mathematics

Article

Asymptotic Behavior of Solutions of the Third Order Nonlinear Mixed Type Neutral Differential Equations

Osama Moaaz [1,†], Dimplekumar Chalishajar [2,*,†] and Omar Bazighifan [3,4,†]

1 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; o_moaaz@mans.edu.eg
2 Department of Applied Mathematics, Virginia Military Institute (VMI) 435 Mallory Hall, Lexington, VA 24450, USA
3 Department of Mathematics, Faculty of Science, Hadhramout University, Hadhramout 50512, Yemen; o.bazighifan@gmail.com
4 Department of Mathematics, Faculty of Education, Seiyun University, Hadhramout 50512, Yemen
* Correspondence: chalishajardn@vmi.edu
† These authors contributed equally to this work.

Received: 29 January 2020; Accepted: 20 March 2020; Published: 1 April 2020

Abstract: The objective of our paper is to study asymptotic properties of the class of third order neutral differential equations with advanced and delayed arguments. Our results supplement and improve some known results obtained in the literature. An illustrative example is provided.

Keywords: oscillation; third order; mixed neutral differential equations

---

1. Introduction

Equations with neutral terms are of particular significance, as they arise in many applications including systems of control, electrodynamics, mixing liquids, neutron transportation, networks and population models; see [1].

Asymptotic properties of solutions of second/third order differential equations have been subject to intensive research in the literature. This problem for differential equations with respective delays has received a great deal of attention in the last years; see for examples, [2–21].

This paper deals with the oscillation and asymptotic behavior of solutions of the class of third-order, nonlinear, mixed-type, neutral differential equations

$$\left(r\left(t\right)\left(z''\left(t\right)\right)^{\alpha}\right)' + q_1\left(t\right)f_1\left(x\left(\sigma_1\left(t\right)\right)\right) + q_2\left(t\right)f_2\left(x\left(\sigma_2\left(t\right)\right)\right) = 0, \tag{1}$$

where

$$z\left(t\right) = x\left(t\right) + p_1\left(t\right)x\left(\tau_1\left(t\right)\right) + p_2\left(t\right)x\left(\tau_2\left(t\right)\right)$$

and we will assume the following assumptions hold:

- (M_1) $r \in C\left(\left[t_0,\infty\right),\left(0,\infty\right)\right)$, $\int_{t_0}^{\infty} r^{-1/\alpha}\left(s\right)\mathrm{d}s = \infty$ and α is a ratio of odd positive integers;
- (M_2) $p_i \in C\left(\left[t_0,\infty\right),\left[0,c_i\right]\right)$ where c_i are constants for $i = 1,2$ and $c_1 + c_2 < 1$;
- (M_3) $\tau_i, \sigma_i \in C\left(\left[t_0,\infty\right),\mathbb{R}\right)$, $\tau_1\left(t\right) < t$, $\sigma_1\left(t\right) < t$, $\tau_2\left(t\right) > t$, $\sigma_2\left(t\right) > t$, $\sigma_i\left(\tau_i\left(t\right)\right) = \tau_i\left(\sigma_i\left(t\right)\right)$ and $\lim_{t\to\infty}\tau_i\left(t\right) = \lim_{t\to\infty}\sigma_i\left(t\right) = \infty$ for $i = 1,2$;
- (M_4) $q_i \in C\left(\left[t_0,\infty\right),\left(0,\infty\right)\right)$ for $i = 1,2$;

(M$_5$) $f_1, f_2 \in C(\mathbb{R},\mathbb{R})$, $f_1(x)/x^{\beta} \geq k_1 > 0$ and $f_2(x)/x^{\gamma} \geq k_2$ for $x \neq 0$ where β and γ are ratios of odd positive integers.

By a solution of Equation (1), we mean a non-trivial real function $x \in C([t_x,\infty))$, $t_x \geq t_0$, with $z(t)$, $z'(t)$ and $r_1(t)(z''(t))^{\alpha}$ being continuously differentiable for all $t \in [t_x,\infty)$, and satisfying (1) on $[t_x,\infty)$. A solution of Equation (1) is called oscillatory if it has arbitrary large zeros; otherwise it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Han et al. in [22] studied the asymptotic properties of the solutions of equation

$$\left(r(t)\left(z''(t)\right)\right)' + q_1(t)\,x(\sigma_1(t)) + q_2(t)\,x(\sigma_2(t)) = 0, \tag{2}$$

where $z(t) = x(t) + p_1(t)\,x(\tau_1(t)) + p_2(t)\,x(\tau_2(t))$.

Baculikova and Dzurina [5] studied the oscillation of the third-order equation

$$\left(r(t)\left(x'(t)\right)^{\alpha}\right)'' + q(t)\,f(x(\tau(t))) + p(t)\,h(x(\sigma(t))) = 0,$$

where $\tau(t) \leq t$ and $\sigma(t) \geq t$.

Thandapani and Rama [23] established some oscillation theorems for equation

$$\left(r(t)\left(z''(t)\right)\right)' + q_1(t)\,x^{\alpha}(\sigma_1(t)) + q_2(t)\,x^{\beta}(\sigma_2(t)) = 0,$$

where $z(t) = x(t) + p_1(t)\,x(\tau_1(t)) + p_2(t)\,x(\tau_2(t))$, and the authors used the Recati technique.

The aim of this paper is to discuss the asymptotic behavior of solutions of a class of third-order, nonlinear, mixed-type, neutral differential equations. We established sufficient conditions to ensure that the solution of Equation (1) is oscillatory or tended to zero. The results of this study basically generalize and improve the previous results. An illustrative example is provided.

2. Auxiliary Lemmas

In order to prove our results, we shall need the next auxiliary lemmas.

Lemma 1. *Assume that* $f(y) = Uy - Vy^{\frac{\eta+1}{\eta}}$*, where* U *and* V *are constants,* $V > 0$ *and* η *is a quotient of odd positive integers. Then* f *imposes its maximum value on* $\mathbb{R}$ *at* $y^* = \left(\frac{U\eta}{V(\eta+1)}\right)^{\eta}$ *and*

$$\max_{y\in\mathbb{R}} f = f(y^*) = \frac{\eta^{\eta}}{(\eta+1)^{\eta+1}}U^{\eta+1}V^{-\eta}.$$

Lemma 2 ([24])**.** *Assume that* $A \geq 0$ *and* $B \geq 0$*. If* $\delta > 1$*, then*

$$(A+B)^{\delta} \leq 2^{\delta-1}\left(A^{\delta}+B^{\delta}\right)$$

Moreover, if $0 < \delta < 1$*, then* $(A+B)^{\delta} \leq \left(A^{\delta}+B^{\delta}\right)$.

