mathematics

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Fixed Point Theorems for Generalized ($\alpha\beta$-ψ)-Contractions in $\mathcal{F}$-Metric Spaces with Applications

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Received: 26 February 2020; Accepted: 8 April 2020; Published: 14 April 2020

Abstract: The purpose of this paper is to define generalized ($\alpha\beta$-ψ)-contraction in the context of $\mathcal{F}$-metric space and obtain some new fixed point results. As applications, we solve a nonlinear neutral differential equation with an unbounded delay $\vartheta'(\iota) = -\rho_1(\iota)\vartheta(\iota) + \rho_2(\iota)\mathcal{L}(\vartheta(\iota - \varsigma(\iota))) + \rho_3(\iota)\vartheta'(\iota - \varsigma(\iota))$, where $\rho_1(\iota), \rho_2(\iota)$ are continuous, $\rho_3(\iota)$ is continuously differentiable and $\varsigma(\iota) > 0$, for all $\iota \in \mathbb{R}$ and is twice continuously differentiable.

Keywords: nonlinear neutral differential equation; fixed point; generalized ($\alpha\beta$-ψ)-contraction; $\mathcal{F}$-metric spaces

MSC: 46S40; 47H10; 54H25

1. Introduction and Preliminaries

In 1906, M. Frechet introduced the notion of metric space [1], which is one pillar of not only mathematics but also physical sciences. Because of its importance and simplicity, this notion has been extended, improved and generalized in many different ways.

In 2018, Jleli et al. [2] introduced a fascinating generalization of metric space as follows:

Let $f \in \mathcal{F}$ and $f : (0, +\infty) \to \mathbb{R}$ be such that:

($\mathcal{F}_1$) $0 < \vartheta < \theta \Longrightarrow f(\vartheta) \le f(\theta)$,
($\mathcal{F}_2$) for $\{\vartheta_n\} \subseteq R^+$, $\lim_{n\to\infty} \vartheta_n = 0 \Longleftrightarrow \lim_{n\to\infty} f(\vartheta_n) = -\infty$.

Definition 1 ([2])**.** *Let $\mathcal{M}$ be a nonempty set, and let $d_\mathcal{F} : \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ be a given mapping. Suppose that there exists $(f, \mathfrak{h}) \in \mathcal{F} \times [0, +\infty)$ such that*

(D_1) $(\vartheta, \theta) \in \mathcal{M} \times \mathcal{M}$, $d_\mathcal{F}(\vartheta, \theta) = 0 \Longleftrightarrow \vartheta = \theta$.
(D_2) $d_\mathcal{F}(\vartheta, \theta) = d_\mathcal{F}(\theta, \vartheta)$, for all $(\vartheta, \theta) \in \mathcal{M} \times \mathcal{M}$,
(D_3) for every $(\vartheta, \theta) \in \mathcal{M} \times \mathcal{M}$, $N \in \mathbb{N}$, $N \ge 2$, and $(\vartheta_i)_{i=1}^N \subset \mathcal{M}$, with $(\vartheta_1, \vartheta_N) = (\vartheta, \theta)$, we get

$$d_\mathcal{F}(\vartheta, \theta) > 0 \text{ implies } f(d_\mathcal{F}(\vartheta, \theta)) \le f(\sum_{i=1}^{N-1} d_\mathcal{F}(\vartheta_i, \vartheta_{i+1})) + \mathfrak{h}.$$

Then $d_\mathcal{F}$ is said to be an $\mathcal{F}$-metric on $\mathcal{M}$, and the pair $(\mathcal{M}, d_\mathcal{F})$ is said to be an $\mathcal{F}$-metric space.

Remark 1. *They showed that any metric space is an $\mathcal{F}$-metric space but the converse is not true in general, which confirms that this concept is more general than the standard metric concept.*

Example 1 ([2]). *The set of real numbers $\mathbb{R}$ is an $\mathcal{F}$-metric Space if we define $d_{\mathcal{F}}$ by*

$$d_{\mathcal{F}}(\vartheta,\theta) = \begin{cases} (\vartheta-\theta)^2 \text{ if } (\vartheta,\theta) \in [0,3]\times[0,3] \\ |\vartheta-\theta| \text{ if } (\vartheta,\theta) \notin [0,3]\times[0,3] \end{cases}$$

with $f(\iota) = \ln(\iota)$ and $\mathfrak{h} = \ln(3)$.

Definition 2 ([2]). *Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$-metric space.*

(i) Let $\{\vartheta_n\}$ be a sequence in $\mathcal{M}$. We say that $\{\vartheta_n\}$ is $\mathcal{F}$-convergent to $\vartheta \in \mathcal{M}$ if $\{\vartheta_n\}$ is convergent to ϑ with respect to the $\mathcal{F}$-metric $d_{\mathcal{F}}$.

(ii) A sequence $\{\vartheta_n\}$ is $\mathcal{F}$-Cauchy, if

$$\lim_{n,m\to\infty} d_{\mathcal{F}}(\vartheta_n,\vartheta_m) = 0.$$

(iii) We say that $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete, if every $\mathcal{F}$-Cauchy sequence in $\mathcal{M}$ is $\mathcal{F}$-convergent to a certain element in $\mathcal{M}$.

Theorem 1 ([2]). *Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$-metric space and $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ be a given mapping. Suppose that the following conditions are satisfied:*

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii) there exists $k \in (0,1)$ such that

$$d_{\mathcal{F}}(\mathcal{H}(\vartheta),\mathcal{H}(\theta)) \leq k d_{\mathcal{F}}(\vartheta,\theta).$$

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\mathcal{H}\vartheta^* = \vartheta^*$ which is unique. Furthermore, for $\vartheta_0 \in \mathcal{M}$, $\{\vartheta_n\} \subset \mathcal{M}$ given by

$$\vartheta_{n+1} = \mathcal{H}(\vartheta_n),$$

for all $n \in \mathbb{N}$, is $\mathcal{F}$-convergent to ϑ^*.

Afterwards, Hussain et al. [3] considered the notion of α-ψ-contraction in the setting of $\mathcal{F}$-metric spaces and proved the following fixed point theorem.

