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Homomorphic Image and Inverse Image of Weak Closure Operations on Ideals of BCK-Algebras

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Abstract: We introduce the notions of meet, semi-prime, and prime weak closure operations. Using homomorphism of BCK-algebras $\varphi : X \rightarrow Y$, we show that every epimorphic image of a non-zero meet element is also non-zero meet and, for mapping $cl_Y : \mathcal{I}(Y) \rightarrow \mathcal{I}(Y)$, we define a map cl_Y^{\leftarrow} on $\mathcal{I}(X)$ by $A \mapsto \varphi^{-1}(\varphi(A)^{cl_Y})$. We prove that, if “ cl_Y ” is a weak closure operation (respectively, semi-prime and meet) on $\mathcal{I}(Y)$, then so is “ cl_Y^{\leftarrow} ” on $\mathcal{I}(X)$. In addition, for mapping $cl_X : \mathcal{I}(X) \rightarrow \mathcal{I}(X)$, we define a map cl_X^{\rightarrow} on $\mathcal{I}(Y)$ as follows: $B \mapsto \varphi(\varphi^{-1}(B)^{cl_X})$. We show that, if “ cl_X ” is a weak closure operation (respectively, semi-prime and meet) on $\mathcal{I}(X)$, then so is “ cl_X^{\rightarrow} ” on $\mathcal{I}(Y)$.

Keywords: zero meet element; meet ideal; relative annihilator; semi-prime map; meet map; (semi-prime, meet) weak closure operation

1. Introduction

Imai and Iseki introduced the concept of BCK-algebra in 1966. BCK-algebra is a generalization of set-theoretic difference. In addition, in [1], Bordbar et al. introduced the concept of weak closure operation as a generalization of closure operation on ideals of a BCK-algebras. They investigated some properties of weak closure operations and provided examples. They defined many different types of weak closure operations on ideals of an arbitrary BCK-algebra such that finite type, (strong) semi-primeness, tender, and naive weak closure operations and investigated their related properties [2–5]. Besides, the notions of commutative, implicative, and positive implicative weak closure operations were investigated by [1]. Moreover, using the notation of weak closure operation, some algebraic structures have been defined and related properties investigated [6–8].

In this article, the notions of semi-prime, meet, and prime weak closure operation are introduced, and their relations and properties investigated. Using an epimorphism $\varphi : X \rightarrow Y$ of lower BCK-semilattices X and Y , mapping $cl_Y : \mathcal{I}(Y) \rightarrow \mathcal{I}(Y)$, and any ideal A of X with $\ker(\varphi) \subseteq A$, we define a new map cl_Y^{\leftarrow} on $\mathcal{I}(X)$ by

$$A \mapsto \varphi^{-1}(\varphi(A)^{cl_Y}).$$

and investigate related properties of this map. We show that, if “ cl_Y ” is a weak closure operation (respectively, semi-prime, meet, and prime) on $\mathcal{I}(Y)$, then so is “ cl_Y^{\leftarrow} ” on $\mathcal{I}(X)$.

Similarly, using a map $cl_X : \mathcal{I}(X) \rightarrow \mathcal{I}(X)$ and any ideal B of Y , we define a new map cl_X^{\rightarrow} on $\mathcal{I}(Y)$ as follows:

$$B \mapsto \varphi(\varphi^{-1}(B)^{cl_X}).$$

We prove that, if “ cl_X ” is a weak closure operation (respectively, semi-prime, meet, and prime) on $\mathcal{I}(X)$, then so is “ cl_X^{\rightarrow} ” on $\mathcal{I}(Y)$.

2. Preliminaries

BCI/BCK-algebra, which was introduced by K. Iséki, is a class of logical algebras. In what follows, we gather some necessary definitions and theorems which we need for further results.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following conditions:

- (I) $(\forall x \in X) (x * x = 0)$;
- (II) $(\forall x, y \in X) ((x * (x * y)) * y = 0)$;
- (III) $(\forall x, y, z \in X) (((x * y) * (x * z)) * (z * y) = 0)$; and
- (IV) $(\forall x, y \in X) (x * y = 0, y * x = 0 \Rightarrow x = y)$.

Moreover, if an arbitrary BCI-algebra X satisfies the following condition, then X is called a BCK-algebra:

- (V) $(\forall x \in X) (0 * x = 0)$.