Lemma 3 ([17])**.** *If the function* y *satisfies* $y^{(i)} > 0$*,* $i = 0, 1, ..., n$*, and* $y^{(n+1)} < 0$*, then*

$$\frac{y(t)}{t^n/n!} \geq \frac{y'(t)}{t^{n-1}/(n-1)!}.$$

Lemma 4 ([23])**.** *Assume that* $u(t) > 0$, $u'(t) > 0$, $u''(t) > 0$ *and* $u'''(t) < 0$ *on* (T, ∞). *Then,*

$$\frac{u(t)}{u'(t)} \geq \frac{t-T}{2} \geq \frac{\mu t}{2}$$

for $t \geq T$ *and some* $\mu \in (0,1)$.

Lemma 5. *Let* x *be a positive solution of Equation (1). Then* z *has only one of the following two properties eventually:*
(**i**) $z(t) > 0$, $z'(t) > 0$ *and* $z''(t) > 0$;
(**ii**) $z(t) > 0$, $z'(t) < 0$ *and* $z''(t) > 0$.

Proof. The proof is similar to that of Lemma 2.1 of [10] and hence the details are omitted. □

Lemma 6. *Let* x *be a positive solution of Equation (1), and* z *has the property* (**ii**). *If* $\beta = \gamma$ *and*

$$\int_{t_0}^{\infty} \int_{v}^{\infty} \left(\frac{1}{r(u)} \int_{u}^{\infty} (k_1 q_1(s) + k_2 q_2(s)) \mathrm{d}s \right)^{1/\alpha} \mathrm{d}u \mathrm{d}v = \infty, \tag{3}$$

then the solution x *of Equation (1) converges to zero as* $t \to \infty$.

Proof. Let x be a positive solution of Equation (1). Since z satisfies the property (**ii**), we get $\lim_{t\to\infty} z(t) = \delta \geq 0$. Next, we will prove that $\delta = 0$. Suppose that $\delta > 0$, then we have for all $\varepsilon > 0$ and t enough large $\delta < z(t) < \delta + \varepsilon$. By choosing $\varepsilon < \frac{1-c_1-c_2}{c_1+c_2}\delta$, we obtain

$$\begin{aligned} x(t) &= z(t) - p_1(t) x(\tau_1(t)) - p_2(t) x(\tau_2(t)) \\ &> \delta - (c_1 + c_2) z(\tau_1(t)) \\ &> \delta - (c_1 + c_2)(\delta + \varepsilon) \\ &> L(\delta + \varepsilon) > Lz(t), \end{aligned}$$

where $L = \frac{\delta - (c_1+c_2)(\delta+\varepsilon)}{\delta+\varepsilon} > 0$. Thus, from (1) and ($M_5$), we have

$$\begin{aligned} 0 &\geq \left(r(t) (z''(t))^{\alpha} \right)' + k_1 q_1(t) x^{\beta}(\sigma_1(t)) + k_2 q_2(t) x^{\beta}(\sigma_2(t)) \\ &\geq \left(r(t) (z''(t))^{\alpha} \right)' + L^{\beta} (k_1 q_1(t) + k_2 q_2(t)) z^{\beta}(\sigma_2(t)), \end{aligned}$$

and so,

$$\left(r(t) (z''(t))^{\alpha} \right)' \leq -L^{\beta} \delta^{\beta} (k_1 q_1(t) + k_2 q_2(t)).$$

By integrating this inequality two times from t to ∞, we get

$$-z'(t) > L^{\beta/\alpha} \delta^{\beta/\alpha} \int_{t}^{\infty} \left(\frac{1}{r(u)} \int_{u}^{\infty} (k_1 q_1(s) + k_2 q_2(s)) \mathrm{d}s \right)^{1/\alpha} \mathrm{d}u.$$

Integrating the last inequality from t_1 to ∞, we have

$$z(t_1) > L^{\beta/\alpha} \delta^{\beta/\alpha} \int_{t_1}^{\infty} \int_{v}^{\infty} \left(\frac{1}{r(u)} \int_{u}^{\infty} (k_1 q_1(s) + k_2 q_2(s)) \mathrm{d}s \right)^{1/\alpha} \mathrm{d}u \mathrm{d}v.$$

Thus, we are led to a contradiction with (3). Then, $\lim_{t\to\infty} z(t) = 0$; moreover, the fact that $x(t) \le z(t)$ implies $\lim_{t\to\infty} x(t) = 0$. □

3. Main Results

In this section, we will establish new oscillation criteria for solutions of the Equation (1). For the sake of convenience, we insert the next notation:

$$R_u(t) := \int_u^t \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s,$$

$$R_u^*(t) := \min_{t \ge t_0} \{R_u(t), R_u(\tau_1(t))\}$$

and

$$q_i^*(t) := \min_{t \ge t_0} \{q_i(t), \ q_i(\tau_1(t)), \ q_i(\tau_2(t))\}, \ i = 1, 2.$$

Theorem 1. *Assume that (M_1)–(M_5) and (3) hold. Let $\beta = \gamma \ge \alpha$, $\sigma_1(t) \le \tau_1(t)$ and $\sigma_1'(t) > 0$. If there exists a positive function $\rho \in C^1([t_0, \infty))$ such that*

$$\limsup_{t\to\infty} \int_{t_0}^t \left(\Theta_1(s) - \left(1 + c_1^\beta + \frac{c_2^\beta}{2^{\beta-1}}\right) \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\rho_+'(s))^{\alpha+1} r(\sigma_1(s))}{(\rho(s)\sigma_1'(s))^\alpha} \right) \mathrm{d}s = \infty, \tag{4}$$

where $\rho_+'(s) = \max\{\rho'(s), 0\}$ and

$$\Theta_1(t) = \frac{\mu^\alpha v^{\beta-\alpha}}{2^{2\beta+\alpha-2}} \rho(t) \sigma_1^\beta(t) (k_1 q_1^*(t) + k_2 q_2^*(t)),$$

then every solution of equation (1) either oscillates or tends to zero as $t \to \infty$.