Theorem 2 ([3]). *Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$-metric space and $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ be β-admissible mapping. Suppose that the following conditions are satisfied: (i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,*

(ii) there exists $\beta : \mathcal{M}\times\mathcal{M} \to [0,+\infty)$ and $\psi \in \Psi$ such that

$$\beta(\vartheta,\theta) d_{\mathcal{F}}(\mathcal{H}(\vartheta),\mathcal{H}(\theta)) \leq \psi(M(\vartheta,\theta)),$$

where

$$M(\vartheta,\theta) = \max\{d_{\mathcal{F}}(\vartheta,\theta), d_{\mathcal{F}}(\vartheta,\mathcal{H}\vartheta), d_{\mathcal{F}}(\theta,\mathcal{H}\theta)\},$$

for all $\vartheta,\theta \in \mathcal{M}$,

(iii) there exists $\vartheta_0 \in \mathcal{M}$ such that $\beta(\vartheta_0,\mathcal{H}(\vartheta_0))$. Then there exists unique $\vartheta^* \in \mathcal{M}$ such that $\mathcal{H}\vartheta^* = \vartheta^*$.

For more details in this direction, we refer the readers to References [4–10].

On the other hand, Samet et al. [11] introduced the concepts of α-ψ-contractive and α-admissible mappings and established various fixed point theorems for such mappings in complete metric spaces.

Denote with Ψ the family of nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(\vartheta) < +\infty$ for all $\vartheta > 0$, where ψ^n is the n-th iterate of ψ.

The following lemma is well known.

Lemma 1. *If $\psi \in \Psi$, then the following hold:*

(i) $(\psi^n(\vartheta))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$ for all $\vartheta \in (0, +\infty)$,

(ii) $\psi(\vartheta) < \vartheta$ for all $\vartheta > 0$,

(iii) $\psi(\vartheta) = 0$ iff $\vartheta = 0$.

Samet et al. [11] defined the notion of α-admissible mappings as follows:

Definition 3 ([11])**.** *Let $\mathcal{H}$ be a self-mapping on $\mathcal{M}$ and $\alpha : \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ be a function. We say that $\mathcal{H}$ is an α-admissible mapping if*

$$\alpha(\vartheta, \theta) \geq 1 \Longrightarrow \alpha(\mathcal{H}\vartheta, \mathcal{H}\theta) \geq 1,$$

for all $\vartheta, \theta \in \mathcal{M}$.

Hussain et al. [12] extended the above notion of α-admissible mapping as follows.

Definition 4 ([12])**.** *Let $\mathcal{H}$ be a self-mapping on $\mathcal{M}$ and $\alpha, \beta : \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ be two functions. We say that $\mathcal{H}$ is an α-admissible mapping with respect to β if*

$$\alpha(\vartheta, \theta) \geq \beta(\vartheta, \theta) \Longrightarrow \alpha(\mathcal{H}\vartheta, \mathcal{H}\theta) \geq \beta(\mathcal{H}\vartheta, \mathcal{H}\theta),$$

for all $\vartheta, \theta \in \mathcal{M}$.

If $\beta(\vartheta, \theta) = 1$, then Definition 4 reduces to Definition 3.

Later on, the authors (see References [13,14]) utilized the above concepts and obtained different fixed point results.

In this paper, we define the notion of generalized ($\alpha\beta$-ψ)-contraction and establish some new fixed point theorems in the context of $\mathcal{F}$-metric spaces. We also furnish a notable example to describe the significance of established results.

2. Results and Discussions

Definition 5. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$-metric space and $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Then $\mathcal{H}$ is said to be generalized ($\alpha\beta$-ψ)-contraction if there exists $\alpha, \beta : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ and $\psi \in \Psi$ such that $\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) \geq \beta(\vartheta, \mathcal{H}\vartheta)\beta(\theta, \mathcal{H}\theta)$ implies*

$$d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq \psi\left(\max\left\{d_{\mathcal{F}}(\vartheta, \theta), \min\left\{d_{\mathcal{F}}(\vartheta, \mathcal{H}\vartheta), d_{\mathcal{F}}(\theta, \mathcal{H}\theta)\right\}\right\}\right), \tag{1}$$

for all $\vartheta, \theta \in \mathcal{M}$.

Theorem 3. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$-metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ be generalized ($\alpha\beta$-ψ)-contraction. Suppose that the following assertions hold:*

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii) $\mathcal{H}$ is an α-admissible mapping with respect to β,

(iii) there exists $\vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq \beta(\vartheta_0, \mathcal{H}\vartheta_0)$,

(v) either $\mathcal{H}$ is continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq \beta(\vartheta_n, \vartheta_{n+1})$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq \beta(\vartheta, \mathcal{H}\vartheta)$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Proof. Let $\vartheta_0 \in \mathcal{M}$ be such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq \beta(\vartheta_0, \mathcal{H}\vartheta_0)$ and construct $\{\vartheta_n\}$ in $\mathcal{M}$ by $\vartheta_{n+1} = \mathcal{H}^n\vartheta_0 = \mathcal{H}\vartheta_n$, $\forall\, n \in \mathbb{N}$. By (ii), we have

$$\alpha(\vartheta_0, \vartheta_1) = \alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq \beta(\vartheta_0, \mathcal{H}\vartheta_0) = \beta(\vartheta_0, \vartheta_1).$$

