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Stability of Unbounded Differential Equations in Menger k-Normed Spaces: A Fixed Point Technique

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Abstract: We attempt to solve differential equations $v'(\nu) = \Gamma(\nu, v(\nu))$ and use the fixed point technique to prove its Hyers–Ulam–Rassias stability in Menger k-normed spaces.

Keywords: integral equation; differential equation; stability; Menger k-normed spaces

MSC: 54H24; 39B62; 47N20

1. Introduction

Let $(S, \xi, *)$ be Menger k-normed space and let I be an open interval. Assume that for any function $\Gamma : I \to S$ satisfying the differential inequality

$$\xi_\tau^{\sum_{j=0}^n a_j(\nu_1)v_1^{(j)}(\nu_1)+h(\nu_1),\cdots,\sum_{j=0}^n a_j a_j(\nu_k)v_k^{(j)}(\nu_k)+h(\nu_k)} \geq 1-\varepsilon$$

for all $\nu \in I$ and for some $\varepsilon \geq 0$, there exists a solution $\gamma_0 : I \to S$ of the differential equation

$$\sum_{j=0}^{n} a_j(\nu)v_1^{(j)}(\nu) + h(\nu) = 0$$

such that $\xi_\tau^{\gamma(\nu_1)-\gamma_0(\nu_1),\cdots,\gamma(\nu_k)-\gamma_0(\nu_k)} \geq 1 - K(\varepsilon)$ for any $\nu \in I$, where $K(\varepsilon)$ is an expression of ε only. Then, we say that the above differential equation has the Hyers–Ulam stability. If the above statement is also true when we replace $1-\varepsilon$ by $\varphi_\tau^{\nu 1,\cdots,\nu_k}$, where $\varphi : J^k \to O^+$ is a distribution function not depending on γ and γ_0 explicitly, then we say that the corresponding differential equation has the Hyers–Ulam–Rassias stability (or the generalized Hyers–Ulam stability). We may apply these terminologies for other differential equations. For more detailed definitions of the Hyers–Ulam stability and the Hyers–Ulam–Rassias stability [1,2].

Obłoza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equation [3,4]. Next, Takahasi, Miura and Miyajima, proved in [5–8] that the Hyers-Ulam stability holds for the Banach space valued differential equation $v'(\nu) = \lambda v(\nu)$. Recently, Miura, Miyajima and Takahasi also proved the Hyers-Ulam stability of linear differential equations of first order, $v'(\nu) + \Psi(\nu)v(\nu) = 0$, where $\Psi(x)$ is a continuous function. In the following, Jung proved the Hyers–Ulam stability of linear differential equations of other type (see [9–13]). In this paper, for a continuous function $\Gamma(\nu, v)$, we will adopt the idea of Cădariu and Radu [14,15] and prove the Hyers–Ulam–Rassias stability as well as the Hyers–Ulam stability of the differential equation of the form

$$v'(\nu) = \Gamma(\nu, v(\nu)) \tag{1}$$

in the Menger k-normed spaces.

2. Preliminaries

Let Ξ^+ be the set of distribution mappings, i.e., the set of all mappings $\rho : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1]$, writing ρ_τ for $\rho(\tau)$, such that ρ is left continuous and increasing on $\mathbb{R}$. $O^+ \subseteq \Xi^+$ includes all mappings $\rho \in \Xi^+$ for which $\ell^-\rho_{+\infty}$ is one and $\ell^-\rho_\tau$ is the left limit of the mapping ρ at the point τ, i.e., $\ell^-\rho_\tau = \lim_{\sigma\to\tau^-} \rho_\sigma$.

In Ξ^+, we define " $\leq$ " as follows:

$$\rho \leq \varrho \iff \rho_\tau \leq \varrho_\tau$$

for each τ in $\mathbb{R}$ (partially ordered). Note that the function ϑ^0 defined by

$$\vartheta^u_s = \begin{cases} 0, & \text{if } s \leq u, \\ 1, & \text{if } s > u, \end{cases}$$

is a element of Ξ^+ and ϑ^0 is the maximal element in this space (for more details, see [16–18]).