Proof. Let x be non-oscillatory solution of Equation (1). Without loss of generality, we assume that $x(t) > 0$; then there exists a $t_1 \ge t_0$ such that $x(t) > 0$, $x(\tau_i(t)) > 0$ and $x(\sigma_i(t)) > 0$ for $t \ge t_1$and $i = 1, 2$. From Lemma 5, we have that z has the property (**i**) or the property (**ii**). From Lemma 6, if $z(t)$ has the property (**ii**), then we obtain $\lim_{t\to\infty} x(t) = 0$. Next, let z have the property (**i**). Using (1) and (M_5), we obtain

$$\left(r(t)(z''(t))^\alpha\right)' + k_1 q_1(t) x^\beta(\sigma_1(t)) + k_2 q_2(t) x^\beta(\sigma_2(t)) \le 0.$$

Thus, we get

$$\begin{aligned} 0 \ \ge\ & \left(r(t)(z''(t))^\alpha\right)' + k_1 q_1(t) x^\beta(\sigma_1(t)) + k_2 q_2(t) x^\beta(\sigma_2(t)) \\ & + c_1^\beta [\left(r(\tau_1(t))(z''(\tau_1(t)))^\alpha\right)' + k_1 q_1(\tau_1(t)) x^\beta(\sigma_1(\tau_1(t))) \\ & + k_2 q_2(\tau_1(t)) x^\beta(\sigma_2(\tau_1(t)))\Big] + \frac{c_2^\beta}{2^{\beta-1}} [\left(r(\tau_2(t))(z''(\tau_2(t)))^\alpha\right)' \\ & + k_1 q_1(\tau_2(t)) x^\beta(\sigma_1(\tau_2(t))) + k_2 q_2(\tau_2(t)) x^\beta(\sigma_2(\tau_2(t)))\Big]. \end{aligned}$$

That is

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' + c_1^{\beta}\left(r(\tau_1(t))\left(z''(\tau_1(t))\right)^{\alpha}\right)' + \frac{c_2^{\beta}}{2^{\beta-1}}\left(r(\tau_2(t))\left(z''(\tau_2(t))\right)^{\alpha}\right)'$$
$$+ k_1 q_1^*(t)\left(x^{\beta}(\sigma_1(t)) + c_1^{\beta}x^{\beta}(\sigma_1(\tau_1(t))) + \frac{c_2^{\beta}}{2^{\beta-1}}x^{\beta}(\sigma_1(\tau_2(t)))\right)$$
$$+ k_2 q_2^*(t)\left(x^{\beta}(\sigma_2(t)) + c_1^{\beta}x^{\beta}(\sigma_2(\tau_1(t))) + \frac{c_2^{\beta}}{2^{\beta-1}}x^{\beta}(\sigma_2(\tau_2(t)))\right) \leq 0. \tag{5}$$

From Lemma 2, we obtain

$$\begin{aligned} z^{\beta}(t) &\leq (x(t) + c_1(t)x(\tau_1(t)) + c_2(t)x(\tau_2(t)))^{\beta} \\ &\leq 4^{\beta-1}\left(x^{\beta}(t) + c_1^{\beta}x^{\beta}(\tau_1(t)) + \frac{c_2^{\beta}}{2^{\beta-1}}x^{\beta}(\tau_2(t))\right), \end{aligned} \tag{6}$$

which with (5) gives

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' + c_1^{\beta}\left(r(\tau_1(t))\left(z''(\tau_1(t))\right)^{\alpha}\right)' + \frac{c_2^{\beta}}{2^{\beta-1}}\left(r(\tau_2(t))\left(z''(\tau_2(t))\right)^{\alpha}\right)'$$
$$+ \frac{k_1}{4^{\beta-1}}q_1^*(t)z^{\beta}(\sigma_1(t)) + \frac{k_2}{4^{\beta-1}}q_2^*(t)z^{\beta}(\sigma_2(t)) \leq 0.$$

This implies that

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' + c_1^{\beta}\left(r(\tau_1(t))\left(z''(\tau_1(t))\right)^{\alpha}\right)' + \frac{c_2^{\beta}}{2^{\beta-1}}\left(r(\tau_2(t))\left(z''(\tau_2(t))\right)^{\alpha}\right)'$$
$$+ \frac{1}{4^{\beta-1}}\left(k_1 q_1^*(t) + k_2 q_2^*(t)\right)z^{\beta}(\sigma_1(t)) \leq 0. \tag{7}$$

Now, we define

$$\omega_1(t) = \rho(t)\frac{r(t)\left(z''(t)\right)^{\alpha}}{\left(z'(\sigma_1(t))\right)^{\alpha}}.$$

Then $\omega_1(t) > 0$. By differentiating, we get

$$\omega_1'(t) = \frac{\rho'(t)}{\rho(t)}\omega_1(t) + \rho(t)\frac{\left(r(t)\left(z''(t)\right)^{\alpha}\right)'}{\left(z'(\sigma_1(t))\right)^{\alpha}} - \alpha\rho(t)\frac{r(t)\left(z''(t)\right)^{\alpha}}{\left(z'(\sigma_1(t))\right)^{\alpha+1}}z''(\sigma_1(t))\sigma_1'(t).$$

Since $\left(r(t)\left(z''(t)\right)^{\alpha}\right)' < 0$ and $\sigma_1(t) < t$, we obtain

$$r(t)\left(z''(t)\right)^{\alpha} \leq r(\sigma_1(t))\left(z''(\sigma_1(t))\right)^{\alpha},$$

and hence

$$\omega_1'(t) \leq \frac{\rho_+'(t)}{\rho(t)}\omega_1(t) - \alpha\frac{\sigma_1'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}(\sigma_1(t))}\omega_1^{\frac{\alpha+1}{\alpha}}(t) + \rho(t)\frac{\left(r(t)\left(z''(t)\right)^{\alpha}\right)'}{\left(z'(\sigma_1(t))\right)^{\alpha}}.$$

