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# Bohr Radius Problems for Some Classes of Analytic Functions Using Quantum Calculus Approach

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Received: 27 January 2020; Accepted: 11 April 2020; Published: 18 April 2020

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**Abstract:** The main purpose of this investigation is to use quantum calculus approach and obtain the Bohr radius for the class of $q$-starlike ($q$-convex) functions of order $\alpha$. The Bohr radius is also determined for a generalized class of $q$-Janowski starlike and $q$-Janowski convex functions with negative coefficients.

**Keywords:** $q$-Bohr radius; $q$-Janowski starlike functions; $q$-Janowski convex functions; $q$-starlike functions of order $\alpha$; $q$-convex functions of order $\alpha$; $q$-derivative (or $q$-difference) operator; quantum calculus approach

**MSC:** 30C45; 30C50; 30C80

## 1. Introduction

Let $\mathbb{D} := \{z :\in \mathbb{C} : |z| < 1\}$ be the open unit disc in $\mathbb{C}$. Suppose $\mathcal{A}$ denote the class of analytic functions in $\mathbb{D}$ normalized by $f(0) = 0 = f'(0) - 1$. Also, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$.

Suppose $\mathcal{H}(\mathbb{D}, \Omega)$ is the class of analytic functions mapping open unit disc $\mathbb{D}$ into a domain $\Omega$. Harald Bohr [1] in 1914 proved that if a function $f$ of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belong to $\mathcal{H}(\mathbb{D}, \mathbb{D})$, then $\sum_{n=0}^{\infty} |a_n z^n| \leq 1$ in the disc $|z| \leq k$, where $k \geq 1/6$. As reported by Bohr in [1], Riesz, Schur and Wiener discovered that $|z| \leq k$ is actually true for $0 \leq k \leq 1/3$ and that $1/3$ is the best possible. The number $1/3$ is commonly called the "Bohr radius" for the class of analytic self-maps $f$ in $\mathbb{D}$, while the inequality $\sum_{n=0}^{\infty} |a_n z^n| \leq 1$ is known as the "Bohr inequality". Later on, extensions of Bohr inequality and their proofs were given in [2–4]. Note that Bohr Radius is somewhat whimsical, for physicists consider the Bohr Radius $a_0$ of the hydrogen atom to be a fundamental constant, that is, $4\pi\epsilon h^2/m_e e^2$, or about 0.529A. The physicists Bohr Radius is named for Niels Bohr, a founder of the Quantum Theory and 1922 recipient of the Nobel Prize for physics.

The Bohr inequality has emerged as an active area of research after Dixon [5] used it to disprove a conjecture in Banach algebra. Using the Euclidean distance, denoted by $d$, the Bohr inequality $\sum_{n=0}^{\infty}|a_n z^n| \leq 1$ for a function $f$ of the form $f(z)=\sum_{n=0}^{\infty} a_n z^n$ can be written as

$$\begin{aligned}\sum_{n=0}^{\infty}|a_n z^n| \leq 1 \Leftrightarrow \sum_{n=1}^{\infty}|a_n z^n| &\leq 1-|a_0| \\ &\Leftrightarrow d\left(\sum_{n=0}^{\infty}|a_n z^n|, |a_0|\right) = \sum_{n=1}^{\infty}|a_n z^n| \leq 1-|a_0| = 1-|f(0)| \\ &\Leftrightarrow d\left(\sum_{n=0}^{\infty}|a_n z^n|, |a_0|\right) \leq d(f(0), \partial\mathbb{D}).\end{aligned}$$

where $\partial\mathbb{D}$ is the boundary of the disc $\mathbb{D}$. Thus, the concept of the Bohr inequality for a function $f(z)=\sum_{n=0}^{\infty} a_n z^n$, defined in $\mathbb{D}$, can be generalized by

$$d\left(\sum_{n=0}^{\infty}|a_n z^n|, |f(0)|\right) = \sum_{n=1}^{\infty}|a_n z^n| \leq d(f(0), \partial f(\mathbb{D})). \quad (1)$$

Accordingly, the Bohr radius for a class $\mathcal{M}$ consisting of analytic functions $f$ of the form $f(z)=\sum_{n=0}^{\infty} a_n z^n$ in the disc $\mathbb{D}$ is the largest $r^* > 0$ such that every function $f \in \mathcal{M}$ satisfies the inequality (1) for all $|z| = r \leq r^*$. In this case, the class $\mathcal{M}$ is said to satisfy a Bohr phenomenon.