Continuing in this way, we get

$$\alpha(\vartheta_{n-1}, \vartheta_n) = \alpha(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1}) \geq \beta(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1}) = \beta(\vartheta_{n-1}, \vartheta_n), \tag{2}$$

for all $n \in \mathbb{N}$. Then

$$\alpha(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1})\alpha(\vartheta_n, \mathcal{H}\vartheta_n) \geq \beta(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1})\beta(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1}), \tag{3}$$

for all $n \in \mathbb{N}$. Clearly, if there exists $n_0 = 1, 2, ...$ for which $\vartheta_{n_0+1} = \vartheta_{n_0}$, then $\mathcal{H}\vartheta_{n_0} = \vartheta_{n_0}$ and the proof is completed. Hence, we suppose that $\vartheta_{n+1} \neq \vartheta_n$ or $d_{\mathcal{F}}(\mathcal{H}\vartheta_{n-1}, \mathcal{H}\vartheta_n) > 0$ for every $n \in \mathbb{N}$. Now as $\mathcal{H}$ is generalized ($\alpha\beta$-ψ)-contraction, so we have

$$\begin{aligned} d_{\mathcal{F}}(\vartheta_n, \vartheta_n) &= d_{\mathcal{F}}(\mathcal{H}\vartheta_{n-1}, \mathcal{H}\vartheta_n) \\ &\leq \psi\left(\max\left\{d_{\mathcal{F}}(\vartheta_{n-1}, \vartheta_n), \min\left\{d_{\mathcal{F}}(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1}), d_{\mathcal{F}}(\vartheta_n, \mathcal{H}\vartheta_n)\right\}\right\}\right), \end{aligned}$$

for all $n \in \mathbb{N}$. Now if $d_{\mathcal{F}}(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1}) < d_{\mathcal{F}}(\vartheta_n, \mathcal{H}\vartheta_n)$, then

$$\max\left\{d_{\mathcal{F}}(\vartheta_{n-1}, \vartheta_n), \min\left\{d_{\mathcal{F}}(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1}), d_{\mathcal{F}}(\vartheta_n, \mathcal{H}\vartheta_n)\right\}\right\} = d_{\mathcal{F}}(\vartheta_{n-1}, \vartheta_n),$$

for all $n \in \mathbb{N}$. If $d_{\mathcal{F}}(\vartheta_n, \mathcal{H}\vartheta_n) < d_{\mathcal{F}}(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1})$, then

$$\max\left\{d_{\mathcal{F}}(\vartheta_{n-1}, \vartheta_n), \min\left\{d_{\mathcal{F}}(\vartheta_{n-1}, \mathcal{H}\vartheta_{n-1}), d_{\mathcal{F}}(\vartheta_n, \mathcal{H}\vartheta_n)\right\}\right\} = d_{\mathcal{F}}(\vartheta_{n-1}, \vartheta_n),$$

for all $n \in \mathbb{N}$. Thus in all case, we have

$$d_{\mathcal{F}}(\vartheta_n, \vartheta_n) \leq \psi\left(d_{\mathcal{F}}(\vartheta_{n-1}, \vartheta_n)\right), \tag{4}$$

for all $n \in \mathbb{N}$. Continuing in this way, we get

$$d_{\mathcal{F}}(\vartheta_n, \vartheta_n) \leq \psi^n\left(d_{\mathcal{F}}(\vartheta_0, \vartheta_1)\right), \tag{5}$$

for all $n \in \mathbb{N}$. Suppose $f \in F$ and $\mathfrak{h} \in [0, +\infty)$ are such that the assertion (D_3) hold and suppose $\epsilon > 0$. Now from ($\mathcal{F}_2$), there exists $\delta > 0$ such that

$$0 < \iota < \delta \Longrightarrow f(\iota) < f(\delta) - \mathfrak{h}. \tag{6}$$

Let $n(\epsilon) \in \mathbb{N}$ be such that $0 < \sum_{n \geq n(\epsilon)} \psi^n(d_{\mathcal{F}}(\vartheta_0, \vartheta_1)) < \delta$. Hence, by (5), ($\mathcal{F}_1$) and ($\mathcal{F}_2$), we have

$$f(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\vartheta_i, \vartheta_{i+1})) \leq f(\sum_{i=n}^{m-1} \psi^i(d_{\mathcal{F}}(\vartheta_0, \vartheta_1))) \leq f(\sum_{n \geq n(\epsilon)} \psi^n(d_{\mathcal{F}}(\vartheta_0, \vartheta_1))) < f(\epsilon) - \mathfrak{h}, \tag{7}$$

for $m > n \geq n(\epsilon)$. Using (D_3) and (7), we obtain $d_{\mathcal{F}}(\vartheta_n, \vartheta_m) > 0$, $m > n \geq n(\epsilon)$ implies

$$f(d_{\mathcal{F}}(\vartheta_n, \vartheta_m)) \leq f(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\vartheta_i, \vartheta_{i+1})) + \mathfrak{h} < f(\epsilon).$$

By ($\mathcal{F}_1$), we have $d_{\mathcal{F}}(\vartheta_n, \vartheta_m) < \epsilon$, $m > n \geq n(\epsilon)$. This proves that $\{\vartheta_n\}$ is $\mathcal{F}$-Cauchy. Since $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete, so $\exists\, \vartheta^* \in \mathcal{M}$ such that

$$\lim_{n \to \infty} d_{\mathcal{F}}(\vartheta_n, \vartheta^*) = 0. \tag{8}$$

Secondly as $\vartheta_n \to \vartheta^*$ and $\alpha(\vartheta_n, \vartheta_{n+1}) \geq \beta(\vartheta_n, \vartheta_{n+1})$, then $\alpha(\vartheta^*, \mathcal{H}\vartheta^*) \geq \beta(\vartheta^*, \mathcal{H}\vartheta^*)$. Thus

$$\alpha(\vartheta^*, \mathcal{H}\vartheta^*)\alpha(\vartheta_n, \mathcal{H}\vartheta_n) \geq \beta(\vartheta^*, \mathcal{H}\vartheta^*)\beta(\vartheta_n, \mathcal{H}\vartheta_n).$$

We start with contradiction by supposing that $d_{\mathcal{F}}(\mathcal{H}(\vartheta^*), \vartheta^*) > 0$. By $(\mathcal{F}_1)$ and (D_3), we get

$$\begin{aligned} f(d_{\mathcal{F}}(\mathcal{H}(\vartheta^*), \vartheta^*)) &\leq f(d_{\mathcal{F}}(\mathcal{H}(\vartheta^*), \mathcal{H}(\vartheta_n)) + d_{\mathcal{F}}(\mathcal{H}(\vartheta_n), \vartheta^*)) + \mathfrak{h} \\ &\leq f(d_{\mathcal{F}}(\mathcal{H}(\vartheta^*), \mathcal{H}(\vartheta_n)) + d_{\mathcal{F}}(\mathcal{H}(\vartheta_n), \vartheta^*)) + \mathfrak{h}. \end{aligned}$$