Using Lemma 1 with

$$\eta=\alpha,\ U=\frac{\rho_{+}'(t)}{\rho(t)},V=\alpha\frac{\sigma_1'(t)}{\rho^{1/\alpha}(t)\,r^{1/\alpha}(\sigma_1(t))}\text{ and } y=\omega_1,$$

we obtain

$$\omega_1'(t)\le\rho(t)\frac{\left(r(t)\left(z''(t)\right)^{\alpha}\right)'}{\left(z'(\sigma_1(t))\right)^{\alpha}}+\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\rho_{+}'(t)\right)^{\alpha+1}r(\sigma_1(t))}{\left(\rho(t)\,\sigma_1'(t)\right)^{\alpha}}. \tag{8}$$

Further, we define the function

$$\omega_2(t)=\rho(t)\frac{r(\tau_1(t))\left(z''(\tau_1(t))\right)^{\alpha}}{\left(z'(\sigma_1(t))\right)^{\alpha}}.$$

Then $\omega_2(t)>0$. By differentiating ω_2 and using $\sigma_1(t)\le\tau_1(t)$, we find

$$\omega_2'(t)\le\frac{\rho'(t)}{\rho(t)}\omega_2(t)+\rho(t)\frac{\left(r(\tau_1(t))\left(z''(\tau_1(t))\right)^{\alpha}\right)'}{\left(z'(\sigma_1(t))\right)^{\alpha}}-\alpha\frac{\sigma_1'(t)}{\rho^{1/\alpha}(t)\,r^{1/\alpha}(\sigma_1(t))}\omega_2^{\frac{\alpha+1}{\alpha}}(t).$$

Using Lemma 1, we obtain

$$\omega_2'(t)\le\rho(t)\frac{\left(r(\tau_1(t))\left(z''(\tau_1(t))\right)^{\alpha}\right)'}{\left(z'(\sigma_1(t))\right)^{\alpha}}+\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\rho_{+}'(t)\right)^{\alpha+1}r(\sigma_1(t))}{\left(\rho(t)\,\sigma_1'(t)\right)^{\alpha}}. \tag{9}$$

Next, we define another function

$$\omega_3(t)=\rho(t)\frac{r(\tau_2(t))\left(z''(\tau_2(t))\right)^{\alpha}}{\left(z'(\sigma_1(t))\right)^{\alpha}}.$$

Thus $\omega_3(t)>0$. By differentiating, and similar to (9) we have

$$\omega_3'(t)\le\rho(t)\frac{\left(r(\tau_2(t))\left(z''(\tau_2(t))\right)^{\alpha}\right)'}{\left(z'(\sigma_1(t))\right)^{\alpha}}+\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\rho_{+}'(t)\right)^{\alpha+1}r(\sigma_1(t))}{\left(\rho(t)\,\sigma_1'(t)\right)^{\alpha}}. \tag{10}$$

From (8)–(10), we get

$$\begin{aligned}\omega_1'(t)+c_1^{\beta}\omega_2'(t)+\frac{c_2^{\beta}}{2^{\beta-1}}\omega_3'(t)\le{}&\frac{\rho(t)}{\left(z'(\sigma_1(t))\right)^{\alpha}}\Bigg(\left(r(t)\left(z''(t)\right)^{\alpha}\right)'+\\&+c_1^{\beta}\left(r(\tau_1(t))\left(z''(\tau_1(t))\right)^{\alpha}\right)'+\frac{c_2^{\beta}}{2^{\beta-1}}\left(r(\tau_2(t))\left(z''(\tau_2(t))\right)^{\alpha}\right)'\Bigg)\\&+\left(1+c_1^{\beta}+\frac{c_2^{\beta}}{2^{\beta-1}}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\rho_{+}'(t)\right)^{\alpha+1}r(\sigma_1(t))}{\left(\rho(t)\,\sigma_1'(t)\right)^{\alpha}},\end{aligned}$$

which with (7) gives

$$\begin{aligned}\omega_1'(t)+c_1^{\beta}\omega_2'(t)+\frac{c_2^{\beta}}{2^{\beta-1}}\omega_3'(t)\le{}&-\frac{\rho(t)}{4^{\beta-1}}\left(k_1q_1^{*}(t)+k_2q_2^{*}(t)\right)\frac{z^{\beta}(\sigma_1(t))}{\left(z'(\sigma_1(t))\right)^{\alpha}}\\&+\left(1+c_1^{\beta}+\frac{c_2^{\beta}}{2^{\beta-1}}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\rho_{+}'(t)\right)^{\alpha+1}r(\sigma_1(t))}{\left(\rho(t)\,\sigma_1'(t)\right)^{\alpha}}.\end{aligned} \tag{11}$$

Using Lemma 4, we have, for some $\mu \in (0,1)$,

$$\frac{z(\sigma_1(t))}{z'(\sigma_1(t))} \geq \frac{\mu}{2}\sigma_1(t).$$

From property (**i**), we get

$$\begin{aligned} z(t) &= z(t_1) + \int_{t_1}^{t} z'(s)\,ds \\ &\geq (t-t_1)\,z'(t_1) \geq \frac{v}{2}t, \end{aligned} \tag{12}$$

for some $v > 0$ and for t enough large. Therefore, for some $\mu \in (0,1)$ and $v > 0$, we find

$$\frac{z^{\beta}(\sigma_1(t))}{(z'(\sigma_1(t)))^{\alpha}} \geq \frac{\mu^{\alpha} v^{\beta-\alpha}}{2^{\alpha}}\sigma_1^{\beta}(t).$$

Combining the last inequality with (11), we obtain

$$\begin{aligned} \omega_1'(t) + c_1^{\beta}\omega_2'(t) + \frac{c_2^{\beta}}{2^{\beta-1}}\omega_3'(t) &\leq -\Theta(t) \\ &+ \left(1 + c_1^{\beta} + \frac{c_2^{\beta}}{2^{\beta-1}}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\rho_+'(t))^{\alpha+1} r(\sigma_1(t))}{\left(\rho(t)\,\sigma_1'(t)\right)^{\alpha}}. \end{aligned}$$

