have

$$f(x_\varepsilon) + \sum_{n=0}^{+\infty} \delta_n \, v(x_\varepsilon - x_n) < f(x) + \sum_{n=0}^{+\infty} \delta_n \, v(x - x_n).$$

Proof. Set

$$\mathcal{S}(x_0) = \Big\{x \in C;\ f(x) + \delta_0 \, v(x - x_0) \leq f(x_0)\Big\}.$$

Clearly $\mathcal{S}(x_0)$ is nonempty, as $x_0 \in \mathcal{S}(x_0)$, and is v-closed because f is v-lower semi-continuous, v satisfies the Fatou property and C is v-closed. Pick $x_1 \in \mathcal{S}(x_0)$ such that

$$f(x_1) + \delta_0 \, v(x_1 - x_0) \leq \inf_{x\in\mathcal{S}(x_0)} \Big\{f(x) + \delta_0 \, v(x - x_0)\Big\} + \frac{\varepsilon\,\delta_1}{2\delta_0}.$$

Next set

$$\mathcal{S}(x_1) = \Big\{x \in \mathcal{S}(x_0);\ f(x) + \sum_{i=0}^{1} \delta_i \, v(x - x_i) \leq f(x_1) + \delta_0 \, v(x_1 - x_0)\Big\}.$$

Arguing, as in the case of $\mathcal{S}(x_0)$, it is easily concluded that $\mathcal{S}(x_1)$ is nonempty and v-closed. We assume that $\{x_0, x_1, \cdots, x_n\}$ and $\{\mathcal{S}(x_0), \mathcal{S}(x_1), \cdots, \mathcal{S}(x_n)\}$ are constructed. Then we pick $x_{n+1} \in \mathcal{S}(x_n)$ such that

$$f(x_{n+1}) + \sum_{i=0}^{n} \delta_i \, v(x_{n+1} - x_i) \leq \inf_{x\in\mathcal{S}(x_n)} \Big\{f(x) + \sum_{i=0}^{n} \delta_i \, v(x - x_i)\Big\} + \frac{\varepsilon\,\delta_n}{2^n\delta_0}.$$

We define the set

$$\mathcal{S}(x_{n+1}) = \Big\{x \in \mathcal{S}(x_n);\ f(x) + \sum_{i=0}^{n+1} \delta_i \, v(x - x_i) \leq f(x_{n+1}) + \sum_{i=0}^{n} \delta_i \, v(x_{n+1} - x_i)\Big\}.$$

By induction, we build the sequences $\{x_n\}$ and $\{\mathcal{S}(x_n)\}$. We fix $n \in \mathbb{N}$. Let $z \in \mathcal{S}(x_n)$. Then

$$f(z) + \sum_{i=0}^{n} \delta_i \, v(z - x_i) \leq f(x_n) + \sum_{i=0}^{n-1} \delta_i \, v(x_n - x_i),$$

which implies

$$\begin{array}{rcl}\delta_n\, v(z-x_n) & \leq & f(x_n)+\sum\limits_{i=0}^{n-1}\delta_i\, v(x_n-x_i)-\Big[f(z)+\sum\limits_{i=0}^{n-1}\delta_i\, v(z-x_i)\Big]\\ & \leq & f(x_n)+\sum\limits_{i=0}^{n-1}\delta_i\, v(x_n-x_i)-\inf\limits_{x\in\mathcal{S}(x_{n-1})}\Big[f(x)+\sum\limits_{i=0}^{n-1}\delta_i\, v(x-x_i)\Big]\\ & \leq & \dfrac{\varepsilon\,\delta_n}{2^n\delta_0}.\end{array}$$

As $\{\mathcal{S}(x_n)\}$ is decreasing with $x_n\in\mathcal{S}(x_n)$, for any $n\in\mathbb{N}$, we conclude that

$$v(x_{n+h}-x_n)\leq\frac{\varepsilon}{2^n\delta_0},$$

for any $n,h\in\mathbb{N}$. In other words, we have proved that $\{x_n\}$ is v-Cauchy. As $\ell_{p(\cdot)}$ is v-complete the v-limit x_ε of $\{x_n\}$ exists and $\bigcap\limits_{n\in\mathbb{N}}\mathcal{S}(x_n)=\{x_\varepsilon\}$ holds. Note that, since $x_{n+1}\in\mathcal{S}(x_n)$, we have

$$f(x_{n+1})+\sum_{i=0}^{n}\delta_i\, v(x_{n+1}-x_i)\leq f(x_n)+\sum_{i=0}^{n-1}\delta_i\, v(x_n-x_i),$$

i.e., the sequence $\{f(x_n)+\sum\limits_{i=0}^{n-1}\delta_i\, v(x_n-x_i)\}$ is decreasing. Next, let $x\neq x_\varepsilon$. Then there exists $m\in\mathbb{N}$ such that x is not in $\mathcal{S}(x_n)$, for any $n\geq m$, i.e.,

$$f(x_n)+\sum_{i=0}^{n-1}\delta_i\, v(x_n-x_i)<f(x)+\sum_{i=0}^{n}\delta_i\, v(x-x_i).$$