Definition 1. *([16,19]) A continuoustriangular norm (shortly, a ct-norm) is a continuous binary operation $*$ from $I = [0,1]^2$ to I such that*

(a) $\varsigma * \tau = \tau * \varsigma$ *and* $\varsigma * (\tau * v) = (\varsigma * \tau) * v$ *for all* $\varsigma, \tau, v \in [0,1]$;
(b) $\varsigma * 1 = \varsigma$ *for all* $\varsigma \in I$;
(c) $\varsigma * \tau \leq v * \iota$ *whenever* $\varsigma \leq v$ *and* $\tau \leq \iota$ *for all* $\varsigma, \tau, v, \iota \in I$.

Some examples of the t-norms are as follows:

(1) $\varsigma *_P \tau = \varsigma\tau$ (: the product t-norm);
(2) $\varsigma *_M \tau = \min\{\varsigma, \tau\}$ (: the minimum t-norm);
(3) $\varsigma *_L \tau = \max\{\varsigma + \tau - 1, 0\}$ (: the Lukasiewicz t-norm).

Definition 2. *([20,21]) Suppose that $*$ is a ct-norm, S is a linear space and ξ is a mapping from S^k to O^+. In this case, the ordered tuple $(S, \xi, *)$ is called a Menger k-normed linear space (in short, M-k-NLS) if the following conditions are satisfied:*

(ξ1) $\xi^{s_1,\ldots,s_k}_\tau = \vartheta^0_\tau$ *for* $\tau \geq 0$ *if and only if* $s_1, \ldots, s_n$ *are linearly dependent;*
(ξ2) $\xi^{s_1,\ldots,s_k}_\tau$ *is invariant under any permutation of* $s_1, \ldots, s_k \in S$;
(ξ3) $\xi^{\alpha s_1,\ldots,s_k}_\tau = \xi^{s_1,\ldots,s_k}_{\frac{\tau}{|\alpha|}}$ *if* $\alpha \neq 0$;
(ξ4) $\xi^{s_0+s_1,s_2,\ldots,s_k}_{\tau+\varsigma} \geq \xi^{s_0,s_2,\ldots,s_k}_\varsigma * \xi^{s_1,s_2,\ldots,s_k}_\tau$.

For more details see [22–28].

Example 1. *Let $(S, \|.,\ldots,.\|)$ be a linear k-normed space. Then*

$$\xi^{s_1,\ldots,s_k}_\tau = \begin{cases} 0, & \text{if } \tau \leq 0, \\ \exp(-\|s_1,\ldots,s_k\|/\tau), & \text{if } \tau > 0, \end{cases}$$

define a Menger norm and the ordered tuple $(S, \xi, *_M)$ is a M-k-NLS.

Note that, a $[0, \infty]$-valued metric is called a generalized metric.

Theorem 1 ([29]). *Consider a complete generalized metric space (Σ, δ) and a strictly contractive function $\Lambda : \Sigma \to \Sigma$ with Lipschitz constant $\beta < 1$. So, for every given element $\sigma \in \Sigma$, either*

$$\delta(\Lambda^n \sigma, \Lambda^{n+1}\sigma) = \infty$$

for each $n \in \mathbb{N}$ or there is $n_0 \in \mathbb{N}$ such that

(1) $\delta(\Lambda^n \sigma, \Lambda^{n+1}\sigma) < \infty, \qquad \forall n \geq n_0$;
(2) *the fixed point s^* of Λ is the convergent point of sequence $\{\Lambda^n \sigma\}$;*
(3) *in the set $V = \{s \in T \mid \delta(\Lambda^{n_0}\sigma, s) < \infty\}$, s^* is the unique fixed point of Λ;*
(4) $(1-\beta)\delta(s, s^*) \leq \delta(s, \Lambda s)$ *for every $s \in V$.*

3. Hyers–Ulam–Rassias Stability in M-k-NLS

Recently, Cădariu and Radu [14] applied the fixed point method to the investigation of the Jensen's functional equation. Using such an idea, they could present a proof for the Hyers–Ulam stability of that equation (see [11,15,30]). In this section, by using the idea of Cădariu and Radu, we will prove the Hyers–Ulam–Rassias stability of the differential Equation (1). Hereinafter we suppose that $* = *_M = \bigwedge$.