Integrating the above inequality from t_1 to t, we have

$$\begin{aligned} &\int_{t_1}^{t}\left(\Theta(s) - \left(1 + c_1^{\beta} + \frac{c_2^{\beta}}{2^{\beta-1}}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\rho_+'(s))^{\alpha+1} r(\sigma_1(s))}{\left(\rho(s)\,\sigma_1'(s)\right)^{\alpha}}\right) ds \\ &\qquad \leq \omega_1(t_1) + c_1^{\beta}\omega_2(t_1) + \frac{c_2^{\beta}}{2^{\beta-1}}\omega_3(t_1). \end{aligned}$$

Taking the superior limit as $t \to \infty$, we get a contradiction with (4). The proof is complete. □

Remark 1. *In the Theorem 1, if $\sigma_1(t) \geq \tau_1(t)$ and $\tau_1'(t) > 0$, then the assumption (4) is replaced by*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\Theta_1(s) - \left(1 + c_1^{\beta} + \frac{c_2^{\beta}}{2^{\beta-1}}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\rho_+'(s))^{\alpha+1} r(\tau_1(s))}{\left(\rho(s)\,\tau_1'(s)\right)^{\alpha}}\right) ds = \infty.$$

Theorem 2. *Assume that (M_1)–(M_5) and (3) hold. Let $\beta = \gamma \geq \alpha$ and $r'(t) > 0$. If there exists a positive function $\rho \in C^1([t_0,\infty))$ such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\Theta_2(s) - \left(1 + c_1^{\beta} + \frac{c_2^{\beta}}{2^{\beta-1}}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\rho_+'(t))^{\alpha+1}}{\left(\rho(t)\,R_{t_0}^{*}(t)\right)^{\alpha}}\right) ds = \infty, \tag{13}$$

where

$$\Theta_2(t) = \frac{v^{\beta-\alpha}}{2^{3\beta-\alpha-2}t^{2\alpha}}\rho(t)\,\sigma_1^{\beta+\alpha}(t)\,(k_1 q_1^{*}(t) + k_2 q_2^{*}(t)),$$

then every solution of Equation (1) either oscillates or tends to zero as $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 1, we have that (7) holds. Since $\left(r(t)(z''(t))^{\alpha}\right)' < 0$, we obtain

$$\begin{aligned} z'(t) &= z'(t_1) + \int_{t_1}^{t} \frac{\left[r(s)(z''(s))^{\alpha}\right]^{1/\alpha}}{r^{1/\alpha}(s)} ds \\ &\geq \left[r(t)(z''(t))^{\alpha}\right]^{1/\alpha} R_{t_1}(t). \end{aligned} \tag{14}$$

Now, we define

$$\omega_1(t) = \rho(t) \frac{r(t)(z''(t))^{\alpha}}{z^{\alpha}(t)}.$$

Then $\omega_1(t) > 0$. By differentiating ω_1 and using (14), we get

$$\omega_1'(t) \leq \frac{\rho_+'(t)}{\rho(t)} \omega_1(t) - \alpha \frac{R_{t_1}(t)}{\rho^{1/\alpha}(t)} \omega_1^{\frac{\alpha+1}{\alpha}}(t) + \rho(t) \frac{\left(r(t)(z''(t))^{\alpha}\right)'}{z^{\alpha}(t)}.$$

Using Lemma 1 with $\eta = \alpha$, $U = \frac{\rho_+'(t)}{\rho(t)}$, $V = \alpha \frac{R_{t_1}(t)}{\rho^{1/\alpha}(t)}$ and $y = \omega_1$, we obtain

$$\omega_1'(t) \leq \rho(t) \frac{\left(r(t)(z''(t))^{\alpha}\right)'}{z^{\alpha}(t)} + \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\rho_+'(t))^{\alpha+1}}{(\rho(t) R_{t_1}(t))^{\alpha}}. \tag{15}$$

Next, we define a function

$$\omega_2(t) = \rho(t) \frac{r(\tau_1(t))(z''(\tau_1(t)))^{\alpha}}{z^{\alpha}(t)}. \tag{16}$$

Then $\omega_2(t) > 0$. Since $z''(t) > 0$ and $\tau_1(t) < t$, we obtain $z'(t) > z'(\tau_1(t))$. Hence, from (14), we find

$$z'(t) > \left[r(\tau_1(t))(z''(\tau_1(t)))^{\alpha}\right]^{1/\alpha} R_{t_1}(\tau_1(t)). \tag{17}$$

for $t \geq t_2 \geq t_1$. By differentiating (16) and using (17), we get

$$\omega_2'(t) \leq \frac{\rho'(t)}{\rho(t)} \omega_2(t) - \alpha \frac{R_{t_1}(\tau_1(t))}{\rho^{1/\alpha}(t)} \omega_2^{\frac{\alpha+1}{\alpha}}(t) + \rho(t) \frac{\left(r(\tau_1(t))(z''(\tau_1(t)))^{\alpha}\right)'}{z^{\alpha}(t)}.$$

By using Lemma 1, we obtain

$$\omega_2'(t) \leq \rho(t) \frac{\left(r(\tau_1(t))(z''(\tau_1(t)))^{\alpha}\right)'}{z^{\alpha}(t)} + \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\rho_+'(t))^{\alpha+1}}{(\rho(t) R_{t_1}(\tau_1(t)))^{\alpha}}. \tag{18}$$

Additionally, we define another function

$$\omega_3(t) = \rho(t) \frac{r(\tau_2(t))(z''(\tau_2(t)))^{\alpha}}{z^{\alpha}(t)}. \tag{19}$$

Thus $\omega_3(t) > 0$. Using $\left(r(t)\left(z''(t)\right)^{\alpha}\right)' < 0$, $\tau_2(t) > t$ and (14), we note that

$$z'(t) > \left[r(\tau_2(t))\left(z''(\tau_2(t))\right)^{\alpha}\right]^{1/\alpha} R_{t_1}(t). \tag{20}$$

By differentiating (19) and using (20) and Lemma 1, we get

$$\omega_3'(t) \le \rho(t)\frac{\left(r(\tau_2(t))\left(z''(\tau_2(t))\right)^{\alpha}\right)'}{z^{\alpha}(t)} + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\rho_+'(t)\right)^{\alpha+1}}{\left(\rho(t)R_{t_1}(t)\right)^{\alpha}}. \tag{21}$$