Quantum calculus (or $q$-calculus) is an approach or a methodology that is centered on the idea of obtaining $q$-analogues without the use of limits. This approach has a great interest due to its applications in various branches of mathematics and physics, such as, the areas of ordinary fractional calculus, optimal control problems, $q$-difference, $q$-integral equations and $q$-transform analysis. Jackson [6] intoduced the $q$-derivative (or $q$-difference, or Jackson derivative) denoted by $D_q$, $q \in (0,1)$, which is defined in a given subset of $\mathbb{C}$ by

$$(D_q f)(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & \text{if } z \neq 0 \\ f'(0), & \text{if } z = 0 \end{cases} \quad (2)$$

provided $f'(0)$ exists. If $f$ is a function defined in a subset of the complex plane $\mathbb{C}$, then (2) yields

$$\lim_{q\to 1^-}(D_q f)(z) = \lim_{q\to 1^-}\frac{f(z)-f(qz)}{(1-q)z} = f'(z).$$

It is easy to see that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then by using (2) we have

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty}[n]_q a_n z^{n-1},$$

$$D_q(zD_q f(z)) = 1 + \sum_{n=2}^{\infty}[n]_q^2 a_n z^{n-1},$$

$$D_q^2 f(z) = D_q(D_q f(z)) = \sum_{n=2}^{\infty}[n]_q^2 a_n z^{n-2},$$

where $[n]_q$ is given by

$$[n]_q = \frac{1-q^n}{1-q}, q \in (0,1).$$

It is a routine to check that

$$D_q(zD_q f(z)) = D_q f(z) + zD_q^2 f(z).$$

In 1869, Thomae introduced the particular $q$-integral [7] which is defined as

$$\int_0^1 f(t)d_qt = (1-q)\sum_{n=0}^{\infty} q^n f(q^n),$$

provided the $q$-series converges. Later on, Jackson [8] defined the general $q$-integral as follows:

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt,$$

where

$$\int_0^a f(t)d_qt = a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n),$$

provided the $q$-series converges. Also note that

$$D_q \int_0^x f(t)d_qt = f(x) \text{ and } \int_0^x D_qf(t)d_qt = f(x) - f(0),$$

where the second equality holds if $f$ is continuous at $x = 0$.

The $q$-calculus plays an important role in the investigation of several subclasses of $\mathcal{A}$. A firm footing of the $q$-calculus in the context of geometric function theory and its usages involving the basic (or $q$-) hypergeometric functions in geometric function theory was actually made in a book chapter by Srivastava (see, for details [9]; see also [10]). In 1990, Ismail et al. [11] introduced a connection between starlike (convex) functions and the $q$-calculus by introducing a $q$-analog of starlike (convex) functions. They generalized a well-known class of starlike functions, called the class of $q$-starlike functions denoted by $\mathcal{S}_q^*$, consisting of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left|\frac{z(D_qf)(z)}{f(z)} - \frac{1}{1-q}\right| \le \frac{1}{1-q}, z \in \mathbb{D}.$$

Baricz and Swaminathan [12] introduced a $q$-analog of convex functions, denoted by $\mathcal{C}_q$, satisfying the relation

$$f \in \mathcal{C}_q \quad \text{if and only if} \quad z(D_qf) \in \mathcal{S}_q^*.$$

Recently Srivastava et al. [13] (see also [14]) successfully combined the concept of Janowski [15] and the above mentioned $q$-calculus and introduced the class $\mathcal{S}_q^*[A,B]$ and $\mathcal{C}_q[A,B]$, $-1 \le B < A \le 1$, $q \in (0,1)$, given by

$$\mathcal{S}_q^*[A,B] := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{(A+1)z + 2 + (A-1)qz}{(B+1)z + 2 + (B-1)qz} \right\},$$

and

$$\mathcal{C}_q[A,B] := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{(A+1)z + 2 + (A-1)qz}{(B+1)z + 2 + (B-1)qz} \right\}$$

respectively, where $\prec$ denotes subordination. As $q \to 1^-$, $\mathcal{S}_q^*[A,B]$ and $\mathcal{C}_q[A,B]$ yield respectively the classes $\mathcal{S}^*[A,B]$ and $\mathcal{C}[A,B]$ defined by Janowski [15]. For various choices of $A$ and $B$, these classes reduce to well-known subclasses of $q$-starlike and $q$-convex functions. For instance, with $0 \le \alpha < 1$, $\mathcal{S}_q^*(\alpha) := \mathcal{S}_q^*[1-2\alpha, -1]$ is the class of $q$-starlike functions of order $\alpha$, introduced by Agrawal and

Sahoo [16]. Motivated by the authors in [16], Agrawal [17] defined a $q$-analog of convex functions of order $\alpha$, $0 \leq \alpha < 1$, $\mathcal{C}_q(\alpha) := \mathcal{C}_q[1-2\alpha, -1]$, satisfying

$$f \in \mathcal{C}_q(\alpha) \quad \text{if and only if} \quad z(D_q f) \in \mathcal{S}_q^*(\alpha). \tag{3}$$

Note that $\mathcal{S}_q^*[1,-1] \equiv \mathcal{S}_q^*$ and $\mathcal{C}_q[1,-1] \equiv \mathcal{C}_q$.

