About Aczél Inequality and Some Bounds for Several Statistical Indicators

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Abstract: In this paper, we will study a refinement of the Cauchy–Buniakowski–Schwarz inequality and a refinement of the Aczél inequality by the technique of the monotony of a sequence. In the final part, we present some properties of bounds of several statistical indicators of variation.

Keywords: Cauchy–Buniakowski–Schwarz inequality; statistical indicators

1. Introduction

Many mathematicians have studied in their papers the inequality Cauchy–Buniakowski–Schwarz and Aczél's inequality (see [1–3]). We find the first inequality in numerical and complex analysis, Hilbert spaces theory, probability and statistics and it is defined as follows:

Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two sequences of real numbers, then

$$\left(\sum_{i=1}^{n} a_i^2 \right) \cdot \left(\sum_{i=1}^{n} b_i^2 \right) \geq \left(\sum_{i=1}^{n} a_i b_i \right)^2, \tag{1}$$

with equality if and only if the sequences $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are proportional.

Mărghidanu, Díaz-Barrero and Rădulescu in [4], proved a refinement of the inequality Cauchy–Buniakowski–Schwarz, given by the following: for arbitrary real sequences $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, $b_i \neq 0$, $i = \overline{1, n}$, we have:

$$\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i \right)^2 \geq \max_{i,j \in \{1, \ldots, n\}} \frac{(a_i b_j - a_j b_i)^2}{b_i^2 + b_j^2} \cdot \sum_{i=1}^{n} b_i^2, \tag{2}$$

for any $n \geq 2$, with equality if and only if the sequences $\mathbf{a}$ and $\mathbf{b}$ are proportional. In [5], Pop showed an improvement of inequality (2).

Two consequences can be obtained from above inequality, namely: for arbitrary sequence $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $n \geq 2$, we have:

$$n \sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i \right)^2 \geq n \cdot \max_{i,j \in \{1, \ldots, n\}} (a_i - a_j)^2, \tag{3}$$

and

$$n \sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i \right)^2 \geq \max_{i,j \in \{1, \ldots, n\}} \frac{(a_i - a_j)^2}{a_i^2 + a_j^2} \sum_{i=1}^{n} a_i^2, \tag{4}$$

with $a_i \neq 0, i = \overline{1,n}$.

In 1956, Aczél studied in [6], several methods in the theory of functional equations in one variable and showed the following inequality:

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be two sequences of positive real numbers such that:

$$A^2 - a_1^2 - \ldots - a_n^2 > 0 \text{ and } B^2 - b_1^2 - \ldots - b_n^2 > 0,$$

where A and B are positive real numbers. Then

$$\left(A^2 - a_1^2 - \ldots - a_n^2\right)\left(B^2 - b_1^2 - \ldots - b_n^2\right) \leq \left(AB - a_1 b_1 - \ldots - a_n b_n\right)^2, \tag{5}$$

with equality if and only if the sequences $\mathbf{a}$ and $\mathbf{b}$ are proportional, known as Aczél's inequality. This has many applications in the theory of functional equations in non-Euclidean geometry. Popoviciu [7], presented a generalized form of the inequality of Aczél:

Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n), \mathbf{b} = (b_1, b_2, \ldots, b_n)$ be two sequences of positive real numbers such that:

$$A^p - a_1^p - \ldots - a_n^p > 0 \text{ and } B^q - b_1^q - \ldots - b_n^q > 0,$$

where A and B are positive real numbers. Then

$$\left(A^p - a_1^p - \ldots - a_n^p\right)^{1/p}\left(B^q - b_1^q - \ldots - b_n^q\right)^{1/q} \leq AB - a_1 b_1 - \ldots - a_n b_n, \tag{6}$$

with equality if and only if the sequences $\mathbf{a}$ and $\mathbf{b}$ are proportional.

In the special case $p = q = 2$, we deduce the classical Aczél inequality.

In Section 2, we study a refinement of the Cauchy–Buniakowski–Schwarz inequality and a refinement of the Aczél inequality by the technique of the monotony of a sequence. In Section 3, there are presented new bounds for the statistical indicators related to variance, standard deviation and coefficient of variation.

2. Refinements of the Cauchy–Buniakowski–Schwarz Inequality and of the Aczél Inequality

Next, we present a refinement of Cauchy–Buniakowski–Schwarz inequality, thus:

Theorem 1. *For arbitrary real sequences* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$, *we have:*

$$\sqrt{\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2} - \sum_{i=1}^{n} a_i b_i \geq \frac{1}{2} \min\{A, B\} \sum_{i=1}^{n} (a_i - b_i)\left(\frac{a_i}{A} - \frac{b_i}{B}\right) \geq 0, \tag{7}$$

where $A = \sqrt{\sum_{i=1}^{n} a_i^2}$ *and* $B = \sqrt{\sum_{i=1}^{n} b_i^2}$ *with equality if and only if the sequences* $\mathbf{a}$ *and* $\mathbf{b}$ *are proportional.*