By (1), we have

$$\begin{aligned} f(d_{\mathcal{F}}(\mathcal{H}(\vartheta^*), \vartheta^*)) &\leq f(d_{\mathcal{F}}(\mathcal{H}(\vartheta^*), \mathcal{H}(\vartheta_n)) + d_{\mathcal{F}}(\mathcal{H}(\vartheta_n), \vartheta^*)) + \mathfrak{h} \\ &\leq f(\psi\,(\max\{d_{\mathcal{F}}(\vartheta^*, \vartheta_n), \min(d_{\mathcal{F}}(\vartheta^*, \mathcal{H}(\vartheta^*)), d_{\mathcal{F}}(\vartheta_n, \mathcal{H}(\vartheta_n))\}) \\ &\quad + d_{\mathcal{F}}(\vartheta_{n+1}, \vartheta*)) + \mathfrak{h}, \\ &= f(\psi\,(\max\{D(\vartheta^*, \vartheta_n), \min(d_{\mathcal{F}}(\vartheta^*, \mathcal{H}(\vartheta^*)), d_{\mathcal{F}}(\vartheta_n, \vartheta_{n+1}))\}) \\ &\quad + d_{\mathcal{F}}(\vartheta_{n+1}, \vartheta*)) + \mathfrak{h}, \end{aligned}$$

for all $n \in \mathbb{N}$. Letiing $n \to \infty$ and using $(\mathcal{F}_2)$ and (8), we get

$$\lim_{n\to\infty} f(d_{\mathcal{F}}(\mathcal{H}(\vartheta^*), \vartheta^*)) \leq \lim_{n\to\infty} f(d_{\mathcal{F}}(\vartheta^*, \vartheta_n) + D(\vartheta_{n+1}, \vartheta*)) + \mathfrak{h} = -\infty.$$

This implies that $d_{\mathcal{F}}(\mathcal{H}(\vartheta^*), \vartheta^*) = 0$, which is a contradiction.

Thus $d_{\mathcal{F}}(\mathcal{H}(\vartheta^*), \vartheta^*) = 0$, that is, $\mathcal{H}(\vartheta^*) = \vartheta^*$. As consequence, $\vartheta^* \in \mathcal{M}$ is the fixed point of $\mathcal{H}$. □

Example 2. *Let $\mathcal{M} = \mathbb{R}$ endowed with $\mathcal{F}$-metric $d_{\mathcal{F}}$ given by*

$$d_{\mathcal{F}}(\vartheta, \theta) = \begin{cases} e^{|\vartheta-\theta|}, & \text{if } \vartheta \neq \theta \\ 0, & \text{if } \vartheta = \theta. \end{cases}$$

Then $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete $\mathcal{F}$-metric space with $f(\iota) = \frac{-1}{\iota}$ and $\mathfrak{h} = 1$. Define $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ and $\alpha, \beta : \mathcal{M}\times\mathcal{M} \to [0, \infty)$ by

$$\mathcal{H}(\vartheta) = \begin{cases} 3\vartheta, & \text{if } \vartheta > 1 \\ \frac{\vartheta}{4}, & \text{if } 0 \leq \vartheta \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\alpha(\vartheta, \theta) = \beta(\vartheta, \theta) = \begin{cases} 1, & \text{if } \vartheta, \theta \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\mathcal{H}$ is generalized $(\alpha\beta$-$\psi)$-contraction mapping with $\psi(\vartheta) = \frac{\vartheta}{2}$ for all $\vartheta \geq 0$ that is

$$d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq \psi\left(\max\left\{d_{\mathcal{F}}(\vartheta, \theta), \min\left\{d_{\mathcal{F}}(\vartheta, \mathcal{H}\vartheta), d_{\mathcal{F}}(\theta, \mathcal{H}\theta)\right\}\right\}\right).$$

Moreover, there exists $\vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) = 1 = \beta(\vartheta_0, \mathcal{H}\vartheta_0)$ and $\mathcal{H}$ is an α-admissible mapping with respect to β. Thus all the hypotheses of Theorem 3 are satisfied. Consequently $\mathcal{H}0 = 0$.

Corollary 1. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be a an $\mathcal{F}$-metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii)$\mathcal{H}$ is an α-admissible mapping,

(iii) if for $\vartheta, \theta \in \mathcal{M}$ *and* $\psi \in \Psi$ *such that*

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) \geq 1 \Longrightarrow d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq \psi\left(\max\left\{d_{\mathcal{F}}(\vartheta, \theta), \min\left\{d_{\mathcal{F}}(\vartheta, \mathcal{H}\vartheta), d_{\mathcal{F}}(\theta, \mathcal{H}\theta)\right\}\right\}\right),$$

(iv) there exists $\vartheta_0 \in \mathcal{M}$ *such that* $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(v) either $\mathcal{H}$ *is an continuous or if* $\{\vartheta_n\}$ *is a sequence in* $\mathcal{M}$ *such that* $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, *then* $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Proof. Consider $\beta : \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ as $\beta(\vartheta, \theta) = 1$ for all $\vartheta, \theta \in \mathcal{M}$ in Theorem 3. □

The following corollaries are direct consequences of Theorem 3.

Corollary 2. *Let* $(\mathcal{M}, d_{\mathcal{F}})$ *be a an* $\mathcal{F}$*-metric space and let* $\mathcal{H} : \mathcal{M} \to \mathcal{M}$*. Assume that the following assertions hold:*
(i) $(\mathcal{M}, d_{\mathcal{F}})$ *is* $\mathcal{F}$*-complete,*
(ii)$\mathcal{H}$ *is an* α*-admissible mapping,*
(iii) if for $\vartheta, \theta \in \mathcal{M}$ *and* $\psi \in \Psi$ *such that*

$$(d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta) + l)^{\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta)} \leq \psi\left(\max\left\{d_{\mathcal{F}}(\vartheta, \theta), \min\left\{d_{\mathcal{F}}(\vartheta, \mathcal{H}\vartheta), d_{\mathcal{F}}(\theta, \mathcal{H}\theta)\right\}\right\}\right) + l,$$