In recent years, there is a great development of geometric function theory because of using quantum calculus approach. In particular, Srivastava et al. [18] found distortion and radius of univalence and starlikenss for several subclasses of $q$-starlike functions with negative coefficients. They [19] also determined sufficient conditions and containment results for the different types of $k$-uniformly $q$-starlike functions. Naeem et al. [20] investigated subfamilies of $q$-convex functions and $q$-close to convex functions with respect to the Janowski functions connected with q-conic domain which explored some important geometric properties such as coefficient estimates, sufficiency criteria and convolution properties of these classes. For a survey on the use of quantum calculus approach in mathematical sciences and its role in geometric function theory, one may refer to [21]. In addition, one may refer to a survey-cum-expository article written by Srivastava [22] where he explored the mathematical application of q-calculus, fractional q- calculus and fractional q-differential operators in geometric function theory.

In this paper, we investigate Bohr radius problems for the classes $\mathcal{S}_q^*(\alpha)$ and $\mathcal{C}_q(\alpha)$, respectively, in Sections 2 and 3. In Section 4, we define and investigate the Bohr radius problem for a generalized class, $\mathcal{TP}_q(\lambda, A, B)$, of functions with negative coefficients, where $q \in (0,1)$, $\lambda \in [0,1]$ and $-1 \leq B < A \leq 1$. In particular, we also define and obtain sharp Bohr radius for the class of the $q$-Janowski functions with negative coefficients in Section 4.

## 2. The Bohr Radius for the Class $S_q^*(\alpha)$

To find the Bohr radius for the class $S_q^*(\alpha)$, we first need the following four lemmas.

**Lemma 1** ([23] (Theorem 2.5, p. 1511))**.** *For $q \in (0,1)$, suppose $a, b, c$ are non-negative real numbers satisfying $0 \leq 1 - aq \leq 1 - cq$ and $0 < 1 - b \leq 1 - c$. Then there exists a non-decreasing function $\mu : [0,1] \to [0,1]$ with $\mu(1) - \mu(0) = 1$ such that*

$$\frac{w\phi(q, q, q^2, q, w)}{\phi(q^0, q, q^2, q, w)} = \int_0^1 \frac{w}{1 - tw} d\mu(t),$$

*where $\phi(a, b; c; q, z)$ is a hypergeometric function (see [24,25]) given by*

$$\phi(a, b; c; q, z) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} z^n$$

*and $(a;q)_0 = 1, (a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$, which is analytic in the cut-plane $\mathbb{C} \setminus [1,\infty]$ and maps both the unit disc and the half-plane $\{z \in \mathbb{C} : \mathrm{Re}\, z < 1\}$ univalently onto domains convex in the direction of the imaginary axis.*

**Lemma 2** ([16] (Theorem 1.1, p. 17))**.** *If $f \in \mathcal{A}$, then $f \in \mathcal{S}_q^*(\alpha)$ if and only if there exists a probability measure $\mu$ supported on the circle such that*

$$\frac{zf'(z)}{f(z)} = 1 + \int_{|\sigma|=1} \sigma z F'_{q,\alpha}(\sigma z) d\mu(\sigma),$$

*where*

$$F_{q,\alpha}(z) = \sum_{n=1}^{\infty} \frac{-2}{1-q^n} \ln\left(\frac{q}{1-\alpha(1-q)}\right) z^n, \quad z \in \mathbb{D}.$$

**Lemma 3** (Distortion theorem). *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n = zh(z) \in \mathcal{S}_q^*(\alpha)$. Then*

$$\exp(F_{q,\alpha}(-r)) \leq |h(z)| \leq \exp(F_{q,\alpha}(r)).$$

**Proof.** Let $f \in \mathcal{S}_q^*(\alpha)$. By Lemma 2, there exists a probability measure $\mu$ supported on the unit circle such that

$$\frac{zf'(z)}{f(z)} = 1 + \int_{|\sigma|=1} \sigma z F'_{q,\alpha}(\sigma z) d\mu(\sigma),$$

where

$$F_{q,\alpha}(z) = \sum_{n=1}^{\infty} \frac{-2\ln\left(\frac{q}{1-\alpha(1-q)}\right)}{1-q^n} z^n, \quad z \in \mathbb{D}.$$

Integrating and then taking exponential on both sides, we have

$$f(z) = z \exp\left(\int_{|\sigma|=1} F_{q,\alpha}(\sigma z) d\mu(\sigma)\right).$$

Since $f(z) = zh(z) \in S_q^*(\alpha)$, it follows that

$$|h(z)| = \exp\left(\operatorname{Re}\int_{|\sigma|=1} F_{q,\alpha}(\sigma z) d\mu(\sigma)\right).$$