Proof. We have the following:

$$\sum_{i=1}^{n}(a_i - b_i)\left(\frac{a_i}{A} - \frac{b_i}{B}\right) = \sum_{i=1}^{n}\left(\frac{a_i^2}{A} - \frac{a_i b_i}{B} - \frac{a_i b_i}{A} + \frac{b_i^2}{B}\right)$$

$$= A + B - \frac{A+B}{AB}\sum_{i=1}^{n} a_i b_i = \frac{A+B}{AB}\left(AB - \sum_{i=1}^{n} a_i b_i\right) \geq 0.$$

Therefore, we obtain:

$$AB - \sum_{i=1}^{n} a_i b_i = \frac{AB}{A+B}\sum_{i=1}^{n}(a_i - b_i)\left(\frac{a_i}{A} - \frac{b_i}{B}\right). \tag{8}$$

But, we have the inequality $\frac{AB}{A+B} \geq \frac{1}{2}min\{A,B\}$, and using the equality (8), we deduce the inequality of the statement. □

Corollary 1. *For arbitrary sequence* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $n \geq 2$, *and* $b > 0$, *we have:*

$$\sqrt{n\sum_{i=1}^{n} a_i^2} - \sum_{i=1}^{n} a_i \geq \frac{1}{2A\sqrt{n}} \min\{A, b\sqrt{n}\} \sum_{i=1}^{n} (\frac{a_i}{b} - 1)(a_i\sqrt{n} - A) \geq 0, \tag{9}$$

where $A = \sqrt{\sum_{i=1}^{n} a_i^2}$.

Proof. If we take in inequality (7), $b_1 = b_2 = \ldots = b_n = b > 0$, then we obtain:

$$b\sqrt{n\sum_{i=1}^{n} a_i^2} - b\sum_{i=1}^{n} a_i \geq \frac{1}{2} \min\{A, b\sqrt{n}\} \sum_{i=1}^{n} (a_i - b)\left(\frac{a_i}{A} - \frac{b}{b\sqrt{n}}\right) \geq 0.$$

We divide with b above relation and we obtain:

$$\sqrt{n\sum_{i=1}^{n} a_i^2} - \sum_{i=1}^{n} a_i \geq \frac{1}{2} \min\{A, b\sqrt{n}\} \sum_{i=1}^{n} (\frac{a_i}{b} - 1)\left(\frac{a_i\sqrt{n} - A}{A\sqrt{n}}\right) \geq 0,$$

which implies the inequality of the statement. □

Theorem 2. *For arbitrary sequences of positive real numbers* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ *such that:*

$$A^2 - a_1^2 - \ldots - a_n^2 > 0 \text{ and } B^2 - b_1^2 - \ldots - b_n^2 > 0, n \geq 1,$$

where A and B are positive real numbers, we have:

$$\left(AB - \sum_{i=1}^{n} a_ib_i\right)^2 - \left(A^2 - \sum_{i=1}^{n} a_i^2\right)\left(B^2 - \sum_{i=1}^{n} b_i^2\right)$$

$$\geq \max_{i\in\{1,\ldots,n\}} \frac{(Ab_i - Ba_i)^2}{B^2 - b_i^2}\left(B^2 - \sum_{i=1}^{n} b_i^2\right) \geq 0, \tag{10}$$

with equality if and only if the sequences $\mathbf{a}$ *and* $\mathbf{b}$ *are proportional.*

Proof. We use the technique of the monotony, as in [8] and [9], to the sequence $G_n(\mathbf{a}, \mathbf{b})$, which is defined as follows:

$$G_n(\mathbf{a}, \mathbf{b}) = \frac{\left(AB - \sum_{i=1}^{n} a_ib_i\right)^2}{B^2 - \sum_{i=1}^{n} b_i^2} - A^2 + \sum_{i=1}^{n} a_i^2, n \geq 1.$$

for $k \leq n$, we have:

$$G_{k+1}(\mathbf{a}, \mathbf{b}) - G_k(\mathbf{a}, \mathbf{b}) = \frac{\left(AB - \sum_{i=1}^{k} a_ib_i - a_{k+1}b_{k+1}\right)^2}{B^2 - \sum_{i=1}^{k} b_i^2 - b_{k+1}^2} + a_{k+1}^2 - \frac{\left(AB - \sum_{i=1}^{k} a_ib_i\right)^2}{B^2 - \sum_{i=1}^{k} b_i^2}.$$

Using Bergström's inequality for two terms, we have:

$$\frac{\left(AB - \sum\limits_{i=1}^{k} a_i b_i - a_{k+1} b_{k+1}\right)^2}{B^2 - \sum\limits_{i=1}^{k} b_i^2 - b_{k+1}^2} + \frac{(a_{k+1} b_{k+1})^2}{b_{k+1}^2} \geq \frac{\left(AB - \sum\limits_{i=1}^{k} a_i b_i\right)^2}{B^2 - \sum\limits_{i=1}^{k} b_i^2}.$$