Theorem 2. *Let $p < q$ and $\rho = q - p$. Let $J = [p, q]$ and choose $m \in J$. Consider the constants β with $0 < \rho\beta < 1$. Let the continuous map $\Gamma : J \times \mathbb{R} \to \mathbb{R}$ satisfies in the Lipschitz condition*

$$\xi_{\rho\beta t}^{\Gamma(\nu_1, \upsilon_1) - \Gamma(\nu_1, \vartheta_1), \ldots, \Gamma(\nu_k, \upsilon_k) - \Gamma(\nu_k, \vartheta_k)} \geq \xi_t^{\upsilon_1 - \vartheta_1, \ldots, \upsilon_k - \vartheta_k} \tag{2}$$

for any $\nu_j \in J$, $\upsilon_j, \vartheta_j \in \mathbb{R}$, $(j = 1, 2, \ldots, k)$ and $t > 0$. If a continuous differentiable function $\upsilon : J \to \mathbb{R}$ satisfies

$$\xi_t^{\upsilon(\nu_1) - \upsilon(m) - \int_m^{\nu_1} \Gamma(\tau, \upsilon(\tau)) d\tau, \ldots, \upsilon(\nu_k) - \upsilon(m) - \int_m^{\nu_k} \Gamma(\tau, \upsilon(\tau)) d\tau} \geq \varphi_t^{\nu_1, \ldots, \nu_k} \tag{3}$$

for all $\nu_j \in J$, $(j = 1, 2, \ldots, k)$ and $t > 0$, where $\varphi : J^k \to O^+$ is a distribution function with

$$\inf_{\vartheta_j \in [m, \nu_j]} \varphi_t^{\vartheta_1, \ldots, \vartheta_k} \geq \varphi_t^{\nu_1, \ldots, \nu_k} \tag{4}$$

for all $\nu_j \in J$, $(j = 1, 2, \ldots, k)$ and $t > 0$. So, there is a unique continuous map $\upsilon_0 : J \to \mathbb{R}$ such that

$$\upsilon_0(\nu) = \upsilon(m) + \int_m^{\nu} \Gamma(\tau, \upsilon_0(\tau)) d\tau \tag{5}$$

(consequently, υ_0 is a solution to (1)) and

$$\xi_t^{\upsilon(\nu_1) - \upsilon_0(\nu_1), \ldots, \upsilon(\nu_k) - \upsilon_0(\nu_k)} \geq \varphi_{(1-\rho\beta)t}^{\nu_1, \ldots, \nu_k} \tag{6}$$

for all $\nu_j \in J$, $(j = 1, 2, \ldots, k)$ and $t > 0$.

Proof. We show the set of all continuous map $\sigma : J \to \mathbb{R}$ by

$$\Sigma = \{\sigma : J \to \mathbb{R}\} \tag{7}$$

and define the function δ on Σ,

$$\delta(\sigma, \varrho) = \inf\{M > 0 : \xi_{Mt}^{\sigma(\nu_1) - \varrho(\nu_1), \ldots, \sigma(\nu_k) - \varrho(\nu_k)} \geq \varphi_t^{\nu_1, \ldots, \nu_k}, \ \forall \nu_j \in J, \ t > 0\} \tag{8}$$

In [31], Miheţ and Radu proved that (Σ, δ) is a complete generalized metric space (see also [32]).

Now, we consider the linear map $\Lambda : \Sigma \to \Sigma$ is defined by

$$(\Lambda v)(\nu_j) = v(m) + \int_m^{\nu_j} \Gamma(\tau, v(\tau))d\tau \qquad (\nu_j \in J) \tag{9}$$

for all $v \in \Sigma$.

We show that the strict contractivity of Λ. Assume that $\sigma, \varrho \in \Sigma$ and $\varepsilon = \varepsilon_{\sigma,\varrho} > 0$ with $\delta(\sigma, \varrho) \leq \varepsilon$, so, we have

$$\xi_{\varepsilon t}^{\sigma(\nu_1)-\varrho(\nu_1),...,\sigma(\nu_k)-\varrho(\nu_k)} \geq \varphi_t^{\nu_1,...,\nu_k} \tag{10}$$

for any $\nu_j \in J, (j = 1, 2, ..., k)$ and $t > 0$.