Thus

$$\begin{aligned}
\ln|h(z)| &= \operatorname{Re}\int_{|\sigma|=1} F_{q,\alpha}(\sigma z) d\mu(\sigma) \\
&= -2\ln\left(\frac{q}{1-\alpha(1-q)}\right) \operatorname{Re}\int_{|\sigma|=1} \sum_{n=1}^{\infty} \frac{(\sigma z)^n}{1-q^n} d\mu(\sigma) \\
&= \frac{-2}{1-q} \ln\left(\frac{q}{1-\alpha(1-q)}\right) \operatorname{Re}\int_{|\sigma|=1} (\sigma z \phi(q,q,q^2,q,\sigma z)) d\mu(\sigma) \\
&= \frac{-2}{1-q} \ln\left(\frac{q}{1-\alpha(1-q)}\right) \operatorname{Re}\int_0^{2\pi} ((e^{i\theta}z)\phi(q,q,q^2,q,e^{i\theta}z)) d\mu(\theta) \\
&= \frac{-2}{1-q} \ln\left(\frac{q}{1-\alpha(1-q)}\right) \operatorname{Re}\int_0^{2\pi} (w\phi(q,q,q^2,q,w)) d\mu(\theta), \quad w = e^{i\theta}z \in \mathbb{D} \\
&= \frac{-2}{1-q} \ln\left(\frac{q}{1-\alpha(1-q)}\right) \operatorname{Re}\int_0^{2\pi} \frac{w\phi(q,q,q^2,q,w)}{\phi(q^0,q,q^2,q,w)} d\mu(\theta),
\end{aligned} \tag{4}$$

where $\phi(a,b;c;q,z)$ is the hypergeometric function defined in Lemma 1. By Lemma 1, we have

$$\frac{w\phi(q,q,q^2,q,w)}{\phi(q^0,q,q^2,q,w)} = \int_0^1 \frac{w}{1-tw} d\mu(t). \tag{5}$$

Let

$$\begin{aligned}
g(re^{i\psi}) &= \operatorname{Re}\frac{w}{1-tw}, w = re^{i\psi} \\
&= \operatorname{Re}\frac{r(\cos\psi + i\sin\psi)}{1-tr(\cos\psi + i\sin\psi)} \\
&= \frac{r\cos\psi(1-tr\cos\psi) - tr^2\sin^2\psi}{1+r^2t^2-2tr\cos\psi}.
\end{aligned}$$

A routine calculation shows that

$$\min_{\psi} g(re^{i\psi}) = g(-r) \quad \text{and} \quad \max_{\psi} g(re^{i\psi}) = g(r).$$

Thus

$$\min_{|w|\leq r} \operatorname{Re} \frac{w}{1-tw} = \frac{-r}{1+rt} \quad \text{and} \quad \max_{|w|\leq r} \operatorname{Re} \frac{w}{1-tw} = \frac{r}{1-rt}. \tag{6}$$

By (4)–(6), it follows that

$$\begin{aligned}
\ln|h(z)| &\geq \frac{-2}{1-q} \ln\left(\frac{q}{1-\alpha(1-q)}\right) \int_{|\sigma|=1} (-r\phi(q,q,q^2,q,-r))d\mu(\sigma) \\
&\geq \frac{-2}{1-q} \ln\left(\frac{q}{1-\alpha(1-q)}\right) (-r\phi(q,q,q^2,q,-r)) \\
&= F_{q,\alpha}(-r)
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
\ln|h(z)| &\leq \int_{|\sigma|=1} F_{q,\alpha}(r)d\mu(\sigma) \\
&= F_{q,\alpha}(r).
\end{aligned} \tag{8}$$

By (7) and (8), we have $\exp(F_{q,\alpha}(-r)) \leq |h(z)| \leq \exp(F_{q,\alpha}(r))$. □

**Remark 1.** *As $q \to 1^-$, Lemma 3 yields the corresponding distortion theorem [26] (Theorem 8, p. 117) for the class $S^*(\alpha)$.*

**Lemma 4** ([16] (Theorem 1.3, p. 8))**.** *Let*

$$G_{q,\alpha}(z) = z\exp(F_{q,\alpha}(z)) = z + \sum_{n=2}^{\infty} c_n z^n.$$

*Then $G_{q,\alpha}(z) \in \mathcal{S}_q^*(\alpha)$. However, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_q^*(\alpha)$, then $|a_n| \leq c_n$ with equality holding for all $n$ if and only if $f$ is a rotation of $G_{q,\alpha}$.*

**Theorem 1.** *Let $\phi(z) = \sum_{n=1}^{\infty} \phi_n z^n$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z\exp(\phi(z)) \in \mathcal{S}_q^*(\alpha)$. Then*

$$|z| + \sum_{n=2}^{\infty} |a_n||z|^n \leq d(0, \partial f(\mathbb{D}))$$

*for $|z| \leq r^*$, where $r^* \in (0,1)$ is the unique root of the equation*

$$r\exp(F_{q,\alpha}(r)) = \exp(F_{q,\alpha}(-1)).$$

*The radius is sharp.*

**Proof.** Let $f \in \mathcal{S}_q^*(\alpha)$. Proceeding as in proof of [16] (Theorem 1.3, p. 8), it is easy to see that coefficients bound for the function $\phi(z) = \sum_{n=1}^{\infty} \phi_n z^n$ are given by

$$|\phi_n| \leq \frac{-2\ln\left(\frac{q}{1-\alpha(1-q)}\right)}{1-q^n}. \tag{9}$$