This means that $G_{k+1}(\mathbf{a}, \mathbf{b}) - G_k(\mathbf{a}, \mathbf{b}) \geq 0$, so the sequence $G_k(\mathbf{a}, \mathbf{b})$ is increasing. Therefore, we obtain $G_n(\mathbf{a}, \mathbf{b}) \geq G_{n-1}(\mathbf{a}, \mathbf{b}) \geq \ldots \geq G_2(\mathbf{a}, \mathbf{b}) \geq G_1(\mathbf{a}, \mathbf{b})$. But,

$$G_1(\mathbf{a}, \mathbf{b}) = \frac{(AB - a_1 b_1)^2}{B^2 - b_1^2} - A^2 + a_1^2 = \frac{(Ab_1 - Ba_1)^2}{B^2 - b_1^2},$$

and taking into account that we can rearrange the terms of the two sequences, we have the inequality:

$$G_n(\mathbf{a}, \mathbf{b}) \geq \max_{i \in \{1, \ldots, n\}} \frac{(Ab_i - Ba_i)^2}{B^2 - b_i^2}.$$

Multiplying by $B^2 - \sum\limits_{i=1}^{n} b_i^2 > 0$, we deduce the inequality of the statement. □

Remark 1. *In paper [10], F. Qi, P. Cerone, S. Dragomir and H.M. Srivastava have developed alternative proofs for the monotony.*

Theorem 3. *Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be two sequences of positive real numbers such that:*

$$A^p - a_1^p - \ldots - a_n^p > 0 \text{ and } B^q - b_1^q - \ldots - b_n^q > 0, n \geq 1,$$

where A and B are positive real numbers. Then, we have:

$$AB - a_1 b_1 - \ldots - a_n b_n - \left(A^p - a_1^p - \ldots - a_n^p\right)^{1/p} \left(B^q - b_1^q - \ldots - b_n^q\right)^{1/q}$$
$$\geq \max_{i \in \{1, \ldots, n\}} \left\{AB - a_i b_i - \left(A^p - a_i^p\right)^{1/p} \left(B^q - b_i^q\right)^{1/q}\right\} \geq 0, \tag{11}$$

with equality if and only if the sequences $\mathbf{a}$ and $\mathbf{b}$ are proportional.

Proof. We consider the sequence:

$$H_n(\mathbf{a}, \mathbf{b}) = AB - a_1 b_1 - \ldots - a_n b_n - \left(A^p - a_1^p - \ldots - a_n^p\right)^{1/p} \left(B^q - b_1^q - \ldots - b_n^q\right)^{1/q}.$$

We study the monotony of the sequence $H_k(\mathbf{a}, \mathbf{b}), k \leq n$. Since

$$H_{k+1}(\mathbf{a}, \mathbf{b}) - H_k(\mathbf{a}, \mathbf{b}) = \left(A^p - a_1^p - \ldots - a_k^p\right)^{1/p} \left(B^q - b_1^q - \ldots - b_k^q\right)^{1/q}$$

$$-a_{k+1} b_{k+1} - \left(A^p - a_1^p - \ldots - a_{k+1}^p\right)^{1/p} \left(B^q - b_1^q - \ldots - b_{k+1}^q\right)^{1/q}.$$

which involves,

$$H_{k+1}(\mathbf{a}, \mathbf{b}) - H_k(\mathbf{a}, \mathbf{b}) = MN - a_{k+1} b_{k+1} - \left(M^p - a_{k+1}^p\right)^{1/p} \left(N^q - b_{k+1}^q\right)^{1/q},$$

where we have $M^p = A^p - a_1^p - \ldots - a_k^p$ and $N^q = B^q - b_1^q - \ldots - b_k^q$. But using inequality (6) for two terms, we deduce:

$$\left(M^p - a_{k+1}^p\right)^{1/p} \left(N^q - b_{k+1}^q\right)^{1/q} \leq MN - a_{k+1}b_{k+1},$$

which involves the inequality $H_{n+1}(\mathbf{a}, \mathbf{b}) - H_n(\mathbf{a}, \mathbf{b}) \geq 0$. Therefore the sequence H_n is increasing, so, we obtain:

$$H_n(\mathbf{a}, \mathbf{b}) \geq H_{n-1}(\mathbf{a}, \mathbf{b}) \geq \ldots \geq H_2(\mathbf{a}, \mathbf{b}) \geq H_1(\mathbf{a}, \mathbf{b})$$

$$= AB - a_1 b_1 - \left(A^p - a_1^p\right)^{1/p} \left(B^q - b_1^q\right)^{1/q}.$$

Consequently, we deduce the statement. $\square$

Remark 2. *In the particular case $p = q = 2$, we deduce another refinement of the classical Aczél inequality, given by:*

$$AB - a_1 b_1 - \ldots - a_n b_n - \left(A^2 - a_1^2 - \ldots - a_n^2\right)^{1/2} \left(B^2 - b_1^2 - \ldots - b_n^2\right)^{1/2}$$
$$\geq \max_{i \in \{1,\ldots,n\}} \{AB - a_i b_i - \left(A^2 - a_i^2\right)^{1/2} \left(B^2 - b_i^2\right)^{1/2}\} \geq 0, \tag{12}$$

with equality if and only if the sequences $\mathbf{a}$ and $\mathbf{b}$ are proportional.