From (7), (15), (18) and (21), we find

$$\begin{aligned}\omega_1'(t) + c_1^{\beta}\omega_2'(t) + \frac{c_2^{\beta}}{2^{\beta-1}}\omega_3'(t) &\le -\frac{\rho(t)}{4^{\beta-1}}\left(k_1q_1^*(t) + k_2q_2^*(t)\right)\frac{z^{\beta}(\sigma_1(t))}{z^{\alpha}(t)} \\ &+ \left(1 + c_1^{\beta} + \frac{c_2^{\beta}}{2^{\beta-1}}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\rho_+'(t)\right)^{\alpha+1}}{\left(\rho(t)R_{t_1}^*(t)\right)^{\alpha}}.\end{aligned} \tag{22}$$

Using (12) and Lemma 6, we have

$$\frac{z^{\beta}(\sigma_1(t))}{z^{\alpha}(t)} \ge \frac{\upsilon^{\beta-\alpha}}{2^{\beta-\alpha}t^{2\alpha}}\sigma_1^{\beta+\alpha}(t).$$

As in the proof of Theorem 1, we are led to a contradiction with (13). This completes the proof. □

In the following Theorems, we are concerned with the oscillation of solutions of Equation (1) when $\alpha = 1$ and $r(t) = 1$.

Theorem 3. *Assume that (M_1)-(M_5) and (3) hold. Let $0 < \beta < 1 < \gamma$ and τ_i^{-1} exists for $i = 1, 2$. If the inequalities*

$$y'''(t) + \left(\frac{k_1}{\lambda_1}\right)^{\lambda_1}\left(\frac{k_2}{4^{\gamma-1}\lambda_2}\right)^{\lambda_2}\frac{\left(q_1^*(t)\right)^{\lambda_1}\left(q_2^*(t)\right)^{\lambda_2}}{\left(1 + c_1^{\beta} + c_2^{\beta}\right)}y\left(\tau_i^{-1}(\sigma_j(t))\right) \le 0, \tag{23}$$

where $i, j = 1, 2$, $i \ne j$, $\lambda_1 = \frac{\gamma-1}{\gamma-\beta}$ and $\lambda_2 = \frac{1-\beta}{\gamma-\beta}$, have oscillatory solutions, then every solution of Equation (1) is oscillatory.

Proof. Let x non-oscillatory solution of Equation (1). Without loss of generality we assume that $x > 0$; then, there exists a $t_1 \ge t_0$ such that $x(t) > 0$, $x(\tau_i(t)) > 0$ and $x(\sigma_i(t)) > 0$ for $t \ge t_1$and $i = 1, 2$. By Lemma 6, we get that $z(t) > 0$, $z''(t) > 0$ and $z'''(t) < 0$. Now, we define a function

$$y(t) = z(t) + c_1^{\beta}z(\tau_1(t)) + c_2^{\beta}z(\tau_2(t)). \tag{24}$$

Thus $y(t) > 0$ and $y''(t) > 0$. From (1) and (M_5), we obtain

$$z'''(t) \le -k_1q_1(t)x^{\beta}(\sigma_1(t)) - k_2q_2(t)x^{\gamma}(\sigma_2(t)). \tag{25}$$

Combining (24) with (25), we get

$$\begin{aligned} y'''(t) &= z'''(t) + c_1^{\beta} z'''(\tau_1(t)) + c_2^{\beta} z'''(\tau_2(t)) \\ &\leq -k_1 q_1(t) x^{\beta}(\sigma_1(t)) - k_2 q_2(t) x^{\gamma}(\sigma_2(t)) \\ &\quad -c_1^{\beta}\left(-k_1 q_1(t) x^{\beta}(\sigma_1(\tau_1(t))) - k_2 q_2(t) x^{\gamma}(\sigma_2(\tau_1(t)))\right) \\ &\quad -c_2^{\beta}\left(-k_1 q_1(t) x^{\beta}(\sigma_1(\tau_2(t))) - k_2 q_2(t) x^{\gamma}(\sigma_2(\tau_2(t)))\right), \end{aligned}$$

and so,

$$\begin{aligned} y'''(t) &\leq -k_1 q_1^*(t)\left(x^{\beta}(\sigma_1(t)) + c_1^{\beta} x^{\beta}(\sigma_1(\tau_1(t))) + c_2^{\beta} x^{\beta}(\sigma_1(\tau_2(t)))\right) \\ &\quad -k_2 q_2^*(t)\left(x^{\gamma}(\sigma_2(t)) + c_1^{\beta} x^{\gamma}(\sigma_2(\tau_1(t))) + c_2^{\beta} x^{\gamma}(\sigma_2(\tau_2(t)))\right). \end{aligned}$$

By Lemma 2, since $c_1 + c_2 < 1$ and $\beta < 1 < \gamma$, we obtain

$$\begin{aligned} &y'''(t) + k_1 q_1^*(t) z^{\beta}(\sigma_1(t)) \\ &\quad + k_2 q_2^*(t)\left(x^{\gamma}(\sigma_2(t)) + c_1^{\gamma} x^{\gamma}(\sigma_2(\tau_1(t))) + \frac{c_2^{\gamma}}{2^{\gamma-1}} x^{\gamma}(\sigma_2(\tau_2(t)))\right) \leq 0. \end{aligned}$$

This implies

$$y'''(t) + k_1 q_1^*(t) z^{\beta}(\sigma_1(t)) + \frac{k_2}{4^{\gamma-1}} q_2^*(t) z^{\gamma}(\sigma_2(t)) \leq 0. \tag{26}$$

Using Lemma 6, we have two cases for $z'(t)$. If $z'(t) > 0$, we find

$$y'''(t) + k_1 q_1^*(t) z^{\beta}(\sigma_1(t)) + \frac{k_2}{4^{\gamma-1}} q_2^*(t) z^{\gamma}(\sigma_1(t)) \leq 0. \tag{27}$$