For $|z| = r \leq r^*$, using Lemma 3 and inequality (9), it follows that

$$\begin{aligned} d(0, \partial f(\mathbb{D})) = \lim_{|z| \to 1^-} \inf |f(z) - f(0)| = \lim_{|z| \to 1^-} \inf \frac{|f(z)|}{|z|} &\geq \exp F_{q,\alpha}(-1) \\ &\geq r \exp F_{q,\alpha}(r) \\ &= r \exp \left( \sum_{n=1}^{\infty} \frac{-2 \ln \left( \frac{q}{1-\alpha(1-q)} \right)}{1-q^n} r^n \right) \\ &\geq |z| + \sum_{n=2}^{\infty} |a_n||z|^n \end{aligned}$$

if and only if

$$r \exp(F_{q,\alpha}(r)) \leq \exp F_{q,\alpha}(-1).$$

In order to prove that the radius is sharp, let

$$G_{q,\alpha}(z) := z \exp(F_{q,\alpha}(z)),$$

where

$$F_{q,\alpha}(z) = \sum_{n=1}^{\infty} \frac{-2}{1-q^n} \ln \left( \frac{q}{1-\alpha(1-q)} \right) z^n, \quad z \in \mathbb{D}.$$

By Lemma 4, it follows that $G_{q,\alpha} \in \mathcal{S}_q^*(\alpha)$. For $|z| = r^*$, we obtain

$$\begin{aligned} |z| + \sum_{n=2}^{\infty} |a_n||z|^n &= r^* \exp \left( \sum_{n=1}^{\infty} \frac{-2}{1-q^n} \ln \left( \frac{q}{1-\alpha(1-q)} \right) (r^*)^n \right) \\ &= r^* \exp F_{q,\alpha}(r^*) \\ &= \exp F_{q,\alpha}(-1) \\ &= \lim_{|z| \to 1^-} \inf \frac{|G_{q,\alpha}(z)|}{|z|} \\ &= \lim_{|z| \to 1^-} \inf |G_{q,\alpha}(z) - f(0)| \\ &= d(0, G_{q,\alpha}(\mathbb{D})). \quad \square \end{aligned}$$

**Remark 2.** *For $\alpha = 0$, Theorem 1 yields the corresponding results found in [27] for the class $S_q^*$.*

**Remark 3.** *Theorem 1 with letting $q \to 1^-$ leads to the Bohr radius for the class of starlike functions of order $\alpha$, $0 \leq \alpha < 1$. Bhowmik and Das [28] (Theorem 3, p. 1093) found the Bohr radius for $S^*(\alpha)$ with $\alpha \in [0, 1/2]$.*

## 3. The Bohr Radius for the Class $C_q(\alpha)$

In the present section, we obtain the sharp Bohr radius for the class of $q$-convex functions of order $\alpha$, $0 \leq \alpha < 1$.

**Lemma 5** ([17] (Theorem 2.9, p. 5))**.** *Let*

$$E_q(z) := \int_0^z \exp(F_{q,\alpha}(t)) d_q t = z + \sum_{n=2}^{\infty} \left( \frac{1-q}{1-q^n} c_n z^n \right),$$

*where $c_n$ is the nth coefficient of the function $z \exp(F_{q,\alpha}(z))$. Then $E_q \in \mathcal{C}_q(\alpha)$ for $0 \leq \alpha < 1$. Moreover, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}_q(\alpha)$, then $|a_n| \leq ((1-q)/(1-q^n))c_n$, with equality holding for all n if and only if f is a rotation of $E_q$.*

**Theorem 2.** *The Bohr radius for the class $\mathcal{C}_q(\alpha)$ is $r^*$, where $r^* \in (0,1]$ is the unique root of the equation*

$$\int_0^r \exp(F_{q,\alpha}(t))d_qt = \int_0^1 \exp(F_{q,\alpha}(-t))d_qt.$$

*The radius is sharp.*

**Proof.** Let $f \in \mathcal{C}_q(\alpha)$. Then, by (3), $z(D_qf)(z) \in \mathcal{S}_q^*(\alpha)$. It follows from Lemma 3 that

$$\exp(F_{q,\alpha}(-r)) \le |(D_qf)(z)| \le \exp(F_{q,\alpha}(r)).$$

Taking $q$-integral of all the inequalities, we have

$$\int_0^r \exp(F_{q,\alpha}(-t))d_qt \le |f(z)| \le \int_0^r \exp(F_{q,\alpha}(t))d_qt. \tag{10}$$

Since $f(z) = z + \sum_{n=2}^{\infty} a_nz^n \in \mathcal{C}_q(\alpha)$, Lemma 5 yields the coefficients bound for the function $f$ given by

$$|a_n| \le \frac{1-q}{1-q^n}c_n, \tag{11}$$

where inequality holds for all $n$ if and only if $f$ is a rotation of

$$E_q(z) = \int_0^z \exp(F_{q,\alpha}(t))d_qt = z + \sum_{n=2}^{\infty}\left(\frac{1-q}{1-q^n}\right)c_nz^n$$

and where $c_n$ is the $nth$ coefficient of $z\exp(F_{q,\alpha}(z))$.

