Article

Inertial Krasnoselskii–Mann Method in Banach Spaces

Yekini Shehu [1],[†] and **Aviv Gibali** [2,3,*,†]

[1]   Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China; yekini.shehu@unn.edu.ng
[2]   Department of Mathematics, ORT Braude College, 2161002 Karmiel, Israel
[3]   The Center for Mathematics and Scientific Computation, University of Haifa, Mt. Carmel, 3498838 Haifa, Israel
*   Correspondence: avivg@braude.ac.il
†   These authors contributed equally to this work.

check for updates

Abstract: In this paper, we give a general inertial Krasnoselskii–Mann algorithm for solving inclusion problems in Banach Spaces. First, we establish a weak convergence in real uniformly convex and q-uniformly smooth Banach spaces for finding fixed points of nonexpansive mappings. Then, a strong convergence is obtained for the inertial generalized forward-backward splitting method for the inclusion. Our results extend many recent and related results obtained in real Hilbert spaces.

Keywords: Krasnoselskii–Mann; nonexpansive mapping; forward-backward splitting method; q-uniformly smooth; Banach spaces

1. Introduction

Let X be a real Banach space and given a single and set-valued operators $A : X \rightarrow X$ and $B : X \rightarrow 2^X$, respectively. We consider the following inclusion problem:

$$\text{find } \hat{x} \in X \text{ such that } 0 \in A\hat{x} + B\hat{x}. \tag{1}$$

Such inclusion problems are quite general since it include as special cases various problems such as: non-smooth convex optimization problems, variational inequalities and convex-concave saddle-point problems, just to name a few. (see, e.g., [1–5]).

A known and popular method for solving problem (1) is the forward-backward splitting method [6,7], which is defined in the following manner: $x_1 \in X$ and

$$x_{n+1} = J_r^B(x_n - rAx_n), \ n \geq 1, \tag{2}$$

where provided that B is maximally monotone and A is co-coercive (or equivalent assumptions) and $J_r^B := (I + rB)^{-1}$, $r > 0$, is called the "resolvent of B". The forward-backward splitting method (2) includes the proximal point algorithm, (see, e.g., [8–12]), and the gradient method (see, for example, [2,13]). It has been shown that (2) in general only converges weakly to a zero of (1) (see, for example, [3,6,14,15]).

The following method was introduced in [16] (see also [14]) for finding zero of (1) when $A = 0$ and B is maximal monotone operator: $x_0, x_1 \in H$;

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = J_{r_n}^B(y_n), \ n \geq 1. \end{cases} \tag{3}$$

Alvarez and Attouch [16] established the weak convergence of (3) under some appropriate conditions on $\{\theta_n\}$ and $\{r_n\}$. Several other modifications of (2) with inertial extrapolation step have been considered in Hilbert spaces by many authors, see, for example, [17–21].

Based on the above mentioned results [19,22–26], our main contribution in this paper is the following. We extend the results of [17] concerning the inertial Krasnoselskii–Mann iteration for fixed point of nonexpansive mappings to uniformly convex and q-uniformly smooth Banach space. We also extend the forward-backward splitting method with inertial extrapolation step for solving (1) from Hilbert spaces to Banach spaces. While the mentioned results establish only weak convergence, we also provide strong convergence analysis in Banach spaces.

The outline of the paper is as follows. We first recall some basic definitions and results in Section 2. Our algorithms are presented and analysed in Section 3. In Section 4 an infinite dimensional example is presented and final remarks and conclusions are given in Section 5.