Using arithmetic-geometric mean inequality with $u_1 = \frac{k_1}{\lambda_1} q_1^*(t) z^{\beta}(\sigma_1(t))$ and $u_2 = \frac{k_2}{4^{\gamma-1}\lambda_2} q_2^*(t) z^{\gamma}(\sigma_1(t))$, we get

$$\begin{aligned} \lambda_1 u_1 + \lambda_2 u_2 &\geq u_1^{\lambda_1} u_2^{\lambda_2} \\ &= \left(\frac{k_1}{\lambda_1}\right)^{\lambda_1}\left(\frac{k_2}{4^{\gamma-1}\lambda_2}\right)^{\lambda_2}(q_1^*(t))^{\lambda_1}(q_2^*(t))^{\lambda_2} z(\sigma_1(t)). \end{aligned} \tag{28}$$

Since $\tau_1(t) < t < \tau_2(t)$, we note that

$$y(t) \leq \left(1 + c_1^{\beta} + c_2^{\beta}\right) z(\tau_2(t)).$$

Hence, from (28), (27) becomes

$$y'''(t) + \left(\frac{k_1}{\lambda_1}\right)^{\lambda_1}\left(\frac{k_2}{4^{\gamma-1}\lambda_2}\right)^{\lambda_2} \frac{(q_1^*(t))^{\lambda_1}(q_2^*(t))^{\lambda_2}}{\left(1 + c_1^{\beta} + c_2^{\beta}\right)} y\left(\tau_2^{-1}(\sigma_1(t))\right) \leq 0. \tag{29}$$

Then, the condition (23) implies (29) has oscillatory solution, which contradicts $y(t) > 0$.

Let $z'(t) < 0$. As in the previous case, we get

$$y'''(t) + \left(\frac{k_1}{\lambda_1}\right)^{\lambda_1} \left(\frac{k_2}{4^{\gamma-1}\lambda_2}\right)^{\lambda_2} \frac{(q_1^*(t))^{\lambda_1} (q_2^*(t))^{\lambda_2}}{\left(1 + c_1^{\beta} + c_2^{\beta}\right)} y\left(\tau_1^{-1}(\sigma_2(t))\right) \leq 0. \tag{30}$$

Hence, the condition (23) implies (30) has oscillatory solution, which contradicts $y(t) > 0$. This contradiction completes the proof. □

Remark 2. *There are numerous results concerning the oscillation of the equation*

$$y'''(t) + q(t) y(\sigma(t)) = 0,$$

(see [2,18,20,21]), which include Hille and Nehari types, Philos type, etc.

Assume that

$$\tau_i(t) = t + (-1)^i \widetilde{\tau}_i, \ \sigma_i(t) = t - (-1)^i \widetilde{\sigma}_i, \tag{31}$$

where $\widetilde{\tau}_i, \widetilde{\sigma}_i$ are positive constants for $i = 1, 2$. It is well known (see [9]) that the differential inequalities (29) and (30) are oscillatory if

$$\liminf_{t\to\infty} \int_{t-(\widetilde{\tau}_2+\widetilde{\sigma}_1)/3}^{t} (\widetilde{\tau}_2 + \widetilde{\sigma}_1)^2 (q_1^*(t))^{\lambda_1} (q_2^*(t))^{\lambda_2} > \frac{9}{2e} \left(\frac{\lambda_1}{k_1}\right)^{\lambda_1} \left(\frac{4^{\gamma-1}\lambda_2}{k_2}\right)^{\lambda_2} \tag{32}$$

and

$$\liminf_{t\to\infty} \int_{t}^{t+\widetilde{\tau}_1+\widetilde{\sigma}_2} (s-t)^2 (q_1^*(t))^{\lambda_1} (q_2^*(t))^{\lambda_2} > 2 \left(\frac{\lambda_1}{k_1}\right)^{\lambda_1} \left(\frac{4^{\gamma-1}\lambda_2}{k_2}\right)^{\lambda_2}, \tag{33}$$

respectively. Hence, we conclude the following theorem:

Theorem 4. *Assume that $0 < \beta < 1 < \gamma$ and (31) hold. If (32) and (33) hold, then every solution of Equation (1) is oscillatory.*

Remark 3. *In the case where $\alpha = 1$, $r(t) = 1$ and $p_i(t) = 0$, Equation (1) becomes*

$$x'''(t) + q_1(t) f_1(x(\sigma_1(t))) + q_2(t) f_2(x(\sigma_2(t))) = 0. \tag{34}$$

Baculikova and Dzurina [5] proved that every nonoscillatory solution x of (34) satisfies $x' < 0$. Thus, Theorems 3 and 4 improve the results in [5].

Remark 4. *A manner similar to the Theorem 3, we can study the oscillation of solutions of Equation (1) when $0 < \gamma < 1 < \beta$.*

Remark 5. *If $\alpha = 1$, $f_1(x) = x^{\beta}$, $f_2(x) = x^{\gamma}$, $\tau_1(t) = t - \widetilde{\tau}_1$, $\sigma_1(t) = t - \widetilde{\sigma}_1$, $\tau_2(t) = t + \widetilde{\tau}_2$, $\sigma_2(t) = t + \widetilde{\sigma}_2$ and $\widetilde{\tau}_i$, $\widetilde{\sigma}_i$ are positive constants, then Theorem 1 extends Theorem 2.5 and 2.7 in [23].*

Remark 6. *The results of Theorem 3 can be extended to the third-order differential equation*

$$\left((z(t))^{\alpha}\right)''' + q_1(t) f_1(x(\sigma_1(t))) + q_2(t) f_2(x(\sigma_2(t))) = 0;$$

the details are left to the reader.