By (10) and (11), we have

$$\begin{aligned} r + \sum_{n=2}^{\infty}|a_n|r^n &\le r + \sum_{n=2}^{\infty}\frac{1-q}{1-q^n}c_nr^n \\ &= \int_0^r \exp(F_{q,\alpha}(t))d_qt \le \int_0^1 \exp(F_{q,\alpha}(-t))d_qt \le d(0,\partial f(\mathbb{D})) \end{aligned}$$

if and only if

$$\int_0^r \exp(F_{q,\alpha}(t))d_qt \le \int_0^1 \exp(F_{q,\alpha}(-t))d_qt.$$

Now, consider the function

$$E_q(z) := \int_0^z \exp(F_{q,\alpha}(t))d_qt = z + \sum_{n=2}^{\infty}\left(\frac{1-q}{1-q^n}\right)c_nz^n.$$

It follows from Lemma 5 that the function $E_q(z) \in \mathcal{C}_q(\alpha)$. At $|z| = r^*$, we have

$$\begin{aligned} r^* + \sum_{n=2}^{\infty}|a_n|(r^*)^n &= r^* + \sum_{n=2}^{\infty}\frac{1-q}{1-q^n}c_n(r^*)^n \\ &= \int_0^{r^*} \exp(F_{q,\alpha}(t))d_qt = \int_0^1 \exp(F_{q,\alpha}(-t))d_qt = d(0,\partial E_q(\mathbb{D})) \end{aligned}$$

which shows that the Bohr radius $r^*$ is sharp for the class $\mathcal{C}_q(\alpha)$. $\square$

Putting $\alpha = 0$ in Theorem 2, we obtain the Bohr radius for the class $\mathcal{C}_q$ of $q$-convex functions.

**Corollary 1** ([27] (Theorem 2, p. 111))**.** *The Bohr radius for the class $\mathcal{C}_q$ is $r^*$, where $r^* \in (0,1]$ is the unique root of*

$$\int_0^r \exp(F_{q,0}(t))d_q t = \int_0^1 \exp(F_{q,0}(-t))d_q t.$$

*The radius is sharp.*

If $q \to 1^-$, then Corollary 1 yields the Bohr radius for the class $\mathcal{C}$ of convex functions, that is , $r^* = 1/3$. The same Bohr radius for general convex functions had been earlier obtained by Aizenberg in [29] (Thoerem 2.1).

## 4. The Bohr Radius Problems for the Class $\mathcal{TP}_q(\lambda, A, B)$

In 1975, Silverman [30] investigated two new subclasses of the family $\mathcal{T}$, where

$$\mathcal{T} = \{f \in \mathcal{S} : f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, z \in \mathbb{D}\}.$$

Recently, Altıntaş and Mustafa [31] introduced a generalized class, $\mathcal{TP}_q(\lambda, A, B), q \in (0,1), \lambda \in [0,1], -1 \le B < A \le 1$, given by

$$\mathcal{TP}_q(\lambda, A, B) = \left\{ f \in \mathcal{T} : \frac{zD_q f(z) + \lambda z^2 D_q^2 f(z)}{\lambda z D_q f(z) + (1-\lambda) f(z)} \prec \frac{1+Az}{1+Bz}, z \in \mathbb{D} \right\}.$$

For $\lambda = 0$, this class reduces to the class $\mathcal{TS}_q^*[A, B]$ of $q$-Janowski starlike functions with negative coefficients defined by

$$\mathcal{TS}_q^*[A, B] = \left\{ f \in \mathcal{T} : \frac{zD_q f(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, z \in \mathbb{D} \right\}.$$

On the other hand, the case $\lambda = 1$ yields the class $\mathcal{TC}_q[A, B]$ of $q$-Janowski convex functions, defined by

$$\mathcal{TC}_q[A, B] = \left\{ f \in \mathcal{T} : 1 + \frac{zD_q^2 f(z)}{D_q f(z)} \prec \frac{1+Az}{1+Bz}, z \in \mathbb{D} \right\}.$$

As $q \to 1^-$, $\mathcal{TS}_q^*[A, B]$ and $\mathcal{TC}_q[A, B]$ reduce respectively to $\mathcal{TS}^*[A, B]$ and $\mathcal{TC}[A, B]$ studied initially in [32]. Note that the classes $\mathcal{TS}^*(\alpha) \equiv \lim\limits_{q \to 1^-} \mathcal{TS}_q^*[1-2\alpha, -1]$ and $\mathcal{TC}(\alpha) \equiv \lim\limits_{q \to 1^-} \mathcal{TC}_q[1-2\alpha, -1]$ were defined and studied by Silverman [30] in 1975.

In the present section, we will first investigate the sharp Bohr radius for the class $\mathcal{TP}_q(\lambda, A, B)$, $q \in (0,1), \lambda \in [0,1]$ which in particular gives the Bohr radius for the classes $\mathcal{TS}_q^*[A, B]$ and $\mathcal{TC}_q[A, B]$. However, in order to obtain Bohr radius, we first need some results given here in two lemmas.