Let, $m = \xi_1 < \xi_2 < ... < \xi_k = \nu_j$, $\tau_j \in [\xi_j, \xi_{j+1}]$ and $\triangle s_j = \xi_j - \xi_{j-1} = \frac{\nu_j - m}{k}$, $j = 1, 2, ..., k$. By using, (2), (3), (4), (8) and (10), we have

$$\begin{aligned}
& \xi_{\varepsilon\rho\beta t}^{(\Lambda\sigma)(\nu_1)-(\Lambda\varrho)(\nu_1),...,(\Lambda\sigma)(\nu_k)-(\Lambda\varrho)(\nu_k)} \\
= \ & \xi_{\varepsilon\rho\beta t}^{\int_m^{\nu_1}\{\Gamma(\tau,\sigma(\tau))-\Gamma(\tau,\varrho(\tau))\}d\tau,...,\int_m^{\nu_k}\{\Gamma(\tau,\sigma(\tau))-\Gamma(\tau,\varrho(\tau))\}d\tau} \\
= \ & \xi_{\varepsilon\rho\beta t}^{\lim_{\|\Delta s_j\|\to 0}\sum_{j=1}^k\{\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j))\}\Delta s_j,...,\lim_{\|\Delta s_j\|\to 0}\sum_{j=1}^k\{\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j))\}\Delta s_j} \\
= \ & \lim_{\|\Delta s_j\|\to 0} \xi_{\varepsilon\rho\beta t}^{\sum_{j=1}^k\{\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j))\}\Delta s_j,...,\sum_{j=1}^k\{\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j))\}\Delta s_j} \\
\geq \ & \lim_{\|\Delta s_j\|\to 0} \bigwedge \xi_{\left(\frac{\varepsilon\rho\beta t}{|\Delta s_j|k}\right)}^{\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j)),...,\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j))} \\
\geq \ & \bigwedge \xi_{\frac{\varepsilon\rho\beta t}{\rho}}^{\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j)),...,\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j))} \\
\geq \ & \inf_{\tau_j\in[m,\nu_j]} \xi_{\varepsilon t}^{\sigma(\tau_j)-\varrho(\tau_j),...,\sigma(\tau_j)-\varrho(\tau_j)} \\
\geq \ & \inf_{\tau_j\in[m,\nu_j]} \varphi_t^{\tau_1,...,\tau_k} \\
\geq \ & \varphi_t^{\nu_1,...,\nu_k}
\end{aligned}$$

for all $\nu_j \in J, (j = 1, 2, ..., k)$ and $t > 0$. So, we have $\delta(\Lambda\sigma, \Lambda\varrho) \leq \varepsilon\rho\beta$. Hence, we can conclude that $\delta(\Lambda\sigma, \Lambda\varrho) \leq \rho\beta\delta(\sigma, \varrho)$ for any $\sigma, \varrho \in \Sigma$, this shows, Λ is a strictly contractive mapping on Σ with Lipschitz constant $\rho\beta \in (0, 1)$. By using (3) and (9), we conclude that $\delta(\Lambda v, v) \leq 1$ and so, $\delta(\Lambda^{n+1}v, \Lambda^n v) \leq (\rho\beta)^n < \infty$.

Theorem 1, implies that, so there is a unique continuous map $v_0 : J \to \mathbb{R}$ such that

(1) A fixed point for Λ, is v_0, i.e.,
$$\Lambda(v_0) = (v_0). \tag{11}$$
(2) $\delta(\Lambda^n v, v_0) \to 0$ as $n \to \infty$.
(3) $\delta(v, v_0) \leq \frac{1}{1-\rho\beta}\delta(v, \Lambda v)$, which implies that

$$\xi_t^{v(\nu_1)-v_0(\nu_1),...,v(\nu_k)-v_0(\nu_k)} \geq \varphi_{(1-\rho\beta)t}^{\nu_1,...,\nu_k}$$

for all $\nu_j \in J, (j = 1, 2, ..., k)$ and $t > 0$. □

In the last theorem, we have investigated the Hyers–Ulam–Rassias stability of the differential Equation (1) in M-k-NLS defined on a bounded and closed interval. We will now prove the theorem for the case of unbounded intervals. More precisely, Theorem 2 is also true if J is replaced by an unbounded interval such as $(-\infty, q]$, $\mathbb{R}$, or $[p, \infty)$ as we see in the following theorem.