2. Preliminaries

Let X be a real Banach space. The *modulus of convexity* of X is defined as the function $\delta : (0, 2] \to [0, 1]$,

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in X, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}. \tag{4}$$

X is said to be *uniformly convex* if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

The *modulus of smoothness* of X is the function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\rho(t) = \sup \left\{ \frac{\|x + ty\| + \|x - ty\|}{2} - 1 : x, y \in X, \|x\| = \|y\| = 1 \right\}. \tag{5}$$

We say X is uniformly smooth if $\lim_{t \to 0} \rho(t)/t = 0$. X is said to be q-uniformly smooth with $1 < q \leq 2$, if there exists a constant $k_q > 0$ such that $\rho(t) \leq k_q t^q$ for $t > 0$. If X is q-uniformly smooth, then it is uniformly smooth (see, e.g., [27]). Suppose that X^* is the dual space of X. The generalized duality mapping $J_q(q > 1)$ of X is defined by $J_q(x) := \{ j_q(x) \in X^* : \langle x, j_q(x) \rangle = \|x\|^q, \|j_q(x)\| = \|x\|^{q-1} \}, \forall x \in X$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. In particular, we call $J_2 := J$, the *normalized duality mapping* on X. Furthermore, (see, e.g., [28] (p. 1128))

$$J_q(x) = \|x\|^{q-2} J(x), \ x \neq 0. \tag{6}$$

It is well known that (see, for example, [27]) X is uniformly smooth if and only if the duality mapping J_q is single-valued and norm-to-norm uniformly continuous on bounded subsets of X.

Let $B : X \to 2^X$. We denote the *domain* of B by $D(B) = \{ x \in X : Bx \neq \emptyset \}$ and its *range* by $R(B) = \bigcup \{ Bz : z \in D(B) \}$. We say that B is *accretive* if, for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that (see, for example, [25])

$$\langle u - v, j(x - y) \rangle \geq 0, \ u \in Bx, \ v \in By. \tag{7}$$

B is said to be *m-accretive* if $R(I + rB) = X$ for all $r > 0$. Given $\alpha > 0$ and $q \in (1, \infty)$, we say that a single-valued accretive operator A is *α-inverse strongly accretive* (α-isa, for short) of order q if, for each $x, y \in D(A)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Ax - Ay, j_q(x - y) \rangle \geq \alpha \|u - v\|^q. \tag{8}$$

We say that A is *α-strongly accretive* of order q if, for each $x, y \in D(A)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Ax - Ay, j_q(x - y) \rangle \geq \alpha \|x - y\|^q. \tag{9}$$

Let $\varnothing \neq C \subset X$ and $T : C \to C$ be a nonlinear mapping. The set of fixed points of T is defined by $Fix(T) := \{x \in C : x = Tx\}$.

For the rest of this paper, we shall adopt the following notation:

$$T_r^{A,B} = J_r^B(I - rA) = (I + rB)^{-1}(I - rA), \ r > 0. \tag{10}$$

Lemma 1 ([29] p. 33). *Let $q > 1$ and X be a real normed space with the generalized duality mapping J_q. Then, for any $x, y \in X$, we have*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y)\rangle \tag{11}$$

for all $j_q(x + y) \in J_q(x + y)$.

Lemma 2 ([28] Cor. $1'$). *Let $1 < q \leq 2$ and X be a smooth Banach space. Then the following statements are equivalent:*

(i) *X is q-uniformly smooth.*

(ii) *There is a constant $k_q > 0$ such that for all $x, y \in X$*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + k_q\|y\|^q. \tag{12}$$

The best constant k_q will be called the q-uniform smoothness coefficient of X.

Lemma 3 ([25] Lem. 3.1, 3.2). *Let X be a Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ an m-accretive operator. Then we have*

(i) *For $r > 0$, $Fix(T_r^{A,B}) = (A + B)^{-1}(0)$.*

(ii) *For $0 < s \leq r$ and $x \in X$, $\|x - T_s^{A,B}x\| \leq 2\|x - T_r^{A,B}x\|$.*

Lemma 4 ([25] Lem. 3.3). *Let X be a uniformly convex and q-uniformly smooth Banach space for some $q \in (1, 2]$. Assume that A is a single-valued α-isa of order q in X. Then, given $r > 0$, there exists a continuous, strictly increasing and convex function $\phi_q : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi_q(0) = 0$ such that, for all $x, y \in B_r$,*

$$\begin{aligned}\|T_r^{A,B}x - T_r^{A,B}y\|^q &\leq \|x - y\|^q - r(\alpha q - r^{q-1}k_q)\|Ax - Ay\|^q \\ &\quad -\phi_q(\|(I - J_r^B)(I - rA)x - (I - J_r^B)(I - rA)y\|),\end{aligned} \tag{13}$$

where k_q is the q-uniform smoothness coefficient of X.