where $l > 0$,
(iv) there exists $\vartheta_0 \in \mathcal{M}$ *such that* $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(v) either $\mathcal{H}$ *is an continuous or if* $\{\vartheta_n\}$ *is a sequence in* $\mathcal{M}$ *such that* $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, *then* $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Corollary 3. *Let* $(\mathcal{M}, d_{\mathcal{F}})$ *be a an* $\mathcal{F}$*-metric space and let* $\mathcal{H} : \mathcal{M} \to \mathcal{M}$*. Assume that the following assertions hold:*
(i) $(\mathcal{M}, d_{\mathcal{F}})$ *is* $\mathcal{F}$*-complete,*
(ii)$\mathcal{H}$ *is an* α*-admissible mapping,*
(iii) if for $\vartheta, \theta \in \mathcal{M}$ *and* $\psi \in \Psi$ *such that*

$$(\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) + 1)^{d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta)} \leq 2^{\psi(\max\{d_{\mathcal{F}}(\vartheta, \theta), \min\{d_{\mathcal{F}}(\vartheta, \mathcal{H}\vartheta), d_{\mathcal{F}}(\theta, \mathcal{H}\theta)\}\})},$$

(iv) there exists $\vartheta_0 \in \mathcal{M}$ *such that* $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,
(v) either $\mathcal{H}$ *is an continuous or if* $\{\vartheta_n\}$ *is a sequence in* $\mathcal{M}$ *such that* $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, *then* $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Corollary 4. *Let* $(\mathcal{M}, d_{\mathcal{F}})$ *be a an* $\mathcal{F}$*-metric space and let* $\mathcal{H} : \mathcal{M} \to \mathcal{M}$*. Assume that the following assertions hold:*
(i) $(\mathcal{M}, d_{\mathcal{F}})$ *is* $\mathcal{F}$*-complete,*
(ii)$\mathcal{H}$ *is an* α*-admissible mapping,*
(iii) if for $\vartheta, \theta \in \mathcal{M}$ *and* $\psi \in \Psi$ *such that*

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta)d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k \max\left\{d_{\mathcal{F}}(\vartheta, \theta), \min\left\{d_{\mathcal{F}}(\vartheta, \mathcal{H}\vartheta), d_{\mathcal{F}}(\theta, \mathcal{H}\theta)\right\}\right\},$$

(iv) there exists $\vartheta_0 \in \mathcal{M}$ *such that* $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,

(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

If $\alpha(\vartheta, \theta) = 1$, then we have the following corollaries.

Corollary 5. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be a an $\mathcal{F}$-metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii)$\mathcal{H}$ is an β-subadmissible mapping,

(iii) if for $\vartheta, \theta \in \mathcal{M}$ and $\psi \in \Psi$ such that

$$\beta(\vartheta, \mathcal{H}\vartheta)\beta(\theta, \mathcal{H}\theta) \leq 1 \Longrightarrow d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq \psi\left(\max\left\{d_{\mathcal{F}}(\vartheta, \theta), \min\left\{d_{\mathcal{F}}(\vartheta, \mathcal{H}\vartheta), d_{\mathcal{F}}(\theta, \mathcal{H}\theta)\right\}\right\}\right),$$

(iv) there exists $\vartheta_0 \in \mathcal{M}$ such that $\beta(\vartheta_0, \mathcal{H}\vartheta_0) \leq 1$,

(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\beta(\vartheta_n, \vartheta_{n+1}) \leq 1$, then $\beta(\vartheta, \mathcal{H}\vartheta) \leq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Corollary 6. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be a an $\mathcal{F}$-metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold: (i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,*

(ii)$\mathcal{H}$ is an β-subadmissible mapping,

(iii) if for $\vartheta, \theta \in \mathcal{M}$ and $\psi \in \Psi$ such that

$$d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta) + l \leq \left[\psi\left(\max\left\{d_{\mathcal{F}}(\vartheta, \theta), \min\left\{d_{\mathcal{F}}(\vartheta, \mathcal{H}\vartheta), d_{\mathcal{F}}(\theta, \mathcal{H}\theta)\right\}\right\}\right) + l\right]^{\beta(\vartheta, \mathcal{H}\vartheta)\beta(\theta, \mathcal{H}\theta)},$$

where $l > 0$,

(iv) there exists $\vartheta_0 \in \mathcal{M}$ such that $\beta(\vartheta_0, \mathcal{H}\vartheta_0) \leq 1$,

(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\beta(\vartheta_n, \vartheta_{n+1}) \leq 1$, then $\beta(\vartheta, \mathcal{H}\vartheta) \leq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Corollary 7. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be a an $\mathcal{F}$-metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii)$\mathcal{H}$ is an β-subadmissible mapping,

(iii) if for $\vartheta, \theta \in \mathcal{M}$ and $\psi \in \Psi$ such that

$$2^{d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta)} \leq (\beta(\vartheta, \mathcal{H}\vartheta)\beta(\theta, \mathcal{H}\theta) + 1)^{\psi(\max\{d_{\mathcal{F}}(\vartheta, \theta), \min\{d_{\mathcal{F}}(\vartheta, \mathcal{H}\vartheta), d_{\mathcal{F}}(\theta, \mathcal{H}\theta)\}\})},$$

(iv) $\exists\ \vartheta_0 \in \mathcal{M}$ such that $\beta(\vartheta_0, \mathcal{H}\vartheta_0) \leq 1$,

(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\beta(\vartheta_n, \vartheta_{n+1}) \leq 1$, then $\beta(\vartheta, \mathcal{H}\vartheta) \leq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Corollary 8. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be a an $\mathcal{F}$-metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii)$\mathcal{H}$ is an β-subadmissible mapping,

(iii) if for $\vartheta, \theta \in \mathcal{M}$ and $\psi \in \Psi$ such that

$$d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq \psi\left(\beta(\vartheta, \mathcal{H}\vartheta)\beta(\theta, \mathcal{H}\theta) \max\left\{d_{\mathcal{F}}(\vartheta, \theta), \min\left\{d_{\mathcal{F}}(\vartheta, \mathcal{H}\vartheta), d_{\mathcal{F}}(\theta, \mathcal{H}\theta)\right\}\right\}\right),$$