Lemma 1. *Suppose that X is a BCI/BCK-algebra. Then, the following axioms are satisfied:*

- (a1) $(\forall x \in X) (x * 0 = x)$;
- (a2) $(\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x)$;
- (a3) $(\forall x, y, z \in X) ((x * y) * z = (x * z) * y)$; and
- (a4) $(\forall x, y, z \in X) ((x * z) * (y * z) \leq x * y)$

We define \leq such that $x * y = 0$. if and only if $x \leq y$, and call it BCK-order.

For a BCK-algebra X , if X concerning the BCK-order is a lower semilattice, then we called it the lower BCK-semilattice.

Definition 1. *Suppose that X is a BCK-algebra and A is a subset of X . A is called an ideal of X when the following assertions hold:*

$$0 \in A, \tag{1}$$

$$(\forall x \in X) (\forall y \in A) (x * y \in A \Rightarrow x \in A). \tag{2}$$

The following condition is satisfied for any ideal A of a BCK-algebra X

$$(\forall x, y \in X) (x \leq y, y \in A \Rightarrow x \in A). \tag{3}$$

Remark 1. *Let A be a subset of X . The intersection of all ideals of X containing A is an ideal of X and it is denoted by $\langle A \rangle$. Moreover, If A is a finite set, then we say that $\langle A \rangle$ is finitely generated ideal of X .*

From now on, X and $\mathcal{I}(X)$ are the lower BCK-semilattice and the set of all ideals of the X , respectively, unless otherwise specified. In addition, for more information regarding the ordered algebra and BCI/BCK-algebras, please refer to the books [9,10].

Definition 2 ([1]). *Suppose that X is a lower BCK-semilattice and $x \in X$. Then, the element x is called a zeromeet element of X if the following condition holds.*

$$(\exists y \in X \setminus \{0\}) (x \wedge y = 0).$$

Otherwise, x is called a non-zeromeet element of X .

The set of all zeromeet elements of X is denoted by $Z(X)$ and defined as follow:

$$Z(X) = \{x \in X \mid x \wedge y = 0 \text{ for some nonzero element } y \in X\}.$$

It is clear that $0 \in Z(X)$ and also, for the greatest element 1, we have $1 \in X \setminus Z(X)$.

Lemma 2 ([1]). *The set $X \setminus Z(X)$ is closed under the \wedge operation.*

Definition 3 ([6]). *Let A and B be nonempty subsets of X ; define $A \wedge B$ as follows:*

$$A \wedge B := \{a \wedge b \mid a \in A, b \in B\};$$

and call it the meet ideal of X generated by A and B . Furthermore, $\{a\} \wedge B$ is denoted by $a \wedge B$ and, similarly, $A \wedge \{b\}$ is denoted by $A \wedge b$.

Theorem 1 ([6]). *If A and B are ideals of X , then the set*

$$A \wedge B = \{a \wedge b \mid a \in A, b \in B\}$$

is an ideal of X based on A and B .

Definition 4 ([8]). *Define a set $(A :_{\wedge} B)$ as follow for nonempty subsets A and B of X and call it the relative annihilator of B with respect to A .*

$$(A :_{\wedge} B) := \{x \in X \mid x \wedge B \subseteq A\}$$

Lemma 3 ([8]). *The relative annihilator $(A :_{\wedge} B)$ of B with respect to A is an ideal of X , where A and B are ideals of X .*

Lemma 4 ([8]). *For ideal A of X , it is clear that $(A :_{\wedge} X) = A$ and $(A :_{\wedge} A) = X$.*

Now, we define a weak closure operation.

Definition 5 ([1]). *A weak closure operation on set of ideals of a BCK-algebra is defined as a mapping $cl : \mathcal{I}(X) \rightarrow \mathcal{I}(X)$, if the following conditions hold.*

$$(\forall A \in \mathcal{I}(X)) (A \subseteq cl(A)), \tag{4}$$

$$(\forall A, B \in \mathcal{I}(X)) (A \subseteq B \Rightarrow cl(A) \subseteq cl(B)). \tag{5}$$

For a weak closure operation cl , ff the condition

$$(\forall A \in \mathcal{I}(X)) (cl(cl(A)) = cl(A)), \tag{6}$$

holds, then we call it a closure operation on $\mathcal{I}(X)$ (see [3]).

For simplicity, we write A^{cl} instead of $cl(A)$.