Lemma 5 ([26] Lem. 3.1). *Let $\{a_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \ n \geq 1, \tag{14}$$

where $\{\delta_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a real sequence. Assume $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(i) *If $b_n \leq \delta_n M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*

(ii) *If $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n \to \infty} b_n/\delta_n \leq 0$, then $\lim_{n \to \infty} a_n = 0$.*

Lemma 6 (Maingé [30]). *Let $\{\varphi_n\}, \{\delta_n\}$ and $\{\theta_n\}$ be sequences in $[0, +\infty)$ such that*

$$\varphi_{n+1} \leq \varphi_n + \theta_n(\varphi_n - \varphi_{n-1}) + \delta_n, \ \forall n \geq 1, \ \sum_{n=1}^{+\infty} \delta_n < +\infty,$$

and there exists a real number θ with $0 \leq \theta_n \leq \theta < 1$ for all $n \in \mathbb{N}$. Then the following hold:

(i) *$\sum_{n=1}^{+\infty} [\varphi_n - \varphi_{n-1}]_+ < +\infty$, where $[t]_+ := \max\{t, 0\}$;*

(ii)   *there exists $\varphi^* \in [0, +\infty)$ such that $\lim_{n\to\infty} \varphi_n = \varphi^*$.*

Lemma 7 (Goebel and Reich [31])**.** *Let E be a uniformly convex Banach space and $C \subset E$ be nonempty, closed and convex and $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demi-closed at 0.*

Lemma 8 (Xu, [28])**.** *Let E be a uniformly convex Banach space. The following statements hold in E:*

$$\|x + y\|^q \le \|x\|^q + q\langle j_q(x), y\rangle + c_q\|y\|^q \forall x, y \in E.$$

Lemma 9 (Xu, [32])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \ \ n \ge 1,$$

where

(a) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $\limsup \sigma_n \le 0$;
(c) $\gamma_n \ge 0 \ (n \ge 1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$.

   Then, $a_n \to 0$ as $n \to \infty$.

   Notations: $x_n \rightharpoonup x, n \to \infty$ means $\{x_n\}$ converges weakly to x and $x_n \to x, n \to \infty$ means $\{x_n\}$ converges strongly to x.

3. The Algorithm

   In this section, we introduce our method and give the convergence analysis. Recall that ℓ_1 is the space of all sequences whose series is absolutely convergent.

   Let E be a uniformly convex Banach space and $T : E \to E$ a nonexpansive mapping and $Fix(T) \ne \emptyset$.

Remark 1. *Observe that since the value of $\|x_n - x_{n-1}\|$ is a priori known before θ_n, then Step (2) in Algorithm 1 is easily implemented. Furthermore, observe that by the assumption that $\{\epsilon_n\}_{n=1}^{\infty} \subset l_1$, we have that $\sum_{n=0}^{\infty} \epsilon_n\|x_n - x_{n-1}\| < \infty$ and $\sum_{n=0}^{\infty} \epsilon_n\|x_n - x_{n-1}\|^q < \infty$.*

Algorithm 1 Inertial Krasnoselskii–Mann iteration.

1: Choose sequence $\{\epsilon_n\} \subset \ell_1$ and pick $\theta \in [0, 1)$. Select $x_0, x_1 \in E$ and set $n := 1$.
2: Given the iterations x_n, x_{n-1}, choose θ_n such that $0 \le \theta_n \le \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_k - x_{k-1}\|^q}, \frac{\epsilon_n}{\|x_k - x_{k-1}\|}\right\}, & x_k \ne x_{k-1} \\ \theta, & \text{otherwise.} \end{cases}$$

3: Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}),$$

   and

$$x_{n+1} = (1 - \lambda_n)w_n + \lambda_n Tw_n.$$

4: Set $n \leftarrow n + 1$, and go to 2.