Note that there is a typing error in the statement of [31] (Theorem 3.1, p. 993) (replace $\alpha$ by $\beta$). The correct statement in Lemma 6 is as follows:

**Lemma 6** ([31] (Theorem 3.1, p. 993))**.** *If $f \in \mathcal{TP}_q(\lambda, A, B)$, $q \in (0,1)$, $\lambda \in [0,1]$, then*

$$r - \frac{1-\beta}{([2]_q - \beta)[1 + ([2]_q - 1)\lambda]} r^2 \le |f(z)| \le r + \frac{1-\beta}{([2]_q - \beta)[1 + ([2]_q - 1)\lambda]} r^2$$

*where* $\beta = (1-A)/(1-B), -1 \le B < A \le 1$, *with equality for the function*

$$f(z) = z - \frac{1-\beta}{([2]_q-\beta)[1+([2]_q-1)\lambda]}z^2, |z| = r.$$

**Lemma 7** ([31] (Theorem 2.8, p. 991)). *If* $f \in \mathcal{TP}_q(\lambda, A, B)$, $q \in (0,1)$, $\lambda \in [0,1]$, *then the following conditions are satisfied:*

$$\sum_{n=2}^{\infty} |a_n| \le \frac{1-\beta}{([n]_q-\beta)(1+([n]_q-1)\lambda)}$$

$$\sum_{n=2}^{\infty} [n]_q |a_n| \le \frac{(1-\beta)[n]_q}{([n]_q-\beta)(1+([n]_q-1)\lambda)}, n = 2, 3, \cdots,$$

*where* $\beta = (1-A)/(1-B), -1 \le B < A \le 1$. *The results obtained here are sharp.*

**Theorem 3.** *If* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in \mathcal{TP}_q(\lambda, A, B)$ *where* $q \in (0,1)$, $\lambda \in [0,1]$, $\beta = (1-A)/(1-B)$ *and* $c = q(\lambda + 1 + q\lambda - \beta\lambda)$, *then*

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| \le d(0, \partial f(\mathbb{D}))$$

*for* $|z| < r^*$, *where*

$$r^* = \frac{2c}{1-\beta+c+\sqrt{4(1-\beta)c+(1-\beta+c)^2}}.$$

*The radius* $r^*$ *is the sharp Bohr radius for class* $\mathcal{TP}_q(\lambda, A, B)$.

**Proof.** It follows from Lemma 6 that the distance between the origin and the boundary of $f(\mathbb{D})$ satisfies the inequality

$$d(0, \partial f(\mathbb{D})) \ge 1 - \frac{1-\beta}{(1+q-\beta)(1+q\lambda)}. \tag{12}$$

The given $r^*$ is the root of the equation

$$r^* + \frac{(1-\beta)(r^*)^2}{(1+q-\beta)(1+q\lambda)} = 1 - \frac{1-\beta}{(1+q-\beta)(1+q\lambda)}.$$

For $0 < r \le r^*$, we have

$$r + \frac{(1-\beta)r^2}{(1+q-\beta)(1+q\lambda)} \le r^* + \frac{(1-\beta)(r^*)^2}{(1+q-\beta)(1+q\lambda)} = 1 - \frac{1-\beta}{(1+q-\beta)(1+q\lambda)}.$$

Using Lemma 7, it is easy to show that

$$\sum_{n=2}^{\infty} |a_n| \le \frac{1-\beta}{(1+q-\beta)(1+q\lambda)}.$$

The above inequality together with inequality (12) yield

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| \le r + \frac{1-\beta}{(1+q-\beta)(1+q\lambda)}r^2 \le 1 - \frac{1-\beta}{(1+q-\beta)(1+q\lambda)} \le d(0, \partial f(\mathbb{D})).$$

For sharpness, consider the function $f : \mathbb{D} \to \mathbb{C}$ defined by

$$f(z) = z - \frac{1-\beta}{(1+q-\beta)(1+q\lambda)}z^2.$$

This function clearly belongs to $\mathcal{TP}_q(\lambda, A, B)$. For $|z| = r^*$, we find

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| = r^* + \frac{1-\beta}{(1+q-\beta)(1+q\lambda)}(r^*)^2 = 1 - \frac{1-\beta}{(1+q-\beta)(1+q\lambda)} = d(0, \partial f(\mathbb{D})). \quad \square$$

Putting $\lambda = 0$ in Theorem 3, we get the sharp Bohr radius for the class $\mathcal{TS}_q^*[A, B]$.

**Theorem 4.** *If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in \mathcal{TS}_q^*[A, B]$, $\beta = (1-A)/(1-B)$ and $-1 \leq B < A \leq 1$, then*

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D}))$$

*for $|z| < r^*$, where*

$$r^* = \frac{2q}{1+q-\beta+\sqrt{1+6q+q^2-2\beta-6q\beta+\beta^2}}.$$

*The radius $r^*$ is sharp.*

Letting $A = 1 - 2\alpha$ and $B = -1$ in Theorem 4, we obtain the sharp Bohr radius for the class of $q$-starlike functions of order $\alpha$, $0 \leq \alpha < 1$, with negative coefficients.