Definition 6 ([5]). *A semi-prime weak closure operation “ cl ” on $\mathcal{I}(X)$ is defined as follows:*

$$(\forall A, B \in \mathcal{I}(X)) (A \wedge B^{cl} \subseteq (A \wedge B)^{cl}). \tag{7}$$

Definition 7 ([5]). *A meet weak closure operation “ cl ” on $\mathcal{I}(X)$ is defined as follows:*

$$(\forall A \in \mathcal{I}(X)) (\forall a \in X \setminus Z(X)) ((a \wedge A)^{cl} = a \wedge A^{cl}). \tag{8}$$

Definition 8. A prime weak closure operation “cl” on $\mathcal{I}(X)$ is a semi-prime and meet weak closure operation on $\mathcal{I}(X)$.

Lemma 5 ([9]). If $\varphi : X \rightarrow Y$ is an epimorphism of lower BCK-semilattices, then

$$(\forall x, y \in X) (\varphi(x \wedge_X y) = \varphi(x) \wedge_Y \varphi(y)). \tag{9}$$

3. Homomorphic (Inverse) Images of (Meet, Semi-Prime, and Prime) Weak Closure Operations

Definition 9. Let $cl : \mathcal{I}(X) \rightarrow \mathcal{I}(X)$ be a map on set of ideals of a BCK-algebra X . This map is said to be meet when the following condition holds:

$$(\forall A \in \mathcal{I}(X)) (\forall a \in X \setminus Z(X)) ((a \wedge A)^{cl} = a \wedge A^{cl}). \tag{10}$$

Example 1. Suppose that $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Then, X is a lower BCK-semilattice with five following ideals: $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2\}$ and $A_4 = X$ and also $Z(X) = \{0, 1, 2\}$. Now, define a map “cl” on $\mathcal{I}(X)$ such that $A_0^{cl} = A_0$, $A_1^{cl} = A_2$, $A_2^{cl} = A_1$, $A_3^{cl} = A_4$ and $A_4^{cl} = A_3$. It is routine to verify that “cl” is a meet map on $\mathcal{I}(X)$. However, “cl” is not a weak closure operation since $A_1 \not\subseteq A_2 = A_1^{cl}$.

By a meet weak closure operation on $\mathcal{I}(X)$ (see [5]), we mean a weak closure operation cl on $\mathcal{I}(X)$ which satisfies the condition in Equation (10).

Definition 10. A map $cl : \mathcal{I}(X) \rightarrow \mathcal{I}(X)$ is called semi-prime if the following condition holds:

$$(\forall A, B \in \mathcal{I}(X)) (A \wedge B^{cl} \subseteq (A \wedge B)^{cl}). \tag{11}$$

Example 2. Suppose that $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	1	0	0	2
3	3	1	1	0	3
4	4	4	4	4	0

Then, X is a lower BCK-semilattice with four ideals: $A_0 = \{0\}$, $A_1 = \{0, 1, 2, 3\}$, $A_2 = \{0, 4\}$ and $A_3 = X$. Now, define a map “cl” on $\mathcal{I}(X)$ such that $A_0^{cl} = A_3$, $A_1^{cl} = A_2$, $A_2^{cl} = A_1$, and $A_3^{cl} = A_0$. Then, “cl” is a semi-prime map on set of ideals of X . However, it is not a weak closure operation because $A_1 \not\subseteq A_2 = A_1^{cl}$.

By a semi-prime weak closure operation on $\mathcal{I}(X)$ (see [5]), we mean a weak closure operation cl on $\mathcal{I}(X)$, which satisfies the condition in Equation (11).

The notions of semi-prime map and meet map are independent. The following examples show it.

Example 3. Let X be a set of $\{0, 1, 2, 3, 4\}$ with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Then, by routine verification, X is a BCK-semilattice and $Z(X) = \{0, 1, 2\}$. In addition, X has five ideals: $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2\}$ and $A_4 = X$. Now, define a map “cl” on $\mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_2$, $A_2^{cl} = A_1$, $A_3^{cl} = A_4$ and $A_4^{cl} = A_4$. Then, “cl” is a meet map on $\mathcal{I}(X)$. However, “cl” is not a semi-prime map on $\mathcal{I}(X)$ since $A_1 \wedge A_2^{cl} = A_1 \wedge A_1 = A_1 \not\subseteq A_0 = A_0^{cl} = (A_1 \wedge A_2)^{cl}$.