Extending Popoviciu's result to three sequences of positive real numbers, we obtain the following refinement:

Theorem 4. *Let $p, q, r > 1$ be such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ three sequences of positive real numbers such that $A^p - a_1^p - \ldots - a_n^p > 0$, $B^q - b_1^q - \ldots - b_n^q > 0$, and $C^r - c_1^r - \ldots - c_n^r > 0$, $n \geq 1$, where A, B and C are positive real numbers. Then, we have:*

$$ABC - a_1 b_1 c_1 - \ldots - a_n b_n c_n - \left(A^p - a_1^p - \ldots - a_n^p\right)^{1/p} \left(B^q - b_1^q - \ldots - b_n^q\right)^{1/q} \left(C^r - c_1^r - \ldots - c_n^r\right)^{1/r}$$
$$\geq \max_{i \in \{1,\ldots,n\}} \{ABC - a_i b_i c_i - \left((AB)^{\frac{r}{r-1}} - (a_i b_i)^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}} \left(C^r - c_i^r\right)^{1/r}\} \geq 0. \tag{13}$$

Proof. From hypothesis, we have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, which becomes $\frac{1}{p} + \frac{1}{q} = \frac{r-1}{r}$, so, we deduce $\frac{1}{\frac{p(r-1)}{r}} + \frac{1}{\frac{q(r-1)}{r}} = 1$. Using Popoviciu's inequality for two sequences, we have:

$$\left[\left(A^p - a_1^p - \ldots - a_n^p\right)^{1/p} \left(B^q - b_1^q - \ldots - b_n^q\right)^{1/q}\right]^{r/(r-1)} =$$

$$\left(A^p - a_1^p - \ldots - a_n^p\right)^{r/p(r-1)} \left(B^q - b_1^q - \ldots - b_n^q\right)^{r/q(r-1)} =$$

$$\left[\left(\left(A^{r/(r-1)}\right)^{p(r-1)/r} - \left(a_1^{r/(r-1)}\right)^{p(r-1)/r} - \ldots - \left(a_n^{r/(r-1)}\right)^{p(r-1)/r}\right)^{r/p(r-1)}\right]^{r/p(r-1)}$$

$$\left[\left(\left(B^{r/(r-1)}\right)^{q(r-1)/r} - \left(b_1^{r/(r-1)}\right)^{q(r-1)/r} - \ldots - \left(b_n^{r/(r-1)}\right)^{q(r-1)/r}\right)^{r/q(r-1)}\right]^{r/q(r-1)}$$

$$\leq (AB)^{r/(r-1)} - (a_1 b_1)^{r/(r-1)} - \ldots - (a_n b_n)^{r/(r-1)},$$

which is equivalent to:

$$\left(A^p - a_1^p - \ldots - a_n^p\right)^{1/p} \left(B^q - b_1^q - \ldots - b_n^q\right)^{1/q}$$
$$\leq \left[(AB)^{r/(r-1)} - (a_1 b_1)^{r/(r-1)} - \ldots - (a_n b_n)^{r/(r-1)}\right]^{(r-1)/r}. \tag{14}$$

But, using above relation and Theorem 3, we have:

$$\left(A^p - a_1^p - \ldots - a_n^p\right)^{1/p} \left(B^q - b_1^q - \ldots - b_n^q\right)^{1/q} \left(C^r - c_1^r - \ldots - c_n^r\right)^{1/r}$$

$$\leq \left[(AB)^{r/(r-1)} - (a_1 b_1)^{r/(r-1)} - \ldots - (a_n b_n)^{r/(r-1)}\right]^{1-1/r} \left(C^r - c_1^r - \ldots - c_n^r\right)^{1/r}$$

$$\leq ABC - a_1 b_1 c_1 - \ldots - a_n b_n c_n - \max_{i \in \{1,\ldots,n\}} \left\{ ABC - a_i b_i c_i - \left((AB)^{\frac{r}{r-1}} - (a_i b_i)^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}} \left(C^r - c_i^r\right)^{1/r} \right\}.$$

Consequently, we deduce the inequality of the statement. □

3. Applications to Several Statistical Indicators

Analysis of social-economic phenomena with the statistical indicators help us to characterize the evolution of these phenomena, (see [11]). Therefore, below we present some of the statistical indicators that characterize a data series.

The mean of individual values of a data series is the expression level synthesis into a single representative of everything is essential, typical and objective in appearance, manifestation and its development.

The arithmetic mean $(\overline{X})$ is used when the phenomenon under investigation register changes approximately constant in arithmetic progression.

Let $X = \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{1 \leq i \leq n}$ be a discrete random variable with probabilities $P(X = x_i) = p_i = \dfrac{1}{n}$ for any

$i = \overline{1, n}$, then expression of simple arithmetic mean is: $\overline{X} = \dfrac{x_1 + x_2 + \ldots + x_n}{n} = \dfrac{\sum\limits_{i=1}^{n} x_i}{n}$.

The geometric mean $(\overline{X}_g)$ is that value which shows that if you replace each individual value, their product would not change and we have the formula: $\overline{X}_g = \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n}$.