(iv) there exists $\vartheta_0 \in \mathcal{M}$ such that $\beta(\vartheta_0, \mathcal{H}\vartheta_0) \leq 1$,

(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\beta(\vartheta_n, \vartheta_{n+1}) \leq 1$, then $\beta(\vartheta, \mathcal{H}\vartheta) \leq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Corollary 9. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be a an $\mathcal{F}$-metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii)$\mathcal{H}$ is an α-admissible mapping,

(ii) if for $\vartheta, \theta \in \mathcal{M}$ and $\psi \in \Psi$ such that

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) \geq 1 \Longrightarrow d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq \psi\left(d_{\mathcal{F}}(\vartheta, \theta)\right),$$

(iv) there exists $\vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,

(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Corollary 10. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be a an $\mathcal{F}$-metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii)$\mathcal{H}$ is an α-admissible mapping,

(ii) if for $\vartheta, \theta \in \mathcal{M}$ and $\psi \in \Psi$ such that

$$(\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) + 1)^{d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta)} \leq 2^{\psi(d_{\mathcal{F}}(\vartheta, \theta))},$$

(iv) there exists $\vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,

(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Corollary 11. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be a an $\mathcal{F}$-metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii)$\mathcal{H}$ is an α-admissible mapping,

(iii) if for $\vartheta, \theta \in \mathcal{M}$ and $\psi \in \Psi$ such that

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta)d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq \psi\left(d_{\mathcal{F}}(\vartheta, \theta)\right),$$

(iv) there exists $\vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,

(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Corollary 12 ([3]). *Let $(\mathcal{M}, d_{\mathcal{F}})$ be a an $\mathcal{F}$-metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ is continuous. Assume that the following assertions hold:*

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii) if for $\vartheta, \theta \in \mathcal{M}$ and $\psi \in \Psi$ such that

$$d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq \psi\left(d_{\mathcal{F}}(\vartheta, \theta)\right).$$

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Proof. Taking $\alpha(\vartheta, \theta) = 1$, for all $\vartheta, \theta \in \mathcal{M}$ in the Corollary 11. □

Corollary 13. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be a an $\mathcal{F}$-metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii)$\mathcal{H}$ is an α-admissible mapping,

(iii) if there exists $k \in (0, 1)$ such that

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta)d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k d_{\mathcal{F}}(\vartheta, \theta),$$

for all $\vartheta, \theta \in \mathcal{M}$

(iv) there exists $\vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,

(v) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Corollary 14 ([2]). *Let $(\mathcal{M}, d_{\mathcal{F}})$ be a an $\mathcal{F}$-metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ be a continuous mapping. Assume that the following assertions hold:*

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii) if there exists $k \in (0, 1)$ such that

$$d_{\mathcal{F}}(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k d_{\mathcal{F}}(\vartheta, \theta),$$

for all $\vartheta, \theta \in \mathcal{M}$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Proof. Taking $\alpha(\vartheta, \theta) = 1$, for all $\vartheta, \theta \in \mathcal{M}$ in the Corollary 13. □

3. Consequences

The following results are direct consequences of main results by taking $f(\iota) = \ln(\iota)$ and $\mathfrak{h} = \ln(1)$.

Theorem 4. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Suppose that the following assertions hold:*

(i) there exist two functions $\alpha, \beta : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ and $k \in [0, 1)$ such that $\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) \geq \beta(\vartheta, \mathcal{H}\vartheta)\beta(\theta, \mathcal{H}\theta)$ implies

$$d(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k \max\left\{d(\vartheta, \theta), \min\left\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\right\}\right\}$$

(ii) $\mathcal{H}$ is an α-admissible mapping with respect to β,

(iii) there exists $\vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq \beta(\vartheta_0, \mathcal{H}\vartheta_0)$,

(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq \beta(\vartheta_n, \vartheta_{n+1})$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq \beta(\vartheta, \mathcal{H}\vartheta)$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Proof. Taking $\psi(\iota) = k\iota$, where $k \in [0,1)$ in Theorem 3. $\square$

Corollary 15. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,

(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) \geq 1$ implies

$$d(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k \max\left\{d(\vartheta, \theta), \min\left\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\right\}\right\},$$

(iii) there exists $\vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,

(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Proof. Taking $\beta(\vartheta, \theta) = 1$ in above corollary. $\square$

Example 3. *Let $\mathcal{M} = [0, \infty)$ be endowed with the usual metric $d(\vartheta, \theta) = |\vartheta - \theta|$ for all $\vartheta, \theta \in \mathcal{M}$ and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ be defined by $\mathcal{H}\vartheta = \frac{\vartheta}{4}$. Also, define $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ by $\alpha(\vartheta, \theta) = 3$ and $\exists k = \frac{1}{2} \in (0,1)$. Clearly,$\mathcal{H}$ is an α-admissible mapping. Also, $\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) = 9 \geq 1$ for all $\vartheta, \theta \in \mathcal{M}$. Hence*

$$d(\mathcal{H}\vartheta, \mathcal{H}\theta) = \frac{1}{4}|\vartheta - \theta| \leq \frac{1}{2} \max\left\{|\vartheta - \theta|, \min\left\{|\vartheta - \frac{\vartheta}{4}|, |\theta - \frac{\theta}{4}|\right\}\right\}.$$

Then the conditions of Corollary 15 hold and $\mathcal{H}$ has a fixed point which is 0.

Corollary 16. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,

(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that

$$(d(\mathcal{H}\vartheta, \mathcal{H}\theta) + l)^{\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta)} \leq k \max\left\{d(\vartheta, \theta), \min\left\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\right\}\right\} + l,$$

where $l > 0$,

(iii) there exists $\vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,

(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Corollary 17. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,

(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that

$$(\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta) + 1)^{d(\mathcal{H}\vartheta, \mathcal{H}\theta)} \leq 2^{k \max\{d(\vartheta, \theta), \min\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\}\}},$$

(iii) there exists $\vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,

(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Corollary 18. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an α-admissible mapping,

(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that

$$\alpha(\vartheta, \mathcal{H}\vartheta)\alpha(\theta, \mathcal{H}\theta)d(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k \max\{d(\vartheta, \theta), \min\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\}\},$$

(iii) there exists $\vartheta_0 \in \mathcal{M}$ such that $\alpha(\vartheta_0, \mathcal{H}\vartheta_0) \geq 1$,

(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\alpha(\vartheta_n, \vartheta_{n+1}) \geq 1$, then $\alpha(\vartheta, \mathcal{H}\vartheta) \geq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

If $\alpha(\vartheta, \theta) = 1$, then we have the following corollaries.