**Corollary 2.** *Let $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in \mathcal{TS}_q^*(\alpha)$. Then*

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D}))$$

*for $|z| < r^*$, where*

$$r^* = \frac{2q}{1+q-\alpha+\sqrt{q^2+6q(1-\alpha)+(1-\alpha)^2}}.$$

When $q \to 1^-$ in Corollary 2, we obtain the following sharp Bohr radius for the class of starlike functions of order $\alpha$, $0 \leq \alpha < 1$, with negative coefficients obtained by Ali et al. [33].

**Corollary 3** ([33] (Theorem 2.3))**.** *If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in \mathcal{TS}^*(\alpha)$, then*

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D}))$$

*for $|z| < r^*$, where*

$$r^* = \frac{2}{2-\alpha+\sqrt{8-8\alpha+\alpha^2}}.$$

*The radius $r^*$ is the Bohr radius for $\mathcal{TS}^*(\alpha)$.*

When $A = 1$ and $B = -1$, Theorem 4 gives the following sharp Bohr radius for the class of $q$-starlike functions with negative coefficients.

**Corollary 4.** *If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in \mathcal{TS}_q^*$, then*

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D}))$$

*for $|z| < r^*$, where*

$$r^* = \frac{2q}{1+q+\sqrt{1+6q+q^2}}.$$

When $A = 1$, $B = -1$ and $q \to 1^-$, Theorem 4 gives the following sharp Bohr radius for the class of starlike functions with negative coefficients obtained by Ali et al. [33].

**Corollary 5** ([33])**.** *The sharp Bohr radius for the class* $\mathcal{TS}^*$ *is* $\sqrt{2} - 1 \simeq 0.414214$.

When $\lambda = 1$, Theorem 3 gives the following sharp Bohr radius for the class of $\mathcal{TC}_q[A, B]$.

**Theorem 5.** *If* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in \mathcal{TC}_q[A, B]$, $\beta = (1 - A)/(1 - B)$ *and* $-1 \leq B < A \leq 1$, *then*

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D}))$$

*for* $|z| < r^*$, *where*

$$r^* = \frac{2q(2 + q - \beta)}{1 + 2q + q^2 - \beta - q\beta + \sqrt{4(1 - \beta)\left(2q + q^2 - q\beta\right) + \left(q\beta - 1 - 2q - q^2 + \beta\right)^2}}.$$

*The result is sharp for the function*

$$f(z) = z - \frac{1 - \beta}{(1 + q - \beta)(1 + q)} z^2.$$

When $A = 1 - 2\alpha$ and $B = -1$, Theorem 5 gives the sharp Bohr radius for the class of $q$-convex functions with negative coefficients.

**Corollary 6.** *The sharp Bohr radius for the class* $\mathcal{TC}_q(\alpha)$ *is*

$$\frac{2q(2 + q - \alpha)}{1 + 2q + q^2 - \alpha - q\alpha + \sqrt{(1 + q)^2(1 + q - \alpha)^2 + 4q(2 + q - \alpha)(1 - \alpha)}}.$$

Letting $q \to 1^-$ in Corollary 6, we get the following sharp Bohr radius for the class of convex functions of order $\alpha$, $0 \leq \alpha < 1$, with negative coefficients obtained by Ali et al. [33].

**Corollary 7** ([33] (Theorem 2.4))**.** *If* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \in \mathcal{TC}(\alpha)$, *then*

$$|z| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{D}))$$

*for* $|z| < r^*$, *where*

$$\frac{3 - \alpha}{2 - \alpha + \sqrt{7 - 8\alpha + 2\alpha^2}}.$$

*The radius* $r^*$ *is the Bohr radius for* $\mathcal{TC}(\alpha)$.

For $A = 1$ and $B = -1$, Theorem 5 yields the sharp Bohr radius for the class of $q$-convex functions with negative coefficients.

**Corollary 8.** *The sharp Bohr radius for the class* $\mathcal{TC}_q$ *is*

$$\frac{2q(2 + q)}{1 + 2q + q^2 + \sqrt{1 + 12q + 10q^2 + 4q^3 + q^4}}.$$

Letting $q \to 1^-$, $A = 1$ and $B = -1$, Theorem 5 gives the sharp Bohr radius for the class of convex functions with negative coefficients by Ali et al. [33].

**Corollary 9** ([33])**.** *The sharp Bohr radius for the class $\mathcal{TC}$ is* $\sqrt{7} - 2 \simeq 0.645751$.

**Author Contributions:** Investigation, S.A. and N.K.J.; Methodology, O.A. and N.K.J.; Resources, O.A. and S.A.; Software, S.A.; Supervision, N.K.J.; Writing original draft, S.A.; Writing, review and editing, O.A. and N.K.J. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Acknowledgments:** The authors are grateful to the referees for their helpful suggestions and insights that helped to improve quality and clarity of this manuscript.

**Conflicts of Interest:** The authors declare no conflict of interest.

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