Example 4. Let X be a set of $\{0, 1, 2, 3, 4\}$ with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	2
3	3	3	2	0	3
4	4	4	4	4	0

Then, by routine verification, X is a BCK-semilattice and $Z(X) = \{0\}$. In addition, X has five ideals: $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2, 3\}$, $A_3 = \{0, 1, 4\}$ and $A_4 = X$. Now, define “cl” on $\mathcal{I}(X)$ by $A_0^{cl} = A_1$, $A_1^{cl} = A_4$, $A_2^{cl} = A_4$, $A_3^{cl} = A_3$, and $A_4^{cl} = A_4$. Then, “cl” is a semi-prime map on $\mathcal{I}(X)$ which is not meet since

$$3 \wedge A_2^{cl} = 3 \wedge A_4 = A_2 \neq A_4 = A_2^{cl} = (3 \wedge A_2)^{cl}.$$

In the following examples, we see that a meet weak closure operation is independent of a semi-prime weak closure operation.

Example 5. Suppose that $X = \{0, 1, 2, 3, 4\}$. Then, X is a lower BCK-semilattice with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	0
3	3	3	3	0	0
4	4	4	4	4	0

We also have $Z(X) = \{0, 1, 2, 3\}$ and X has five ideals: $A_0 = \{0\}$, $A_1 = \{0, 1, 2\}$, $A_2 = \{0, 3\}$, $A_3 = \{0, 1, 2, 3\}$, and $A_4 = X$. Now, define “cl” on $\mathcal{I}(X)$ by $A_0^{cl} = A_1$, $A_1^{cl} = A_3$, $A_2^{cl} = A_3$, $A_3^{cl} = A_3$ and $A_4^{cl} = A_4$. Then, “cl” is a meet weak closure operation on $\mathcal{I}(X)$ which is not a semi-prime map since

$$A_1^{cl} \wedge A_2 = A_3 \wedge A_2 = A_2 \not\subseteq A_1 = A_0^{cl} = (A_1 \wedge A_2)^{cl}.$$

Example 6. Consider $X = \{0, 1, 2, 3, 4\}$ the same as given in Example 4. If we define “cl” on $\mathcal{I}(X)$ by $A_0^{cl} = A_1$, $A_1^{cl} = A_4$, $A_2^{cl} = A_4$, $A_3^{cl} = A_4$ and $A_4^{cl} = A_4$, then “cl” is a weak closure operation which is semi-prime but not meet since

$$4 \wedge A_2^{cl} = 4 \wedge A_4 = A_3 \neq A_4 = (4 \wedge A_2)^{cl}.$$

Definition 11. A prime map $cl : \mathcal{I}(X) \rightarrow \mathcal{I}(X)$ is a map which is both meet and semi-prime.

Example 7. Let $X = \{0, 1, 2, 3\}$ and define the following Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

Then, X is a lower BCK-semilattice and, clearly, $Z(X) = \{0, 1, 2\}$. In addition, X has five ideals: $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2\}$, and $A_4 = X$. Now, define a map “ cl ” on $\mathcal{I}(X)$ by $A_0^{cl} = A_4$, $A_1^{cl} = A_3$, $A_2^{cl} = A_3$, $A_3^{cl} = A_3$ and $A_4^{cl} = A_4$. Then, “ cl ” is a prime map on $\mathcal{I}(X)$.

Remember that, by a *prime weak closure operation*, we mean a weak closure operation cl on set of ideals, which is both meet and semi-prime.

Example 8. Let $X = \{0, 1, 2, 3, 4\}$ with the conditions given in Example 5. Then, X has five ideals: $A_0 = \{0\}$, $A_1 = \{0, 1, 2\}$, $A_2 = \{0, 3\}$, $A_3 = \{0, 1, 2, 3\}$, and $A_4 = X$. In addition, $Z(X) = \{0, 1, 2, 3\}$. If we define “ cl ” on $\mathcal{I}(X)$ by $A_0^{cl} = A_1$, $A_1^{cl} = A_1$, $A_2^{cl} = A_3$, $A_3^{cl} = A_4$, and $A_4^{cl} = A_4$, then “ cl ” is a prime weak closure operation on $\mathcal{I}(X)$.

Lemma 6 ([9]). If $\varphi : X \rightarrow Y$ is an epimorphism of lower BCK-semilattices, then

$$(\forall x, y \in X) (\varphi(x \wedge_X y) = \varphi(x) \wedge_Y \varphi(y)). \tag{12}$$

Theorem 2. Every isomorphic image of a non-zero meet element is also non-zero meet.