Corollary 19. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an β-subadmissible mapping,

(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that $\beta(\vartheta, \mathcal{H}\vartheta)\beta(\theta, \mathcal{H}\theta) \leq 1$ implies

$$d(\mathcal{H}\vartheta, \mathcal{H}\theta) \leq k \max\{d(\vartheta, \theta), \min\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\}\},$$

(iii) there exists $\vartheta_0 \in \mathcal{M}$ such that $\beta(\vartheta_0, \mathcal{H}\vartheta_0) \leq 1$,

(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\beta(\vartheta_n, \vartheta_{n+1}) \leq 1$, then $\beta(\vartheta, \mathcal{H}\vartheta) \leq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Corollary 20. *Let $(\mathcal{M}, d)$ be a complete metric space and let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$. Assume that the following assertions hold:*

(i) $\mathcal{H}$ is an β-subadmissible mapping,

(ii) if for $\vartheta, \theta \in \mathcal{M}$ such that

$$d(\mathcal{H}\vartheta, \mathcal{H}\theta) + l \leq [k \max\{d(\vartheta, \theta), \min\{d(\vartheta, \mathcal{H}\vartheta), d(\theta, \mathcal{H}\theta)\}\} + l]^{\beta(\vartheta, \mathcal{H}\vartheta)\beta(\theta, \mathcal{H}\theta)},$$

where $l > 0$.

(iii) there exists $\vartheta_0 \in \mathcal{M}$ such that $\beta(\vartheta_0, \mathcal{H}\vartheta_0) \leq 1$,

(iv) either $\mathcal{H}$ is an continuous or if $\{\vartheta_n\}$ is a sequence in $\mathcal{M}$ such that $\vartheta_n \to \vartheta$, $\beta(\vartheta_n, \vartheta_{n+1}) \leq 1$, then $\beta(\vartheta, \mathcal{H}\vartheta) \leq 1$.

Then there exists $\vartheta^* \in \mathcal{M}$ such that $\vartheta^* = \mathcal{H}\vartheta^*$.

Remark 2. *One can easily derive the main results of References [11] and [15] from our Corollaries 11 and 14 respectively by taking $f(\iota) = \ln(\iota)$ and $\mathfrak{h} = \ln(1)$.*

4. Applications

In the present section, we solve the following differential equation

$$\vartheta'(\iota) = -\rho_1(\iota)\vartheta(\iota) + \rho_2(\iota)\mathcal{L}(\vartheta(\iota - \varsigma(\iota))) + \rho_3(\iota)\vartheta'(\iota - \varsigma(\iota)). \tag{9}$$

The lemma of Djoudi et al. [16] is very handy in the proof our theorem.

Lemma 2 ([16]). *Suppose that $\varsigma^{/}(\iota) \neq 1\ \forall \iota \in \mathbb{R}$. Then $\vartheta(\iota)$ is a solution of (9) iff*

$$\vartheta(\iota) = \left(\vartheta(0) - \frac{\rho_3(0)}{1-\varsigma^{/}(0)}\vartheta(-\varsigma(0))\right)\rho^{-\int_0^{\iota}\alpha(\varsigma)d\varsigma} + \frac{\rho_3(\iota)}{1-\varsigma^{/}(\iota)}\vartheta(\iota-\varsigma(\iota))$$
$$-\int_0^{\iota}(h(\omega))\vartheta(\omega-\varsigma(\omega))) - \rho_2(\omega)\mathcal{L}\left(\vartheta(\omega-\varsigma(\omega))\right))\rho^{-\int_{\omega}^{\iota}\alpha(\varsigma)d\varsigma}d\omega, \tag{10}$$

where

$$h(\omega) = \frac{\varsigma^{//}(\omega)\rho_3(\omega) + \left(\rho_3^{/}(\omega) + \rho_3(\omega)\rho_1(\omega)\right)(1-\varsigma^{/}(\omega))}{(1-\varsigma^{/}(\omega))^2}. \tag{11}$$

Now assume that $\phi : (-\infty, 0] \to \mathbb{R}$ is a continuous bounded initial function, then $\vartheta(\iota) = \vartheta(\iota, 0, \phi)$ is a solution of (9) if $\vartheta(\iota) = \phi(\iota)$ for $\iota \leq 0$ and assures (9) for $\iota \geq 0$. Assume $\mathfrak{C}$ be the collection of $\pi : \mathbb{R} \to \mathbb{R}$ which are continuous functions. Define $\mathcal{B}_\phi$ by

$$\mathcal{B}_\phi = \{\pi : \mathbb{R} \to \mathbb{R} \text{ such that } \phi(\iota) = \pi(\iota) \text{ if } t \leq 0,\ \pi(\iota) \to 0 \text{ as } \iota \to \infty,\ \pi \in \mathfrak{C}\}.$$

Then $\mathcal{B}_\phi$ is a Banach space equipped with the supremum norm $\|\cdot\|$.

Lemma 3 ([3]). *The space $(\mathcal{B}_\phi, \|\cdot\|)$ provided with d given by*

$$d(\iota, \iota^*) = ||\iota - \iota^*|| = \sup_{\vartheta \in I} |\iota(\vartheta) - \iota^*(\vartheta)|$$

for $\iota, \iota^* \in \mathcal{B}_\phi$, is an $\mathcal{F}$-metric space.