In statistics, by general notion of scattering (variance or dispersion) refer to the individual values of measurable deviations from the central value (typical).

Among the main indicators characterizing the scattering one can include: dispersion, standard deviation and coefficient of variation.

The dispersion (or variance) $(Var(X))$ represents the simple arithmetic or weighted mean of individual deviations squared from their central tendency. is the parameter that characterizes the normal distribution and we have the formula: $Var(X) = \dfrac{\left(x_1 - \overline{X}\right)^2 + \left(x_2 - \overline{X}\right)^2 + \ldots + \left(x_n - \overline{X}\right)^2}{n}$.

The standard deviation $(\sigma_{\overline{X}})$ has a similar role with average linear deviation, but keeping the dispersion characteristics; statistics used this indicator which is calculated as mean of individual deviations squared from their central tendency, and the interval $\left(\overline{X} - \sigma_{\overline{X}}; \overline{X} + \sigma_{\overline{X}}\right)$ is the medium interval of variation and we have the following formula: $\sigma_{\overline{X}} = \sqrt{Var(X)}$.

The coefficient of variation (C_V) is a relative measure of the scattering, which describes the percentage standard deviation from the arithmetic mean, $C_V = \dfrac{\sigma_{\overline{X}}}{\overline{X}}$. The covariance is a measure of how much two random variables X and Y change together at the same time and is defined as $Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$, and is equivalent to the form $Cov(X, Y) = E[XY] - E[X][Y]$. we find the inequality of Cauchy–Schwarz for discrete random variables given by: $|Cov(X, Y)| \leq \sqrt{Var(X)Var(Y)}$.

In 1935, Popoviciu, shows the following inequality:

$$\frac{\left(x_1 - \overline{X}\right)^2 + \left(x_2 - \overline{X}\right)^2 + \ldots + \left(x_n - \overline{X}\right)^2}{n} \leq \frac{1}{4}(Q - q)^2, \tag{15}$$

where $Q = \max\{x_1, x_2, \ldots, x_n\}$ and $q = \min\{x_1, x_2, \ldots, x_n\}$. This inequality suggests an upper bound for indicators: dispersion, standard deviation and coefficient of variation, thus:

$$Var(X) \leq \frac{1}{4}(Q - q)^2, \sigma_{\overline{X}} \leq \frac{1}{2}(Q - q) \text{ and } C_V \leq \frac{Q - q}{2\overline{X}.} \tag{16}$$

In 2000, Bhatia and Davis, show in [12] that the dispersion is less or equal than the product differences between maximum and mean value, respectively mean and minimum value, thus,

$$Var(X) \leq (Q - \overline{X})(\overline{X} - q), \tag{17}$$

which is equivalent with the inequality:

$$\frac{(x_1 - \overline{X})^2 + (x_2 - \overline{X})^2 + \ldots + (x_n - \overline{X})^2}{n} \leq (Q - \overline{X})(\overline{X} - q). \tag{18}$$

Because the term $(Q - \overline{X})(\overline{X} - q)$ is less or equal than the term $\frac{1}{4}(Q - q)^2$, we obtain that the inequality (18) is better than inequality (15). Thus, we obtain upper bounds better than in the relation (16):

$$\begin{aligned} Var(X) &\leq (Q - \overline{X})(\overline{X} - q), \\ \sigma_{\overline{X}} &\leq \sqrt{(Q - \overline{X})(\overline{X} - q)}, \\ C_V &\leq \frac{\sqrt{(Q - \overline{X})(\overline{X} - q)}}{\overline{X}}. \end{aligned} \tag{19}$$

It has been shown in [13] by Mercer that for a sample of real numbers,

$$Var(X) \leq 2Q(\overline{X} - \overline{X}_h), \tag{20}$$

where $\overline{X}_h$ is the harmonic mean of the sample, i.e., $\overline{X}_h = \dfrac{n}{\sum_{i=1}^n \dfrac{1}{x_i}}.$

Next, we will present several improvements of the above inequalities related to variance.

Theorem 5. *For a sample of real number X there is the following inequality:*

$$2q(\overline{X} - \overline{X}_g) \leq Var(X) \leq 2Q(\overline{X} - \overline{X}_g). \tag{21}$$

Proof. In the paper [14], Cartwright and Field, proved the following inequality:

$$\begin{aligned} \frac{1}{2Q} \sum_{i=1}^n \alpha_i \left(x_i - \sum_{i=1}^n \alpha_i x_i\right)^2 &\leq \sum_{i=1}^n \alpha_i x_i - \prod_{i=1}^n x_i^{\alpha_i} \\ &\leq \frac{1}{2q} \sum_{i=1}^n \alpha_i \left(x_i - \sum_{i=1}^n \alpha_i x_i\right)^2, \end{aligned} \tag{22}$$

where $\alpha_i > 0, \forall i = 1, \ldots, n$ and $\sum_{i=1}^n \alpha_i = 1$.