Proof. Let $\varphi : X \rightarrow Y$ be an isomorphism of lower BCK-semilattices. Suppose that z is a non-zero meet element of X . Let $y \in Y$ be such that $y \wedge_Y \varphi(z) = 0$. Then, $y = \varphi(x)$ for some $x \in X$, and thus

$$0 = y \wedge_Y \varphi(z) = \varphi(x) \wedge_Y \varphi(z) = \varphi(x \wedge_X z)$$

by using Lemma 6. Hence, $x \wedge_X z \in \ker(\varphi) = \{0\}$, and thus $x \wedge_X z = 0$. Since z is a non-zero meet element of X , it follows that $x = 0$ and thus $y = \varphi(x) = \varphi(0) = 0$. Therefore, $\varphi(z)$ is a non-zero meet element of Y . \square

Lemma 7 ([9]). 1. Let $\varphi : X \rightarrow Y$ be an epimorphism of BCK-algebras. If A is an ideal of X , then $\varphi(A)$ is an ideal of Y .

2. Let $\varphi : X \rightarrow Y$ be a homomorphism of BCK-algebras. If B is an ideal of Y , then $\varphi^{-1}(B)$ is an ideal of X .

Lemma 8 ([9]). Let $\varphi : X \rightarrow Y$ be a homomorphism of BCK-algebras X and Y and let A be an ideal of X such that $\text{Ker}(\varphi) \subseteq A$. Then, $\varphi^{-1}(A') = A$ where $A' = \varphi(A)$.

Lemma 9. If $\varphi : X \rightarrow Y$ is a homomorphism of lower BCK-semilattices, then

$$\varphi^{-1}(G) \wedge_X \varphi^{-1}(H^{cl_Y}) \subseteq \varphi^{-1}(G \wedge_Y H^{cl_Y}) \tag{13}$$

for all ideals G and H of Y . Moreover, if φ is an isomorphism, then

$$\varphi^{-1}(G \wedge_Y H^{cl_Y}) = \varphi^{-1}(G) \wedge_X \varphi^{-1}(H^{cl_Y}) \tag{14}$$

for all ideals G and H of Y .

Proof. Let G and H be ideals of Y . Assume that $\varphi : X \rightarrow Y$ is a homomorphism of lower BCK-semilattices. If $z \in \varphi^{-1}(G) \wedge_X \varphi^{-1}(H^{cl_Y})$, then there exist $x \in \varphi^{-1}(G)$ and $y \in \varphi^{-1}(H^{cl_Y})$ such that $z = x \wedge_X y$. Hence,

$$\varphi(z) = \varphi(x \wedge_X y) = \varphi(x) \wedge_Y \varphi(y) \in G \wedge_Y H^{cl_Y}.$$

Therefore, $z \in \varphi^{-1}(G \wedge_Y H^{cl_Y})$.

Now, suppose that $\varphi : X \rightarrow Y$ is an isomorphism of lower BCK-semilattices and let $z \in \varphi^{-1}(G \wedge_Y H^{cl_Y})$. Then, $\varphi(z) \in G \wedge_Y H^{cl_Y}$, and thus there exist $g \in G$ and $h \in H^{cl_Y}$ such that $\varphi(z) = g \wedge h$. Since φ is an isomorphism, we have

$$z = \varphi^{-1}(\varphi(z)) = \varphi^{-1}(g \wedge_Y h) = \varphi^{-1}(g) \wedge_X \varphi^{-1}(h) \in \varphi^{-1}(G) \wedge_X \varphi^{-1}(H^{cl_Y}).$$

Hence, $\varphi^{-1}(G \wedge_Y H^{cl_Y}) = \varphi^{-1}(G) \wedge_X \varphi^{-1}(H^{cl_Y})$. \square

Lemma 10. If $\varphi : X \rightarrow Y$ is a isomorphism of lower BCK-semilattices, then $(\forall A, B \in \mathcal{I}(X)) (\varphi(A \wedge_X B) = \varphi(A) \wedge_Y \varphi(B))$.