As $x_\varepsilon\in\mathcal{S}(x_n)$, for any $n\geq m$, we obtain

$$\begin{array}{rcl}f(x_\varepsilon)+\sum\limits_{i=0}^{n}\delta_i\, v(x_\varepsilon-x_i) & \leq & f(x_n)+\sum\limits_{i=0}^{n-1}\delta_i\, v(x_n-x_i)\\ & \leq & f(x_m)+\sum\limits_{i=0}^{m-1}\delta_i\, v(x_m-x_i).\end{array}$$

Letting $n\to+\infty$ in the preceding inequality, it follows that

$$\begin{array}{rcl}f(x_\varepsilon)+\sum\limits_{i=0}^{+\infty}\delta_i\, v(x_\varepsilon-x_i) & \leq & f(x_m)+\sum\limits_{i=0}^{m-1}\delta_i\, v(x_m-x_i)\\ & < & f(x)+\sum\limits_{i=0}^{m}\delta_i\, v(x-x_i)\\ & \leq & f(x)+\sum\limits_{i=0}^{+\infty}\delta_i\, v(x-x_i).\end{array}$$

In conclusion,

$$f(x_\varepsilon)+\sum_{n=0}^{+\infty}\delta_n\, v(x_\varepsilon-x_n)<f(x)+\sum_{n=0}^{+\infty}\delta_n\, v(x-x_n),$$

which completes the proof of the theorem. □

As an application of Theorem 1, we derive an extension of Caristi's fixed point theorem in $\ell_{p(\cdot)}$.

Theorem 2. *Let C be a nonempty* υ*-closed subset of* $\ell_{p(\cdot)}$. *We fix* $\varepsilon > 0$ *and* $\{\delta_n\}$ *such that* $\eta = \sum_{n=0}^{\infty} \delta_n < +\infty$ *and* $\eta > 0$. *Let* $T : C \to C$ *be a mapping such that there exists a proper,* υ*-lower semi-continuous function* $f : C \to \mathbb{R} \cup \{+\infty\}$ *bounded from below, i.e.,* $\inf_{x \in C} f(x) > -\infty$, *such that*

(1) $\upsilon(T(x) - y) - \upsilon(x - y) \leq \upsilon(x - T(x))$, *for any* $x, y \in C$;

(2) $\upsilon(x - T(x)) \leq f(x) - f(T(x))$, *for any* $x \in C$.

Then, T has a fixed point in C.

Proof. As $\eta = \sum_{n=0}^{\infty} \delta_n$ is a nonzero positive number, the function defined by $f^* = \eta\, f$ is also proper, υ-lower semi-continuous and bounded from below. Moreover, we have for any $x \in C$,

$$\eta\, \upsilon(x - T(x)) \leq f^*(x) - f^*(T(x)). \tag{AM}$$

From the inequality $\inf_{x \in \ell_{p(\cdot)}} f^*(x) > -\infty$, one derives the existence of $x_0 \in C$ such that $f^*(x_0) < \inf_{x \in \ell_{p(\cdot)}} f^*(x) + \varepsilon$. Using Theorem 1, one concludes that there exists $\{x_n\}$ which υ-converges to some $x_\varepsilon \in C$, such that

$$f^*(x_\varepsilon) + \sum_{n=0}^{+\infty} \delta_n\, \upsilon(x_\varepsilon - x_n) < f^*(x) + \sum_{n=0}^{+\infty} \delta_n\, \upsilon(x - x_n),$$

for any $x \neq x_\varepsilon$. We claim that $T(x_\varepsilon) = x_\varepsilon$, and assume not. Then, we have $T(x_\varepsilon) \neq x_\varepsilon$. Hence, we have

$$f^*(x_\varepsilon) + \sum_{n=0}^{+\infty} \delta_n\, \upsilon(x_\varepsilon - x_n) < f^*(T(x_\varepsilon)) + \sum_{n=0}^{+\infty} \delta_n\, \upsilon(T(x_\varepsilon) - x_n),$$

which implies

$$\begin{aligned} f^*(x_\varepsilon) - f^*(T(x_\varepsilon)) &< \sum_{n=0}^{+\infty} \delta_n\, \upsilon(T(x_\varepsilon) - x_n) - \sum_{n=0}^{+\infty} \delta_n\, \upsilon(x_\varepsilon - x_n) \\ &= \sum_{n=0}^{+\infty} \delta_n \Big(\upsilon(T(x_\varepsilon) - x_n) - \upsilon(x_\varepsilon - x_n) \Big). \end{aligned}$$