For $\alpha_i = \dfrac{1}{n}$, for all $i = 1, \ldots, n$, in inequality (22), we will obtain:

$$\frac{1}{2Q} \cdot \frac{(x_1 - \overline{X})^2 + (x_2 - \overline{X})^2 + \ldots + (x_n - \overline{X})^2}{n} \leq$$

$$\overline{X} - \overline{X}_g \leq \frac{1}{2q} \cdot \frac{(x_1 - \overline{X})^2 + (x_2 - \overline{X})^2 + \ldots + (x_n - \overline{X})^2}{n},$$

thus, $2q(\overline{X} - \overline{X}_g) \leq Var(X) \leq 2Q(\overline{X} - \overline{X}_g)$. $\square$

Remark 3. *It is easy to see that inequality (21) is a refinement of inequality (20). Thus, we have situations when a bound is better then the other. This inequality provide another upper bound for the dispersion $Var(X)$, which makes us to compare the terms $2Q\left(\overline{X} - \overline{X}_g\right)$ and $(Q - \overline{X})\left(\overline{X} - q\right)$, to see which of them is less. We consider, the real number $a > 1$ and choose the values $x_1 = a - 1$, $x_2 = a$, $x_3 = a + 1$, to compare the bounds $M_1 = (Q - \overline{X})\left(\overline{X} - q\right)$ and $M_2 = 2Q\left(\overline{X} - \overline{X}_g\right)$. For respective values we have $q = a - 1$, $x = a$, $Q = a + 1$, which means that $M_1 = 1$ and $M_2 = 2(a + 1)\left(a - \sqrt[3]{a^3 - a}\right)$. We consider the function $f : (0, +\infty) \to R$, $f(x) = 2(x + 1)\left(x - \sqrt[3]{x^3 - x}\right) - 1$.*

Studying this function, we found that exists $c = 2,428\ldots$, where $f'(c) = 0$, thus for $x \in (1, c)$, we have $M_1 < M_2$ and for $x \in (c, \infty)$, we have $M_1 > M_2$. When the arithmetical mean of three consecutive numbers is greater than c, we have $M_1 > M_2$, and when the arithmetical mean of three consecutive numbers is between 1 and c, then $M_1 < M_2$. Thus, we have situations when a bound is better then the other.

Combining the above inequalities and taking into account Remark 3, we found other bounds for dispersion $Var(X)$, standard deviation and coefficient of variation:

$$2q\left(\overline{X} - \overline{X}_g\right) \le Var(X) \le \min\{2Q\left(\overline{X} - \overline{X}_g\right), (Q - \overline{X})\left(\overline{X} - q\right)\}, \tag{23}$$

$$\sqrt{2q\left(\overline{X} - \overline{X}_g\right)} \le \sigma_{\overline{X}} \le \sqrt{\min\{2Q\left(\overline{X} - \overline{X}_g\right), (Q - \overline{X})\left(\overline{X} - q\right)\}}, \tag{24}$$

and

$$\frac{\sqrt{2q\left(\overline{X} - \overline{X}_g\right)}}{\overline{X}} \le C_V \le \min\{2Q\left(\overline{X} - \overline{X}_g\right), (Q - \overline{X})\left(\overline{X} - q\right)\}. \tag{25}$$

Now, we want to find an upper bound, better than the Bhatia and Davis, for the above indicators. For the beginning, we establish:

Theorem 6. *For a sample of real number X there is the following equality:*

$$(Q - \overline{X})\left(\overline{X} - q\right) - Var(X) = \frac{1}{n}\sum_{i=1}^{n}(Q - x_i)(x_i - q). \tag{26}$$

Proof. We observe that:

$$\frac{1}{n}\sum_{i=1}^{n}(Q - x_i)(x_i - q) = \frac{1}{n}\left[(Q + q)\sum_{i=1}^{n}x_i - \sum_{i=1}^{n}x_i^2 - Qq\right]$$
$$= (Q + q)\overline{X} - \frac{1}{n}\sum_{i=1}^{n}x_i^2 - Qq$$
$$= (Q + q)\overline{X} - Var(X) - \overline{X}^2 - Qq = (Q - \overline{X})\left(\overline{X} - q\right) - Var(X).$$

$\square$

Theorem 7. *For a sample of real number X if $Q = \max\{x_1, x_2, \ldots, x_n\}$, $q = \min\{x_1, x_2, \ldots, x_n\}$, $Q' = \max\{x_i | x_i \neq Q\}$ and $q' = \min\{x_i | x_i \neq q\}$, then there is the following inequality:*

$$Var(X) \le (Q - \overline{X})\left(\overline{X} - q\right) - \max\left\{(q' - q)(Q - \overline{X}), (Q - Q')\left(\overline{X} - q\right)\right\}. \tag{27}$$

Proof. for $Q = \max\{x_1, x_2, \ldots, x_n\}$ and $q = \min\{x_1, x_2, \ldots, x_n\}$, we have:$Q \ge x_i$, and $q \le x_i$, thus,$(Q - x_i)(x_i - q) \ge 0$.

Consequently, the proof of inequality (17) is obvious.