Convergence Analysis

   We start with the weak convergence analysis of Algorithm 1 for nonexpansive mappings. Throughout our analysis we assume that E be a uniformly convex Banach space.

Theorem 1. *Suppose $T : E \to E$ is a nonexpansive mapping and $Fix(T) \neq \emptyset$. Assume that $0 < a \leq \lambda_n \leq b < 1$. Then $\{x_n\}$ generated by Algorithm 1 converges weakly to a point in $F(T)$.*

Proof. Take $z \in F(T)$. Then

$$
\begin{aligned}
\|x_{n+1} - z\|^q &\leq \left((1 - \lambda_n)\|w_n - z\| + \lambda_n\|Tw_n - z\| \right)^q \\
&= \left((1 - \lambda_n)\|w_n - z\| + \lambda_n\|Tw_n - Tz\| \right)^q \\
&\leq \left((1 - \lambda_n)\|w_n - z\| + \lambda_n\|w_n - z\| \right)^q \\
&= \|w_n - z\|^q
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
\|w_n - z\|^q &= \|x_n - z + \theta_n(x_n - x_{n-1})\|^q \\
&\leq \|x_n - z\|^q + q\theta_n\langle x_n - x_{n-1}, j_q(x_n - z)\rangle + c_q\theta_n^q\|x_n - x_{n-1}\|^q.
\end{aligned}
\tag{16}
$$

Observe that

$$
q\langle x_n - x_{n-1}, j_q(x_n - z)\rangle \leq \|x_n - z\|^q - \|x_{n-1} - z\|^q + c_q\|x_n - x_{n-1}\|^q.
\tag{17}
$$

From (16) and (17), we have (noting that $\theta_n^q \leq \theta_n$)

$$
\begin{aligned}
\|w_n - z\|^q &\leq \|x_n - z\|^q + \theta_n(\|x_n - z\|^q - \|x_{n-1} - z\|^q) \\
&\quad + c_q(\theta_n + \theta_n^q)\|x_n - x_{n-1}\|^q \\
&\leq \|x_n - z\|^q + \theta_n(\|x_n - z\|^q - \|x_{n-1} - z\|^q) \\
&\quad + 2c_q\theta_n\|x_n - x_{n-1}\|^q.
\end{aligned}
\tag{18}
$$

It follows from (15) and (18) that

$$
\begin{aligned}
\|x_{n+1} - z\|^q &\leq \|x_n - z\|^q + \theta_n(\|x_n - z\|^q - \|x_{n-1} - z\|^q) \\
&\quad + 2c_q\theta_n\|x_n - x_{n-1}\|^q.
\end{aligned}
\tag{19}
$$

By Lemma 6, we deduce that $\{\|x_n - z\|\}$ is convergent. Thus, $\{x_n\}$ is bounded and $\sum_{n=1}^{\infty}[\|x_{n+1} - z\|^q - \|x_n - z\|^q]_+ < \infty$.

We next show that $\lim_{n \to \infty}\|Tw_n - w_n\| = 0$. From the update of x_{n+1} in Algorithm 1, we get

$$
\begin{aligned}
\|x_{n+1} - z\|^q &= \|(1 - \lambda_n)(w_n - z) + \lambda_n(Tw_n - z)\|^q \\
&\leq (1 - \lambda_n)\|w_n - z\|^q + \lambda_n\|Tw_n - z\|^q \\
&\quad - w_q(\lambda_n)\varphi(\|Tw_n - w_n\|) \\
&\leq \|w_n - z\|^q - w_q(\lambda_n)\varphi(\|Tw_n - w_n\|).
\end{aligned}
\tag{20}
$$