Theorem 3. *Let J be $(-\infty, q]$ or $\mathbb{R}$ or $[p, \infty)$ in which $p, q \in \mathbb{R}$. Put $m = p$ for $I = [p, \infty)$ or $m = q$ for $J = (-\infty, q]$, or if $J = \mathbb{R}$, put $m \in \mathbb{R}$ being fixed. Consider the constant numbers ρ and β such that $0 < \rho\beta < 1$ and continuous map $\Gamma : J \times \mathbb{R} \to \mathbb{R}$ holds (2) for all $\nu_j \in J$ and all $\upsilon_j, \vartheta_j \in \mathbb{R}$. Let $\upsilon : J \to \mathbb{R}$ be continuous differentiable and satisfies (3) for all $\nu_j \in J$, in which $\varphi : J^k \times \infty \to (0, 1]$ be a distribution function satisfying the condition (4) for any $\nu_i \in J$, $(j = 1, 2, ..., k)$, so there is a unique continuous map $\upsilon_0 : J \to \mathbb{R}$ which satisfies (5) and (6) for all $\nu_j \in J$.*

Proof. We prove for $J = \mathbb{R}$ only. Define $J_n = [m - n, m + n]$, for every $n \in \mathbb{N}$. (Put $J_n = [q - n, q]$ for $J = (-\infty, q]$ and $J_n = [p, p + n]$ for $J = [p, \infty)$.) Theorem 2 implies that there is a unique continuous map $\upsilon_n : J_n \to \mathbb{R}$ such that

$$\upsilon_n(\nu) = \upsilon(m) + \int_m^\nu \Gamma(\tau, \upsilon_n(\tau))d\tau \tag{12}$$

and

$$\xi_t^{\upsilon_n(\nu_1)-\upsilon_0(\nu_1),\ldots,\upsilon_n(\nu_k)-\upsilon_0(\nu_k)} \geq \varphi_{(1-\rho\beta)t}^{\nu_1,\ldots,\nu_k} \tag{13}$$

for all $\nu_j \in J_n$. The uniqueness of υ_n implies that if $\nu \in J_n$, then

$$\upsilon_n(\nu) = \upsilon_{n+1}(\nu) = \upsilon_{n+2}(\nu) = ..., \forall \nu \in \mathbb{R}. \tag{14}$$

Define

$$n(\nu) = \bigwedge\{n \in \mathbb{N} \mid \nu \in J_n\}.$$

and $\upsilon_0 : \mathbb{R} \to \mathbb{R}$ by

$$\upsilon_0(\nu) = \upsilon_{n(\nu)}(\nu). \tag{15}$$

The continuity of υ_n implies that υ_0 is continuous. We will now show that υ_0 satisfies (5) and (6) for all $\nu \in \mathbb{R}$. Let $\nu \in \mathbb{R}$, we select $n(\nu) \in \mathbb{N}$. So, we have $\nu \in J_{n(\nu)}$. Using (12), (14) and (15) we have

$$\upsilon_0(\nu) = \upsilon_{n(\nu)}(\nu) = \upsilon(m) + \textstyle\int_m^\nu \Gamma(\tau, \upsilon_{n(\nu)}(\tau))d\tau = \upsilon(m) + \int_m^\nu \Gamma(\tau, \upsilon_0(\tau))d\tau$$

and

$$\upsilon_{n(\nu)}(\tau) = \upsilon_{n(\tau)}(\tau) = \upsilon_0(\tau).$$

Since $\nu \in J_n(\nu)$ for every $\nu \in \mathbb{R}$, by (13) and (15) that

$$\begin{aligned} & \xi_t^{\upsilon(\nu_1)-\upsilon_0(\nu_1),\ldots,\upsilon(\nu_k)-\upsilon_0(\nu_k)} = \xi_t^{\upsilon(\nu_1)-\upsilon_{n(\nu)}(\nu_1),\ldots,\upsilon(\nu_k)-\upsilon_{n(\nu)}(\nu_k)} \\ \geq \; & \varphi_{(1-\rho\beta)t}^{\nu_1,\ldots,\nu_k} \end{aligned}$$

for every $\nu_j \in \mathbb{R}$.