For $Q' = \max\{x_i | x_i \neq Q\}$ and $q' = \min\{x_i | x_i \neq q\}$, if we replace the term $Q - x_i$ with $Q - Q'$ in relation (26), we obtain:

$$(Q - \overline{X})\left(\overline{X} - q\right) - Var(X) \ge \frac{1}{n}\sum_{i=1}^{n}(Q - Q')(x_i - q) = (Q - Q')\left(\overline{X} - q\right),$$

and if we replace $x_i - q$ with $q' - q$, we have:

$$(Q - \overline{X})(\overline{X} - q) - Var(X) \geq (q' - q)(Q - \overline{X}),$$

which means that inequality (17) has been improved by the inequality of the statement. □

Remark 4. *We obtain the next bounds for variance, standard deviation and coefficient of variation:*

$$
\begin{aligned}
2q\left(\overline{X} - \overline{X}_g\right) \;&\leq\; Var(X) \\
&\leq (Q - \overline{X})(\overline{X} - q) - \\
&\quad - \max\left\{(q' - q)(Q - \overline{X}),\ (Q - Q')(\overline{X} - q)\right\},
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
\sqrt{2q\left(\overline{X} - \overline{X}_g\right)} \;&\leq\; \sigma_{\overline{X}} \\
&\leq \sqrt{(Q - \overline{X})(\overline{X} - q) - \max\left\{\begin{array}{c}(q' - q)(Q - \overline{X}), \\ (Q - Q')(\overline{X} - q)\end{array}\right\}},
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
\frac{\sqrt{2q\left(\overline{X} - \overline{X}_g\right)}}{\overline{X}} \;&\leq\; C_V \\
&\leq \frac{\sqrt{(Q - \overline{X})(\overline{X} - q) - \max\left\{(q' - q)(Q - \overline{X}),\ (Q - Q')(\overline{X} - q)\right\}}}{\overline{X}}.
\end{aligned}
\tag{30}
$$

For a discrete random variable $X = \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{1 \leq i \leq n}$, we take $Q \geq \max\{x_1, x_2, \ldots, x_n\}$ and $q \leq \min\{x_1, x_2, \ldots, x_n\}$. we have $x_i \neq x_j, \forall i \neq j, i, j = \overline{1, n}$.

We consider the following discrete random variables: $Q - X = \begin{pmatrix} Q - x_i \\ \frac{1}{n} \end{pmatrix}_{1 \leq i \leq n}$, and $X - q = \begin{pmatrix} x_i - q \\ \frac{1}{n} \end{pmatrix}_{1 \leq i \leq n}$.

It is easy to see that $Var(Q - X) = Var\,X$ and $Var(X - q) = Var\,X$, if $a_i = Q - x_i$ and $b_i = x_i - q$, then we obtain:

$$\sum_{i=1}^{n} a_i b_i = n\left((Q - \overline{X})(\overline{X} - q) - Var(X)\right).$$

Next, we calculate $\sum_{i=1}^{n} a_i^2$ and $\sum_{i=1}^{n} b_i^2$, for a_i and b_i given above. Thus, we deduce:

$$
\begin{aligned}
\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} (Q - x_i)^2 &= n\left(Var(Q - X) + (Q - \overline{X})^2\right) \\
&= n\left(Var(X) + (Q - \overline{X})^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i=1}^{n} b_i^2 = \sum_{i=1}^{n} (x_i - q)^2 &= n\left(Var(X - q) + (\overline{X} - q)^2\right) \\
&= n\left(Var(X) + (\overline{X} - q)^2\right).
\end{aligned}
$$

Therefore, we have:

$$
\begin{aligned}
\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \\
= n^2\left((Var(X) + (Q - \overline{X})^2)(Var(X) + (\overline{X} - q)^2)\right) \\
- ((Q - \overline{X})(\overline{X} - q) - Var(X))^2) \\
= n^2 Var(X) \cdot (Q - q)^2.
\end{aligned}
$$

However, the second term of inequality (2) becomes

$$\max_{i,j\in\{1,\ldots,n\}} \frac{(a_ib_j - a_jb_i)^2}{b_i^2 + b_j^2} \sum_{i=1}^{n} b_i^2$$

$$= n \cdot (Q - q)^2 \max_{i,j\in\{1,\ldots,n\}} \frac{(x_i - x_j)^2}{(x_i - q)^2 + (x_j - q)^2} \cdot \left(Var(X) + (\overline{X} - q)^2\right), n \geq 2.$$

Thus, we obtain:

$$n \cdot Var(X) \geq \left(Var(X) + (\overline{X} - q)^2\right) \max_{i,j\in\{1,\ldots,n\}} \frac{(x_i - x_j)^2}{(x_i - q)^2 + (x_j - q)^2}.$$

For $n \geq 2$, taking into account that $Var(X) \geq 0$, we proved that inequality:

$$Var(X) \geq \frac{(\overline{X} - q)^2}{n} \max_{i,j\in\{1,\ldots,n\}} \frac{(x_i - x_j)^2}{(x_i - q)^2 + (x_j - q)^2}. \tag{31}$$