Proof. Since A and B are ideals, we have $A \wedge_X B = \{a \wedge_X b \mid a \in A \text{ and } b \in B\}$. In addition, since φ is epimorphism, for every $a \in A$ and $b \in B$ by using Lemma 6, we have $\varphi(a \wedge_X b) = \varphi(a) \wedge_Y \varphi(b)$. Thus,

$$\begin{aligned} \varphi(A \wedge_X B) &= \varphi(\{a \wedge_X b \mid a \in A \text{ and } b \in B\}) \\ &= \{\varphi(a) \wedge_Y \varphi(b) \mid a \in A \text{ and } b \in B\} \\ &= \varphi(A) \wedge_Y \varphi(B). \end{aligned}$$

\square

Theorem 3. Let $\varphi : X \rightarrow Y$ be an epimorphism of lower BCK-semilattices. Given a mapping $cl_Y : \mathcal{I}(Y) \rightarrow \mathcal{I}(Y)$ and any ideal A of X with $\ker(\varphi) \subseteq A$, define a map cl_X^{\leftarrow} on $\mathcal{I}(X)$ as follows:

$$cl_X^{\leftarrow} : \mathcal{I}(X) \rightarrow \mathcal{I}(X), A \mapsto \varphi^{-1}(\varphi(A)^{cl_Y}).$$

- (1) If " cl_Y " is a weak closure operation on $\mathcal{I}(Y)$, then " cl_X^{\leftarrow} " is a weak closure operation on $\mathcal{I}(X)$.
- (2) If " cl_Y " is a semi-prime map on $\mathcal{I}(Y)$, then " cl_X^{\leftarrow} " is a semi-prime map on $\mathcal{I}(X)$.
- (3) If " cl_Y " is a meet map on $\mathcal{I}(Y)$ and φ is an isomorphism, then " cl_X^{\leftarrow} " is a meet map on $\mathcal{I}(X)$.

Proof. (1) Suppose that " cl_Y " is a weak closure operation on $\mathcal{I}(Y)$. Let A be an ideal of X . Then, $\varphi(A)$ is an ideal of Y by Lemma 7, and thus $\varphi(A) \subseteq \varphi(A)^{cl_Y}$. It follows from Lemma 8 that

$$A = \varphi^{-1}(\varphi(A)) \subseteq \varphi^{-1}(\varphi(A)^{cl_Y}) = A^{cl_X^{\leftarrow}}.$$

Now, let A and B be ideals of X such that $A \subseteq B$. Then, $\varphi(A) \subseteq \varphi(B)$, and thus $\varphi(A)^{cl_Y} \subseteq \varphi(B)^{cl_Y}$. It follows that

$$A^{cl_X^{\leftarrow}} = \varphi^{-1}(\varphi(A)^{cl_Y}) \subseteq \varphi^{-1}(\varphi(B)^{cl_Y}) = B^{cl_X^{\leftarrow}}.$$

Therefore, " cl_X^{\leftarrow} " is a weak closure operation on $\mathcal{I}(X)$.

(2) Suppose that " cl_Y " is a semi-prime map on $\mathcal{I}(Y)$. For any ideals A and B of X , we have

$$\begin{aligned} \varphi(A) \wedge_Y \varphi(B)^{cl_Y} &\subseteq (\varphi(A) \wedge_Y \varphi(B))^{cl_Y} \\ &= (\{\varphi(a) \wedge_Y \varphi(b) \mid a \in A, b \in B\})^{cl_Y} \\ &= (\{\varphi(a \wedge_X b) \mid a \in A, b \in B\})^{cl_Y} \\ &= \varphi(A \wedge_X B)^{cl_Y} \end{aligned}$$

by Lemma 6, which implies from Lemmas 8 and 9 that

$$\begin{aligned} A \wedge_X B^{cl_Y^{\leftarrow}} &= A \wedge_X \varphi^{-1}(\varphi(B)^{cl_Y}) \\ &= \varphi^{-1}(\varphi(A)) \wedge_X \varphi^{-1}(\varphi(B)^{cl_Y}) \\ &= \varphi^{-1}(\varphi(A) \wedge_Y \varphi(B)^{cl_Y}) \\ &\subseteq \varphi^{-1}(\varphi(A \wedge_X B)^{cl_Y}) \\ &= (A \wedge_X B)^{cl_Y^{\leftarrow}}. \end{aligned}$$

Therefore, “ cl_Y^{\leftarrow} ” is a semi-prime map on $\mathcal{I}(X)$.