To prove uniqueness, consider another continuous map $\vartheta_0 : \mathbb{R} \to \mathbb{R}$ which holds in (5) and (6). Let $\nu \in \mathbb{R}$, since $\upsilon_0 \mid_{J_{n(\nu)}} (= \upsilon_{n(\nu)})$ and $\vartheta_0 \mid_{J_{n(\nu)}}$ both satisfy (5) and (6) for all $\nu \in J_{n(\nu)}$, the uniqueness of $\upsilon_{n(\nu)} = \upsilon_0 \mid_{J_{n(\nu)}}$ implies that

$$\upsilon_0(\nu) = \upsilon_0 \mid_{J_n(\nu)} (\nu) = \vartheta_0 \mid_{J_n(\nu)} (\nu) = \vartheta_0(\nu),$$

as required. □

4. Hyers-Ulam Stability in M-k-NLS

In the following theorem, we prove the Hyers–Ulam stability of the differential Equation (1) defined on a finite and closed interval.

Theorem 4. *Let $m \in \mathbb{R}$, $\rho > 0$ and $J = \{\nu \in \mathbb{R} \mid m - \rho \leq \nu \leq m + \rho\}$. Assume that $\Gamma : J \times \mathbb{R} \to \mathbb{R}$ is a continuous map which satisfies (2) for every $\nu_j \in J$, $v_j, \vartheta_j \in \mathbb{R}$, $(j = 1, 2, ..., k)$ and $t > 0$, where β is a constant with $0 < \rho\beta < 1$.*

Let

$$\xi_t^{v(\nu_1)-v(m)-\int_m^{\nu_1}\Gamma(\tau,v(\tau))d\tau,...,v(\nu_k)-v(m)-\int_m^{\nu_k}\Gamma(\tau,v(\tau))d\tau} \geq 1 - \varepsilon \tag{16}$$

for every $\nu_i \in J$, $(i = 1, ..., k)$, $t > 0$ and for some $\varepsilon > 0$ in which $v : J \to \mathbb{R}$ is a continuous differentiable map. So, there is a unique continuous map $v_0 : J \to \mathbb{R}$ satisfying (5) and

$$\xi_{\frac{t}{1-\rho\beta}}^{v(\nu_1)-v_0(\nu_1),...,v(\nu_k)-v_0(\nu_k)} \geq 1 - \varepsilon \tag{17}$$

for every $\nu_j \in J$, $(j = 1, 2, ..., k)$ and $t > 0$.

Proof. We show the set of all continuous map $\sigma : J \to \mathbb{R}$ by

$$\Sigma = \{\sigma : J \to \mathbb{R}\}$$

and define the function δ on Σ,

$$\delta(\sigma, \varrho) = \inf\{M > 0 : \xi_{Mt}^{\sigma(\nu_1)-\varrho(\nu_1),...,\sigma(\nu_k)-\varrho(\nu_k)} \geq 1 - \varepsilon, \ \ \forall \nu_j \in J, \ \ t > 0\}$$

In [31], Miheţ and Radu proved that (Σ, δ) is a complete generalized metric space (see also [32]). Now, we consider the linear map $\Lambda : \Sigma \to \Sigma$ is defined by

$$(\Lambda v)(\nu_j) = v(m) + \int_m^{\nu_j} \Gamma(\tau, v(\tau))d\tau \qquad (\nu_j \in J) \tag{18}$$

for all $v \in \Sigma$. We show that the strictly contractively of Λ. Assume that $\sigma, \varrho \in \Sigma$ and $\eta = \eta_{\sigma,\varrho} > 0$ with $\delta(\sigma, \varrho) \leq \eta$, so, we have

$$\xi_{\eta t}^{\sigma(\nu_1)-\varrho(\nu_1),...,\sigma(\nu_k)-\varrho(\nu_k)} \geq 1 - \varepsilon \tag{19}$$

for any $\nu_j \in J(j = 1, ..., k)$, and $t > 0$. Let, $m = \xi_1 < \xi_2 < ... < \xi_k = \nu_j$, $\tau_j \in [\xi_j, \xi_{j+1}]$ and $\triangle s_j = \xi_j - \xi_{j-1} = \frac{\nu_j - m}{k}$, $j = 1, 2, ..., k$. By using, (2), (3), (4), (8) and (10), we have