Using (18) in (20), we get

$$
\begin{aligned}
w_q(\lambda_n)\varphi(\|Tw_n - w_n\|) &\leq \|x_n - z\|^q - \|x_{n+1} - z\|^q \\
&\quad + \theta_n(\|x_n - z\|^q - \|x_{n-1} - z\|^q) \\
&\quad + 2c_q\theta_n\|x_n - x_{n-1}\|^q.
\end{aligned}
\tag{21}
$$

Also,

$$
\begin{aligned}
\|w_n - x_n\|^q &= \theta_n^q \|x_n - x_{n-1}\|^q \\
&\leq \theta_n \|x_n - x_{n-1}\|^q \to 0, n \to \infty.
\end{aligned} \tag{22}
$$

Since $\lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\|^q = 0$ and $\lim_{n\to\infty} \|x_n - z\|^q$ exists, we obtain from (21) that $\lim_{n\to\infty} w_q(\lambda_n) \varphi(\|Tw_n - w_n\|) = 0$. Since $\liminf_{n\to\infty} \lambda_n(1 - \lambda_n) > 0$, we get $\lim_{n\to\infty} \varphi(\|Tw_n - w_n\|) = 0$ and by the continuity of φ, we get $\lim_{n\to\infty} \|Tw_n - w_n\| = 0$.

Furthermore, since $\{x_n\}$ is bounded, there exists $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup p \in B$. By (22), we have that $w_{n_k} \rightharpoonup p \in B$. Using the demiclosedness of $I - T$ in Lemma 7, we get that $p \in F(T)$. By the results in [33], we have that $\{x_n\}$ has exactly one weak limit point and hence $\{x_n\}$ is weakly convergent. This ends the proof. □

Remark 2.

(a) We mention here that quasi-nonexpansiveness is a weaker sufficient condition for Theorem 1.

(b) It can also be shown in Theorem 1 that

$$
\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - w_n\| + \|w_n - Tw_n\| + \|Tw_n - Tx_n\| \\
&\leq 2\|x_n - w_n\| + \|w_n - Tw_n\| \to 0, \; n \to \infty.
\end{aligned}
$$

Therefore, Algorithm 1 preserves certain properties of the Krasnoselskii–Mann iteration.

Now taking $T := T_r^{A,B}$ in Algorithm 1, we obtain the following result for inclusion problem (1).

Theorem 2. *Let E be a uniformly convex and q-uniformly smooth Banach Space. Suppose that $A : E \to E$ is $\alpha-$isa of order q and $B : E \to 2^E$ an $m-$accretive operator. Assume that the solution set S of inclusion problem (1) nonempty. Let $r \in \left(0, \left(\frac{\alpha q}{c_q}\right)^{\frac{1}{q-1}}\right)$. Then the sequence $\{x_n\}$ generated by Algorithm 1 with $T := T_r^{A,B}$ converges weakly to a point in S.*

Proof. By Lemma 3 (i) and Lemma 4, we have that $Fix(T_r^{A,B}) = (A + B)^{-1}(0) = S$ and $T_r^{A,B}$ is nonexpansive. Therefore, by Theorem 1, we have that $\{x_n\}$ converges weakly to a point in S and the desired result is obtained. □

We give two instances of strong convergence of the relaxed forward–backward Algorithm 1.

Theorem 3. *Let E be a uniformly convex and q-uniformly smooth Banach Space. Assume that the solution set S of inclusion problem (1) nonempty and $\{\lambda_n\} \subseteq (0,1)$ is such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. Let $r \in \left(0, \left(\frac{\alpha q}{c_q}\right)^{\frac{1}{q-1}}\right)$. Suppose that one of the following holds:*

(i) *A is α-isa of order q, B is β-strongly accretive of order q, and $r \in \left(0, \left(\frac{\alpha q}{c_q}\right)^{\frac{1}{q-1}}\right)$.*

(ii) *$\beta \leq L$, A is β-strongly accretive and L-Lipschitz on E with $r \in \left(0, \frac{2\beta}{L^2}\right)$.*

Then $\{x_n\}$ generated by Algorithm 1 with $T := T_r^{A,B}$ converges strongly to a unique point in S.