(3) Suppose that “ cl_Y ” is a meet map on $\mathcal{I}(Y)$ and φ is an isomorphism. For any ideal A of X and $z \in X \setminus Z(X)$, we know that $\varphi(z)$ is a non-zero meet element of Y by Theorem 2. Using Lemmas 6, 8, and 10, we have

$$\begin{aligned} (z \wedge_X A)^{cl_Y^{\leftarrow}} &= \varphi^{-1}(\varphi(z \wedge_X A)^{cl_Y}) \\ &= \varphi^{-1}((\varphi(z) \wedge_Y \varphi(A))^{cl_Y}) \\ &= \varphi^{-1}(\varphi(z) \wedge_Y \varphi(A)^{cl_Y}) \\ &= \varphi^{-1}(\varphi(z)) \wedge_X \varphi^{-1}(\varphi(A)^{cl_Y}) \\ &= z \wedge_X A^{cl_Y^{\leftarrow}}. \end{aligned}$$

Therefore, “ cl_Y^{\leftarrow} ” is a meet con $\mathcal{I}(X)$. \square

Corollary 1. *If $\varphi : X \rightarrow Y$ is an isomorphism of lower BCK-semilattices and “ cl_Y ” is a prime weak closure operation on $\mathcal{I}(Y)$, then “ cl_Y^{\leftarrow} ” is a prime weak closure operation on $\mathcal{I}(X)$.*

Proof. The proof is straightforward. \square

Theorem 4. *Let $\varphi : X \rightarrow Y$ be an epimorphism of lower BCK-semilattices. Given a mapping $cl_X : \mathcal{I}(X) \rightarrow \mathcal{I}(X)$ and any ideal B of Y , define a map cl_X^{\rightarrow} on $\mathcal{I}(Y)$ as follows:*

$$cl_X^{\rightarrow} : \mathcal{I}(Y) \rightarrow \mathcal{I}(Y), B \mapsto \varphi(\varphi^{-1}(B)^{cl_X}).$$

- (1) *If “ cl_X ” is a weak closure operation on $\mathcal{I}(X)$, then “ cl_X^{\rightarrow} ” is a weak closure operation on $\mathcal{I}(Y)$.*
- (2) *If “ cl_X ” is a semi-prime map on $\mathcal{I}(X)$, then “ cl_X^{\rightarrow} ” is a semi-prime map on $\mathcal{I}(Y)$.*
- (3) *If “ cl_X ” is a meet map on $\mathcal{I}(X)$ and φ is an isomorphism, then “ cl_X^{\rightarrow} ” is a meet map on $\mathcal{I}(Y)$.*

Proof. (1) Suppose that “ cl_X ” is a weak closure operation on $\mathcal{I}(X)$ and let A and B be ideals of Y . Then, $\varphi^{-1}(A)$ and $\varphi^{-1}(B)$ are ideals of X by Lemma 7. Since “ cl_X ” is a weak closure operation on $\mathcal{I}(X)$, we have $\varphi^{-1}(A) \subseteq \varphi^{-1}(A)^{cl_X}$, which implies that

$$A \subseteq \varphi(\varphi^{-1}(A)) \subseteq \varphi(\varphi^{-1}(A)^{cl_X}) = A^{cl_X^{\rightarrow}}.$$

Now, let $A \subseteq B$. Then, $\varphi^{-1}(A) \subseteq \varphi^{-1}(B)$, and thus $\varphi^{-1}(A)^{cl_X} \subseteq \varphi^{-1}(B)^{cl_X}$. It follows that

$$A^{cl_X^{\rightarrow}} = \varphi(\varphi^{-1}(A)^{cl_X}) \subseteq \varphi(\varphi^{-1}(B)^{cl_X}) = B^{cl_X^{\rightarrow}}.$$

Therefore, “ cl_X^{\rightarrow} ” is a weak closure operation on $\mathcal{I}(Y)$.