Example 1. *Consider the equation*

$$\left(x+\frac{1}{3}x\left(\frac{1}{3}t\right)+\frac{1}{3}x\left(2t\right)\right)'''+\frac{q_0}{t^3}x\left(\frac{1}{2}t\right)+\frac{q_1}{t^3}x\left(2t\right)=0, \tag{35}$$

where $q_0>0$*. We note that* $\alpha=\beta=\gamma=1$*,* $r(t)=1$*,* $p_1(t)=p_2(t)=1/3$*,* $\tau_1(t)=1/3t$*,* $\sigma_1(t)=1/2t$*,* $\tau_2(t)=\sigma_2(t)=2/t$ *and* $q^*(t)=q_0/t^3$*. Hence, it is easy to see that*

$$\int_{t_0}^{\infty}\frac{1}{r^{1/\alpha}(s)}\mathrm{d}s=\infty.$$

Now, if we set $\rho(s):=t$ *and* $k_1=k_2=1$*, then we have*

$$\Theta_1(t)=\frac{q_0}{2s}.$$

Thus, we find

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^{t}\left(\Theta_1(s)-\left(1+c_1^{\beta}+\frac{c_2^{\beta}}{2^{\beta-1}}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\rho'_+(s)\right)^{\alpha+1}r\left(\sigma_1(s)\right)}{\left(\rho(s)\sigma_1'(s)\right)^{\alpha}}\right)ds\\=&\limsup_{t\to\infty}\int_{t_0}^{t}\left(\frac{q_0}{2s}-\frac{5}{6s}\right)ds.\end{aligned}$$

Thus, the conditions become

$$q_0>1.66.$$

Thus, by using Theorem 1, Equation (35) is either oscillatory if $q_0>1.66$ *or tends to zero as* $t\to\infty$*.*

Author Contributions: O.M. and O.B.: Writing original draft, and writing review and editing. D.C.: Formal analysis, writing review and editing, funding and supervision. All authors have read and agreed to the published version of the manuscript.

Funding: The authors received no direct funding for this work.

Acknowledgments: The authors thank the reviewers for for their useful comments, which led to the improvement of the content of the paper.

Conflicts of Interest: There are no competing interests for the authors.

References

1. Hale, J.K. *Theory of Functional Differential Equations*; Springer: New York, NY, USA, 1977.
2. Agarwal, R.P.; Bohner, M.; Li, T.; Zhang, C. Oscillation of third-order nonlinear delay differential equations. *Taiwanese J. Math.* **2013**, *17*, 545–558. [CrossRef]
3. Baculikova, B.; Dzurina, J. On the asymptotic behavior of a class of third order nonlinear neutral differential equations. *Cent. Eur. J. Math.* **2010**, *8*, 1091–1103. [CrossRef]
4. Baculikova, B.; Dzurina, J. Oscillation of third-order neutral differential equations. *Math. Comput. Modell.* **2010**, *52*, 215–226. [CrossRef]
5. Baculikova, B.; Dzurina, J. Some Properties of Third-Order Differential Equations with Mixed Arguments. *J. Math.* **2013**, *2013*, 528279. [CrossRef]

6. Bazighifan, O.; Cesarano, C. Some New Oscillation Criteria for Second-Order Neutral Differential Equations with Delayed Arguments. *Mathematics* **2019**, *7*, 619. [CrossRef]
7. Bazighifan, O.; Elabbasy, E.M.; Moaaz, O. Oscillation of higher-order differential equations with distributed delay. *J. Inequal. Appl.* **2019**, *55*, 1–9. [CrossRef]
8. Chatzarakis, G.E.; Elabbasy, E.M.; Bazighifan, O. An oscillation criterion in 4th-order neutral differential equations with a continuously distributed delay. *Adv. Differ. Equ.* **2019**, *336*, 1–9.
9. Das, P. Oscillation criteria for odd order neutral equations. *J. Math. Anal. Appl.* **1994**, *188*, 245–257. [CrossRef]
10. Elabbasy, E.M.; Hassan, T.S.; Moaaz, O. Oscillation behavior of second-order nonlinear neutral differential equations with deviating arguments. *Opusc. Math.* **2012**, *32*, 719–730. [CrossRef]
11. Elabbasy, E.M.; Moaaz, O. On the asymptotic behavior of third-order nonlinear functional differential equations. *Serdica Math. J.* **2016**, *42*, 157–174.
12. Elabbasy, E.M.; Barsoum, M.Y.; Moaaz, O. Boundedness and oscillation of third order neutral differential equations with deviating arguments. *J. Appl. Math. Phys.* **2015**, *3*, 1367–1375. [CrossRef]
13. Elabbasy, E.M.; Moaaz, O. On the oscillation of third order neutral differential equations. *Asian J. Math. Appl.* **2016**, *2016*, 0274.
14. Elabbasy, E.M.; Moaaz, O.; Almehabresh, E.S. Oscillation Properties of Third Order Neutral Delay Differential Equations. *Appl. Math.* **2016**, *7*, 1780–1788. [CrossRef]
15. Bazighifan, O.; Cesarano, C. A Philos-Type Oscillation Criteria for Fourth-Order Neutral Differential Equations. *Symmetry* **2020**, *12*, 379. [CrossRef]
16. Elabbasy, E.M.; Moaaz, O. Oscillation Criteria for third order nonlinear neutral differential equations with deviating arguments. *Int. J. Sci. Res.* **2016**, *5*, 1.
17. Kiguradze, I.T.; Chanturia, T.A. *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*; Kluwer Academic Publishers: Drodrcht, The Netherlands, 1993.
18. Ladas, G.; Sficas, Y.G.; Stavroulakis, I.P. Necessary and sufficent conditions for oscillation of higher order delay differential equations. *Trans. Am. Math. Soc.* **1984**, *285*, 81–90. [CrossRef]
19. Moaaz, O.; Awrejcewicz, J.; Bazighifan, O. A New Approach in the Study of Oscillation Criteria of Even-Order Neutral Differential Equations. *Mathematics* **2020**, *8*, 197. [CrossRef]
20. Saker, S.H.; Dzurina, J. On the oscillation of certain class of third-order nonlinear delay differential equations. *Math. Bohemica* **2010**, *135*, 225–237.
21. Saker, S.H. Oscillation criteria of Hille and Nehari types for third-order delay differential equations. *Comm. Appl. Anal.* **2007**, *11*, 451–468.
22. Han, Z.; Li, T.; Zhang, C.; Sun, S. Oscillatory Behavior of Solutions of Certain Third-Order Mixed Neutral Functional Differential Equations. *Bull. Malays. Math. Sci. Soc.* **2012**, *35*, 611–620.
23. Thandapani, E.; Rama, R. Oscillatory behavior of solutions of certain third order mixed neutral differential equations. *Tamkang J. Math.* **2013**, *44*, 99–112. [CrossRef]
24. Thandapani, E.; Li, T. On the oscillation of third-order quasi-linear neutral functional differential equations. *Arch. Math.* **2011**, *47*, 181–199.