Proof. We first show that the inclusion problem (1) has a unique solution by showing that in each of the cases above $T_r^{A,B}$ is a contraction map on E.

(i) For all $x, y \in E$, we have

$$
\begin{aligned}
\|(I - rA)x - (I - rA)y\|^q &= \|x - y - r(Ax - Ay)\|^q \\
&\leq \|x - y\|^q + c_q r^q \|Ax - Ay\|^q - rq\langle Ax - Ay, j_q(x - y)\rangle \\
&\leq \|x - y\|^q - r(\alpha q - r^{q-1} c_q)\|Ax - Ay\|^q \\
&\leq \|x - y\|^q.
\end{aligned}
$$

Therefore, $I - rA$ is a nonexpansive mapping. Let $x, y, u, v \in E$. Since B is β-strongly accretive of order q, we have that

$$
(u, v) \in (J_r^B x, J_r^B y) \Leftrightarrow (x - u, y - v) \in Bu \times Bv \Rightarrow \langle (x - u) - (y - v), j_q(u - v)\rangle \geq \beta\|u - v\|^q
$$

$$
\Leftrightarrow \langle x - y, j_q(u - v)\rangle \geq (\beta + 1)\|u - v\|^q.
$$

Hence,

$$
\begin{aligned}
(\beta + 1)\|J_r^B x - J_r^B y\|^q &\leq \langle x - y, j_q(J_r^B x - J_r^B y)\rangle \\
&\leq \|x - y\|\|j_q(J_r^B x - J_r^B y)\| \\
&= \|x - y\|\|J_r^B x - J_r^B y\|^{q-1}.
\end{aligned}
$$

Therefore, $\|J_r^B x - J_r^B y\| \leq \frac{1}{\beta+1}\|x - y\|$. So,

$$
\|J_r^B(I - rA)x - J_r^B(I - rA)y\| \leq \frac{1}{\beta + 1}\|x - y\| = \tau\|x - y\|.
$$

(ii) Observe that $r(\beta q - c_q r^{q-1} L^q) \in (0, 1)$ and define $\tau := \left[1 - r(\beta q - c_q r^{q-1} L^q)\right]^{\frac{1}{q}}$. Then for all $x, y \in E$,

$$
\begin{aligned}
\|J_r^B(I - rA)x - J_r^B(I - rA)y\|^q &\leq \|(I - rA)x - (I - rA)y\|^q \\
&= \|x - y - r(Ax - Ay)\|^q \\
&\leq \|x - y\|^q - qr\langle Ax - Ay, j_q(x - y)\rangle + c_q r^q \|Ax - Ay\|^q \\
&\leq \|x - y\|^q - qr\beta\|x - y\|^q\rangle + c_q r^q L^q \|x - y\|^q \\
&= (1 - r(\beta q - c_q r^{q-1} L^q))\|x - y\|^q.
\end{aligned}
$$

Therefore, in both cases (i) and (ii), $T_r^{A,B}$ is a contraction map on E with constant τ.

Each of these cases in (i) and (ii) above implies that the inclusion problem (1) has a unique solution $x^* \in S$. Consequently, using the update of x_{n+1} in Algorithm 1 with $T = T_r^{A,B}$, we get

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1 - \lambda_n)\|w_n - x^*\| + \lambda_n \tau\|w_n - x^*\| \\
&= (1 - \lambda_n(1 - \tau))\|w_n - x^*\| \\
&\leq (1 - \lambda_n(1 - \tau))(\|x_n - x^*\| + \theta_n\|x_n - x_{n-1}\|) \\
&= (1 - \lambda_n(1 - \tau))\|x_n - x^*\| + \theta_n\|x_n - x_{n-1}\|.
\end{aligned} \tag{23}
$$

Observe that by the update of $\bar{\theta}_n$ in Algorithm 1, we have $\sum_{n=1}^{\infty} \theta_n\|x_n - x_{n-1}\| < \infty$, using Lemma 9, we get that $x_n \to x^*, n \to \infty$, and the proof is complete. $\quad\square$

We next present a complexity bound for Algorithm 1 in this result.