(2) Let “ cl_X ” be a semi-prime map on $\mathcal{I}(X)$ and let A and B be ideals of Y . Then, we have

$$\varphi^{-1}(A) \wedge_X \varphi^{-1}(B)^{cl_X} \subseteq (\varphi^{-1}(A) \wedge_X \varphi^{-1}(B))^{cl_X} \subseteq \varphi^{-1}(A \wedge_Y B)^{cl_X}$$

by Lemma 9. Hence,

$$\varphi(\varphi^{-1}(A) \wedge_X \varphi^{-1}(B)^{cl_X}) \subseteq \varphi(\varphi^{-1}(A \wedge_Y B)^{cl_X}). \tag{15}$$

It follows from Lemma 10 that

$$\begin{aligned} A \wedge_Y B^{cl_{\vec{X}}} &= A \wedge_Y \varphi(\varphi^{-1}(B)^{cl_X}) \\ &\subseteq \varphi(\varphi^{-1}(A)) \wedge_Y \varphi(\varphi^{-1}(B)^{cl_X}) \\ &= \varphi(\varphi^{-1}(A) \wedge_X \varphi^{-1}(B)^{cl_X}) \\ &\subseteq \varphi(\varphi^{-1}(A \wedge_Y B)^{cl_X}) \\ &= (A \wedge_Y B)^{cl_{\vec{X}}}. \end{aligned} \tag{16}$$

Therefore, “ $cl_{\vec{X}}$ ” is a semi-prime map on $\mathcal{I}(Y)$.

(3) Let “ cl_X ” be a meet map on $\mathcal{I}(X)$ and assume that φ is an isomorphism. Let $B \in \mathcal{I}(Y)$ and $z \in Y \setminus Z(Y)$. Then,

$$\begin{aligned} (z \wedge_Y B)^{cl_{\vec{X}}} &= \varphi(\varphi^{-1}(z \wedge_Y B)^{cl_X}) \\ &= \varphi((\varphi^{-1}(z) \wedge_X \varphi^{-1}(B))^{cl_X}) \\ &= \varphi(\varphi^{-1}(z) \wedge_X \varphi^{-1}(B)^{cl_X}) \\ &= \varphi(\varphi^{-1}(z)) \wedge_Y \varphi(\varphi^{-1}(B)^{cl_X}) \\ &= z \wedge_Y B^{cl_{\vec{X}}} \end{aligned}$$

Thus, “ $cl_{\vec{X}}$ ” is a meet map on $\mathcal{I}(Y)$. \square

Corollary 2. *If $\varphi : X \rightarrow Y$ is an isomorphism of lower BCK-semilattices and “ cl_X ” is a prime weak closure operation on $\mathcal{I}(X)$, then “ $cl_{\vec{X}}$ ” is a prime weak closure operation on $\mathcal{I}(Y)$.*

Proof. The proof is straightforward. \square

4. Conclusions

By using an epimorphism $\varphi : X \rightarrow Y$ of lower BCK-semilattices X, Y and mapping $cl_Y : \mathcal{I}(Y) \rightarrow \mathcal{I}(Y)$, and any ideal A of X with $ker(\varphi) \subseteq A$, we define a new map cl_Y^{\leftarrow} on $\mathcal{I}(X)$ by

$$A \mapsto \varphi^{-1}(\varphi(A)^{cl_Y}).$$

and, moreover, we show that, if “ cl_Y ” is a semi-prime, meet, and prime weak closure operation on $\mathcal{I}(Y)$, then so is “ cl_Y^{\leftarrow} ” on $\mathcal{I}(X)$.

Similarly, using a map $cl_X : \mathcal{I}(X) \rightarrow \mathcal{I}(X)$ and any ideal B of Y , we define a new map $cl_{\vec{X}}$ on $\mathcal{I}(Y)$ as follows:

$$B \mapsto \varphi(\varphi^{-1}(B)^{cl_X}).$$

We prove that, if “ cl_X ” is a semi-prime, meet, and prime weak closure operation on $\mathcal{I}(X)$, then so is “ $cl_{\vec{X}}$ ” on $\mathcal{I}(Y)$.

These results can be applied to investigate if the properties of closure operations are preserved under the homomorphism or not. In our future research, we will focus on other types of weak closure operations and we will study about homomorphic images and inverse images under homomorphism in different conditions. This kind of results can be useful to study isomorphism operations in different spaces. One of the motivations could be what happens for a decomposable ideal of a BCK-algebra under the homomorphism, or the associated prime ideals of a decomposable ideal under homomorphism. For more information about decomposable ideals and closure operations, please refer to [7].

In addition, other kinds of closure operations such as meet, tender, naive, etc. can be checked for homomorphic image and inverse image. For further information about other kinds of (weak) closure operation, please refer to [3,4]. Beside, for future research, we invite the researchers to join us and apply the results of this paper to concepts in [11–13] about the height of ideals. More precisely, we mean what happens for the height of ideals of BCK-algebra under homomorphisms.

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