$$
\begin{aligned}
& \zeta_{\eta\rho\beta t}^{(\Lambda\sigma)(\nu_1)-(\Lambda\varrho)(\nu_1),\ldots,(\Lambda\sigma)(\nu_k)-(\Lambda\varrho)(\nu_k)} \\
= \ & \zeta_{\eta\rho\beta t}^{\int_m^{\nu_1}\{\Gamma(\tau,\sigma(\tau))-\Gamma(\tau,\varrho(\tau))\}d\tau,\ldots,\int_m^{\nu_k}\{\Gamma(\tau,\sigma(\tau))-\Gamma(\tau,\varrho(\tau))\}d\tau} \\
= \ & \zeta_{\eta\rho\beta t}^{\lim_{\|\Delta s\|\to 0}\sum_{j=1}^k\{\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j))\}\Delta s_j,\ldots,\lim_{\|\Delta s_j\|\to 0}\sum_{j=1}^k\{\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j))\}\Delta s_j} \\
= \ & \lim_{\|\Delta s_j\|\to 0} \zeta_{\eta\rho\beta t}^{\sum_{j=1}^k\{\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j))\}\Delta s_j,\ldots,\sum_{j=1}^k\{\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j))\}\Delta s_j} \\
\geq \ & \lim_{\|\Delta s_j\|\to 0} \bigwedge \zeta_{\left(\frac{\eta\rho\beta t}{|\Delta s_j|k}\right)}^{\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j)),\ldots,\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j))} \\
\geq \ & \bigwedge \zeta_{\frac{\eta\rho\beta t}{\rho}}^{\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j)),\ldots,\Gamma(\tau_j,\sigma(\tau_j))-\Gamma(\tau_j,\varrho(\tau_j))} \\
\geq \ & \inf_{\tau_j\in[m,\nu_j]} \zeta_{\eta t}^{\sigma(\tau_j)-\varrho(\tau_j),\ldots,\sigma(\tau_j)-\varrho(\tau_j)} \\
\geq \ & 1-\varepsilon
\end{aligned}
$$

for all $\nu_j \in J$ and $t > 0$. So, we have $\delta(\Lambda\sigma, \Lambda\varrho) \leq \eta\rho\beta$. Hence, we can conclude that $\delta(\Lambda\sigma, \Lambda\varrho) \leq \rho\beta\delta(\sigma, \varrho)$ for any $\sigma, \varrho \in \Sigma$, this shows, Λ is a strictly contractive map on Σ and $\rho\beta \in (0, 1)$ is Lipschitz constant. By using definition $\delta(\sigma, \varrho)$, we conclude that $\delta(\Lambda v, v) \leq 1$ and so, $\delta(\Lambda^{n+1}v, \Lambda^n v) \leq (\rho\beta)^n < \infty$.

Theorem 1, implies that, there is a unique continuous map $v_0 : J \to \mathbb{R}$ such that

(1) A fixed point for Λ, is v_0, i.e.,

$$\Lambda(v_0) = v_0. \tag{20}$$

(2) $\delta(\Lambda^n v, v_0) \to 0$ as $n \to \infty$.

(3) $\delta(v, v_0) \leq \frac{1}{1-\rho\beta}\delta(v, \Lambda v)$, which implies that

$$\zeta_{\frac{t}{1-\rho\beta}}^{v(\nu_1)-v_0(\nu_1),\ldots,v(\nu_k)-v_0(\nu_k)} \geq 1-\varepsilon$$

for all $\nu_j \in J$ and $t > 0$. □

5. Examples

In this section, we show that there certainly exist functions $v(\nu)$ which satisfy all the conditions given in Theorems 2, 3 and 4.

Example 2. *Consider $0 < \beta < 1$. For a $0 < \varepsilon < 2\beta$ and $\rho = 2\beta - \varepsilon$, let $J = [0, 2\beta - \varepsilon]$. Let $p(\nu)$ be a polynomial, and $v : J \to \mathbb{R}$, a continuously differentiable map, satisfies*

$$\zeta_{(t)}^{v(\nu_1)-v(m)-\int_0^{\nu_1}\{\beta v(\tau)-P(\tau)\}d\tau,\ldots,v(\nu_k)-v(m)-\int_0^{\nu_k}\{\beta v(\tau)-P(\tau)\}d\tau} \geq \exp\left(-\frac{|\nu_1,\ldots,\nu_k|}{t}\right)$$

for all $\nu_j \in J$, and $t > 0$. If we set $\Gamma(\nu, v) = \beta v + P(\nu)$ in which Γ defined here is of the form of that of the Theorem 4 and satisfies (2) and