Using the assumption (1), we find

$$f^*(x_\varepsilon) - f^*(T(x_\varepsilon)) < \sum_{n=0}^{+\infty} \delta_n\, \upsilon(T(x_\varepsilon) - x_\varepsilon) = \eta\, \upsilon(T(x_\varepsilon) - x_\varepsilon).$$

The inequality (AM) implies

$$\eta\, \upsilon(T(x_\varepsilon) - x_\varepsilon) \leq f^*(x_\varepsilon) - f^*(T(x_\varepsilon)) < \eta\, \upsilon(T(x_\varepsilon) - x_\varepsilon).$$

This is the sought contradiction. Therefore, we must have $T(x_\varepsilon) = x_\varepsilon$, as claimed. □

Author Contributions: M.R.A. and M.A.K. contributed equally on the development of the theory and their respective analysis. All authors have read and agreed to the published version of the manuscript.

Funding: King Fahd University of Petroleum & Minerals research project No. IN171032.

Acknowledgments: Both authors take the opportunity to express their gratitude for the support of the deanship of scientific research at King Fahd University of Petroleum & Minerals in funding the presented research effort.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript :

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	linear dichroism

References

1. Orlicz, W. Über konjugierte Exponentenfolgen. *Studia Math.* **1931**, *3*, 200–211. [CrossRef]
2. Bardaro, C.; Musielak, J.; Vinti, G. *Nonlinear Integral Operators and Applications, de Gruyter Series in Nonlinear Analisys*; De Gruyter: Berlin, Germany; New York, NY, USA, 2003.
3. Klee, V. Summability in $\ell(p_{11}, p_{21}, \cdots)$ Spaces. *Studia Math.* **1965**, 277–280. [CrossRef]
4. Nakano, H. Modulared sequence spaces. *Proc. Jpn. Acad.* **1951**, *27*, 508–512. [CrossRef]
5. Sundaresan, K. Uniform convexity of Banach spaces $\ell(\{p_i\})$. *Studia Math.* **1971**, *39*, 227–231. [CrossRef]
6. Waterman, D.; Ito, T.; Barber, F.; Ratti, J. Reflexivity and Summability: The Nakano $\ell(p_i)$ spaces. *Studia Math.* **1969**, *331*, 141–146. [CrossRef]
7. Diening, L.; Harjulehto, P.; Hästö, P.; Ružîĉka, M. *Lebesgue and Sobolev Spaces with Variable Exponents*; Lecture Note in Mathematics 2017; Springer: Berlin, Germany, 2011.
8. Rajagopal, K.; Ružička, M. On the modeling of electrorheological materials. *Mech. Res. Commun.* **1996**, *23*, 401–407. [CrossRef]
9. Ružička, M. *Electrorheological Fluids: Modeling and Mathematical Theory*; Lecture Notes in Mathematics; Springer: Berlin, Germany, 2000; volume 1748.
10. Khamsi, M.A.; Kirk, W.A. *An Introduction to Metric Spaces and Fixed Point Theory*; John Wiley: New York, NY, USA, 2001.
11. Khamsi, M.A.; Kozlowski, W.M. *Fixed Point Theory in Modular Function Spaces*; Birkhauser: New York, NY, USA, 2015.
12. Nakano, H. *Modulared Semi-Ordered Linear Spaces*; Maruzen Co.: Tokyo, Japan, 1950.
13. Kozlowski, W.M. *Modular Function Spaces*; Series of Monographs and Textbooks in Pure and Applied Mathematics; Dekker: New York, NY, USA; Basel, Swizerland, 1988; Volume 122.
14. Musielak, J. *Orlicz Spaces and Modular Spaces*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany; New York, NY, USA; Tokyo, Japan, 1983; Volume 1034.
15. Farkas, C. A Generalized form of Ekeland's variational principle. *Analele Universitatii "Ovidius" Constanta Seria Matematica* **2013**, 20, 101–111. [CrossRef]