Similarly, we deduce:

$$Var(X) \geq \frac{(Q - \overline{X})^2}{n} \max_{i,j\in\{1,\ldots,n\}} \frac{(x_i - x_j)^2}{(Q - x_i)^2 + (Q - x_j)^2}. \tag{32}$$

We study, two cases:

(I) for $A = B = \sqrt{n(Var(X) + (Q - q)^2)}$, we have:

$$AB - \sum_{i=1}^{n} a_ib_i = n(2Var(X) + (Q - q)^2 - (Q - \overline{X})(\overline{X} - q)) > 0,$$

$$A^2 - \sum_{i=1}^{n} a_i^2 = n(\overline{X} - q)(2Q - q - \overline{X}) > 0, \text{ for every } n \geq 2,$$

$$B^2 - \sum_{i=1}^{n} b_i^2 = n(Q - \overline{X})(Q + \overline{X} - 2q) > 0, \text{ for every } n \geq 2.$$

Therefore, applying the Aczél inequality, we proved:

$$(2Var(X) + (Q - q)^2 - (Q - \overline{X})(\overline{X} - q))^2$$
$$\geq (Q - \overline{X})(\overline{X} - q)(Q + \overline{X} - 2q)(2Q - q - \overline{X}),$$

which means that:

$$Var(X) \geq \frac{1}{2}\left(\sqrt{(Q - \overline{X})(\overline{X} - q)(Q + \overline{X} - 2q)(2Q - q - \overline{X})} + (Q - \overline{X})(\overline{X} - q) - (Q - q)^2\right). \tag{33}$$

But, using the improvement of Aczél inequality, from inequality (12), we obtain:

$$Var(X) \geq \frac{1}{2}\left(\sqrt{(Q - \overline{X})(\overline{X} - q)(Q + \overline{X} - 2q)(2Q - q - \overline{X})} + (Q - \overline{X})(\overline{X} - q) - (Q - q)^2\right) + P, \tag{34}$$

where

$$P = \frac{1}{2n} \max_{i\in\{1,\ldots,n\}} \left\{AB - a_ib_i - \left(A^2 - a_i^2\right)^{1/2} \left(B^2 - b_i^2\right)^{1/2}\right\},$$

and $A = B = \sqrt{n(Var(X) + (Q - q)^2)}$.

(II) For

$$A = \sqrt{n((Q - \overline{X})(\overline{X} - q) + (Q - \overline{X})^2)}$$
$$= \sqrt{n(Q - \overline{X})(Q - q)},$$

and

$$B = \sqrt{n((Q - \overline{X})(\overline{X} - q) + (\overline{X} - q)^2)}$$
$$= \sqrt{n(\overline{X} - q)(Q - q)},$$

we deduce:

$$AB - \sum_{i=1}^{n} a_i b_i = n(Q - q)\sqrt{(Q - \overline{X})(\overline{X} - q)}$$
$$- (Q - \overline{X})(\overline{X} - q) + Var(X) > 0.$$

$$A^2 - \sum_{i=1}^{n} a_i^2 = n\left((Q - \overline{X})(Q - q) - Var(X) - (Q - \overline{X})^2\right)$$
$$= n\left((Q - \overline{X})(\overline{X} - q) - Var(X)\right) > 0$$

and

$$B^2 - \sum_{i=1}^{n} b_i^2 = n\left((\overline{X} - q)(Q - q) - Var(X) - (\overline{X} - q)^2\right)$$
$$= n\left((Q - \overline{X})(\overline{X} - q) - Var(X)\right) > 0.$$

Consequently, if we use the Aczél inequality, we find:

$$\left((Q - q)\sqrt{(Q - \overline{X})(\overline{X} - q)} - (Q - \overline{X})(\overline{X} - q) + Var(X)\right)^2$$
$$\geq \left((Q - \overline{X})(\overline{X} - q) - Var(X)\right)^2,$$

$$Var(X) \geq (Q - \overline{X})(\overline{X} - q) - \frac{Q - q}{2}\sqrt{(Q - \overline{X})(\overline{X} - q)}. \tag{35}$$

But, using the Aczél inequality refinement, from inequality (12), we find:

$$Var(X) \geq (Q - \overline{X})(\overline{X} - q) - \frac{Q - q}{2}\sqrt{(Q - \overline{X})(\overline{X} - q)} + P', \tag{36}$$

where

$$P' = \frac{1}{2n} \max_{i \in \{1,...,n\}} \left\{ AB - a_i b_i - \left(A^2 - a_i^2\right)^{1/2} \left(B^2 - b_i^2\right)^{1/2} \right\},$$

and $A = \sqrt{n(Q - \overline{X})(Q - q)}$ and $B = \sqrt{n(\overline{X} - q)(Q - q)}$.

4. Conclusions

Based on the results obtained in this paper regarding the new refinements for the main indicators characterizing the scattering, we will try in the future to find other applications of Aczél inequality respectively Popoviciu's inequality to other statistical indicators. For example, we will extend the inequality from Theorem 4 to m sequences of positive real numbers. In the future we will look for other applications demonstrated in the article and in other fields. we will also study what variants from these results can be extended in the inner product spaces.

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