$$\varphi_t^{\nu_1,\ldots,\nu_k} = \begin{cases} \exp\left(-\frac{|\nu_1,\ldots,\nu_k|}{t}\right) & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Moreover, we obtain

$$\inf_{\vartheta_j\in[0,\nu_j]} \varphi_t^{\vartheta_1,\ldots,\vartheta_k} \geq \varphi_t^{\nu_1,\ldots,\nu_k}$$

for all $\nu_j \in J$ *and* $t > 0$. *Using Theorem 2, implies that there is a unique continuous map* $v_0 : J \to \mathbb{R}$ *such that*

$$v_0(\nu) = v(0) + \int_0^{\nu} \{\beta v_0(\tau) - P(\tau)\} d\tau$$

and

$$\xi_t^{v(\nu_1)-v_0(\nu_1),\dots,v(\nu_k)-v_0(\nu_k)} \geq \exp(-\frac{|\nu_1,\dots,\nu_k|}{(1-\rho\beta)t})$$

for all $\nu_j \in J$ *and* $t > 0$.

Example 3. *Consider* $p > 1$, $0 < \beta < Lna$, $J = [p, \infty)$ *and a polynomial* $P(\nu)$. *Let the continuously differentiable map* $v : J \to \mathbb{R}$ *satisfies*

$$\xi_{(t)}^{v(\nu_1)-v(m)-\int_0^{\nu_1}\{\beta v(\tau)-P(\tau)\}d\tau,\dots,v(\nu_k)-v(m)-\int_0^{\nu_k}\{\beta v(\tau)-P(\tau)\}d\tau} \geq \exp(-\frac{a^{\nu_1,\dots,\nu_k}}{t})$$

for all $\nu_j \in J$, *and* $t > 0$. *If we set* $\Gamma(\nu, v) = \beta v + P(\nu)$ *and*

$$\varphi_{(t)}^{\nu_1,\dots,\nu_k} = \begin{cases} \exp(-\frac{a^{\nu_1,\dots,\nu_k}}{t}) & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Moreover, we obtain

$$\inf_{\vartheta_j \in [0,\nu_j]} \varphi_t^{\vartheta_1,\dots,\vartheta_k} \geq \varphi_t^{\nu_1,\dots,\nu_k}$$

for all $\nu_j \in J$ *and* $t > 0$. *Using Theorem 3, implies that there is a unique continuous function* $v_0 : J \to \mathbb{R}$ *such that*

$$v_0(\nu) = v(0) + \int_0^{\nu} \{\beta v_0(\tau) - P(\tau)\} d\tau$$

and

$$\xi_t^{v(\nu_1)-v_0(\nu_1),\dots,v(\nu_k)-v_0(\nu_k)} \geq \exp(-\frac{a^{\nu_1,\dots,\nu_k}}{(\ln a-\rho\beta)t})$$

for all $\nu_j \in J$ *and* $t > 0$.

Example 4. *Consider constants* $\rho, \beta > 0$ *such that* $0 < \rho\beta < 1$. *Define* $J = \{\nu \in \mathbb{R} \mid m - \rho \leq \nu \leq m + \rho\}$ *for some* $m \in \mathbb{R}$. *Let* $P(\nu)$ *be a polynomial and let the continuously differentiable map* $v : J \to \mathbb{R}$ *satisfies*

$$\xi_t^{v(\nu_1)-v(m)-\int_0^{\nu_1}\{\beta v(\tau)-P(\tau)\}d\tau,\dots,v(\nu_k)-v(m)-\int_0^{\nu_k}\{\beta v(\tau)-P(\tau)\}d\tau} \geq 1-\varepsilon$$

for all $\nu_j \in J$, *and* $t > 0$ *with* $\varepsilon \geq 0$. *Using Theorem 4, implies that, there is a unique continuous map* $v_0 : J \to \mathbb{R}$ *such that*

$$v_0(\nu) = v(m) + \int_0^{\nu} \{\beta v_0(\tau) + P(\tau)\} d\tau$$

and

$$\xi_{\left(\frac{t}{1-\rho\beta}\right)}^{v(\nu_1)-v_0(\nu_1),\dots,v(\nu_k)-v_0(\nu_k)} \geq 1-\varepsilon,$$

for all $\nu_j \in J$ *and* $t > 0$.

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