Theorem 5. *Let $\mathcal{H} : \mathcal{B}_\phi \to \mathcal{B}_\phi$ be a mapping defined by*

$$(\mathcal{H}\pi)(\iota) = \left(\pi(0) - \frac{\rho_3(0)}{1-\varsigma^{/}(0)}\pi(-\varsigma(0))\right)\rho^{-\int_0^{\iota}\alpha(\varsigma)d\varsigma} + \frac{\rho_3(\iota)}{1-\varsigma^{/}(\iota)}\pi(\iota-\varsigma(\iota))$$
$$-\int_0^{\iota}(h(\omega)\pi(\omega-\varsigma(\omega)) - \rho_2(\omega)\mathcal{L}\left(\pi(\omega-\varsigma(\omega))\right))\rho^{-\int_{\omega}^{\iota}\alpha(\varsigma)d\varsigma}d\omega, \iota \geq 0 \tag{12}$$

for all $\pi \in \mathcal{B}_\phi$. Assume that these assertions are satisfied:

(i) there exists $\mu \geq 0$ and $\phi \in \Psi$ so that

$$\int_0^{\iota}|h(\omega)(\pi(\omega-\varsigma(\omega))) - \varrho(\omega-\varsigma(\omega))|\rho^{-\int_{\omega}^{\iota}\alpha(\varsigma)d\varsigma}$$
$$\leq \frac{\mu}{2}\phi\left(\max\left\{\|\pi-\varrho\|, \min\left\{\|\pi-\mathcal{H}\pi\|, \|\varrho-\mathcal{H}\varrho\|\right\}\right\}\right) \tag{13}$$

and

$$\int_0^{\iota}|(\rho_2(\omega))\mathcal{L}(\pi(\omega-\varsigma(\omega))) - \mathcal{L}(\varrho(\omega-\varsigma(\omega)))|\rho^{-\int_{\omega}^{\iota}\alpha(\varsigma)d\varsigma}$$
$$\leq \frac{\mu}{2}\phi\left(\max\left\{\|\pi-\varrho\|, \min\left\{\|\pi-\mathcal{H}\pi\|, \|\varrho-\mathcal{H}\varrho\|\right\}\right\}\right) \tag{14}$$

for all $\pi, \varrho \in \mathcal{B}_\phi$.

(ii) Then $\mathcal{H}$ has a fixed point.

Proof. Define $\alpha : \mathfrak{C} \times \mathfrak{C} \to \mathbb{R}$ by

$$\alpha(\pi, \varrho) = \beta(\pi, \varrho) = \begin{cases} 1, \text{ if } \pi, \varrho \in \mathcal{B}_\phi, \\ 0, \text{ otherwise.} \end{cases}$$

Now for $\pi, \varrho \in \mathcal{B}_\phi$ such that $\alpha(\pi, \varrho) = \beta(\pi, \varrho) \geq 1$. It follows from (12) that $\mathcal{H}(\pi), \mathcal{H}(\varrho) \in \mathcal{B}_\phi$. Therefore $\alpha(\mathcal{H}(\pi), \mathcal{H}(\varrho)) = \beta(\mathcal{H}(\pi), \mathcal{H}(\varrho)) \geq 1$. Since, (13)–(15) hold, then for $\pi, \varrho \in \mathcal{B}_\phi$, we have

$$
\begin{aligned}
|(\mathcal{H}\pi)(\iota) - (\mathcal{H}\varrho)(\iota)| &\leq \left|\frac{\rho_3(\iota)}{1-\varsigma'(\iota)}\right| \|\pi - \varrho\| \\
&\quad + \int_0^\iota |h(\omega)(\pi(\omega - \varsigma(\omega))) - \varrho(\omega - \varsigma(\omega))| \rho^{-\int_\omega^\iota \alpha(\varsigma) d\varsigma} \\
&\quad \int_0^\iota |(\rho_2(\omega))\mathcal{L}(\pi(\omega - \varsigma(\omega))) - \mathcal{L}(\varrho(\omega - \varsigma(\omega)))| \rho^{-\int_\omega^\iota \alpha(\varsigma) d\varsigma} \\
&\leq \left|\frac{\rho_3(\iota)}{1-\varsigma'(\iota)}\right| \|\pi - \varrho\| + \mu\phi\left(\max\left\{\|\pi - \varrho\|, \min\left\{\|\pi - \mathcal{H}\pi\|, \|\varrho - \mathcal{H}\varrho\|\right\}\right\}\right) \\
&\leq \left\{\left|\frac{\rho_3(\iota)}{1-\varsigma'(\iota)}\right| + \mu\right\} \phi\left(\max\left\{\|\pi - \varrho\|, \min\left\{\|\pi - \mathcal{H}\pi\|, \|\varrho - \mathcal{H}\varrho\|\right\}\right\}\right) \\
&\leq \phi\left(\max\left\{\|\pi - \varrho\|, \min\left\{\|\pi - \mathcal{H}\pi\|, \|\varrho - \mathcal{H}\varrho\|\right\}\right\}\right).
\end{aligned}
$$

Hence,

$$
d_{\mathcal{F}}(\mathcal{H}\pi, \mathcal{H}\varrho) \leq \psi\left(\max\left\{d_{\mathcal{F}}(\pi, \varrho), \min\left\{d_{\mathcal{F}}(\pi, \mathcal{H}\pi), d_{\mathcal{F}}(\varrho, \mathcal{H}\varrho)\right\}\right\}\right)
$$

implies that $\mathcal{H}$ is generalized ($\alpha\beta$-ψ)-contraction. Thus by Theorem 3, $\mathcal{H}$ has a unique fixed point in $\mathcal{B}_\phi$ which solves (9). □

5. Conclusions

In this paper, we defined generalized ($\alpha\beta$-ψ)-contraction in the setting of $\mathcal{F}$-metric space and obtained some new fixed point results. As consequence of main results, we derived some fixed point results in metric spaces. We investigated the existence of solution for the following nonlinear neutral differential equation with an unbounded delay as application of our main results.

Author Contributions: Methodology and investigation, S.A.A.-M., write and editing, J.A. and formal analysis, G.M. All authors have read and agree to the published version of the manuscript.

Funding: Deanship of Scientific Research (DSR), University of Jeddah, Jeddah. Grant No. UJ-02-001-DR.

Acknowledgments: This work was funded by the University of Jeddah, Saudi Arabia, under grant No. UJ-02-001-DR. The authors, therefore, acknowledge with thanks the University technical and financial support. Moreover the co-author Giuseppe Marino inserts this paper into the research activity carried out under the auspices of GNAMPA.

Conflicts of Interest: The authors declare that they have no competing interests.

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