Theorem 4. *Suppose that either of condition (i) or (ii) in Theorem 3 is satisfied and let $x^* \in S$ be the unique solution of the inclusion problem (1). Let $\lambda_n = \lambda$ and $\epsilon_n = \epsilon$ be constant. Then, given $\rho \in (0, \lambda(1 - \tau))$, for any*

$$n \geq \bar{n} := \left\lfloor \log_{(1-\rho)} \left(\left(\frac{\epsilon}{\|x_0 - x^*\|} \right) \left(\frac{1 - \lambda(1 - \tau)}{\lambda(1 - \tau) - \rho} \right) \right) \right\rfloor,$$

assuming $\bar{n} \geq 0$, it holds that

$$\|x_n - x^*\| \leq \epsilon \left(\frac{1 - \lambda(1 - \tau)}{\lambda(1 - \tau) - \rho} + 1 \right), \tag{24}$$

where

(i) $\tau := \frac{1}{\beta r + 1}$ *if A is $\alpha - isa$ of order q, B is β-strongly accretive of order q, and $r \in (0, 2\alpha)$ and*

(ii) $\tau := \sqrt{1 - r(2\beta - rL^2)}$ *if $\beta \leq L$, A is β-strongly accretive and L-Lipschitz on E with $r \in \left(0, \frac{2\beta}{L^2} \right)$.*

Proof. From the proof of Theorem 3, for any $n \geq 1$ we get

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1 - \lambda(1 - \tau))(\|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\|) \\
&\leq (1 - \lambda(1 - \tau))(\|x_n - x^*\| + \epsilon).
\end{aligned} \tag{25}
$$

Without the loss of generality, we assume that for every $n < \bar{n}$ we have

$$\|x_n - x^*\| \geq \epsilon \frac{1 - \lambda(1 - \tau)}{\lambda(1 - \tau) - \rho}. \tag{26}$$

Concatenating (25) and (26) we obtain, for every $k < \bar{k}$,

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1 - \lambda(1 - \tau)) \left(1 + \frac{\lambda(1 - \tau) - \rho}{1 - \lambda(1 - \tau)} \right) \|x_n - x^*\| \\
&= (1 - \rho) \|x_n - x^*\|.
\end{aligned} \tag{27}
$$

Therefore, by the definition of $\bar{n}$, it holds that

$$\|x_{\bar{n}} - x^*\| \leq (1 - \rho)^{\bar{n}} \|x_0 - x^*\| \leq \epsilon \frac{1 - \lambda(1 - \tau)}{\lambda(1 - \tau) - \rho}.$$

For any $n > \bar{n}$ there are two possibilities. If

$$\|x_{n-1} - x^*\| \leq \epsilon \frac{1 - \lambda(1 - \tau)}{\lambda(1 - \tau) - \rho},$$

then, by (25) and recalling that $(1 - \lambda(1 - \tau)) \leq 1$, we obtain that x_n satisfies (24). Otherwise, if

$$\epsilon \frac{1 - \lambda(1 - \tau)}{\lambda(1 - \tau) - \rho} \leq \|x_{n-1} - x^*\| \leq \epsilon \left(\frac{1 - \lambda(1 - \tau)}{\lambda(1 - \tau) - \rho} + 1 \right),$$

then

$$\|x_n - x^*\| \leq (1 - \rho) \|x_{n-1} - x^*\| \leq \|x_{n-1} - x^*\|,$$

and the desired result holds.　□

Remark 3. *We observe that, in contradiction with the assumptions of Theorem 2, in Theorem 4 the summability of $\{\epsilon_n\}$ is not required. However if one wants a good bound in (24) then a small value of ϵ must be set, but, in this case, small values of θ_n are allowed.*

To summarize and emphasize the novelty and major advantages of our proposed scheme, we list next several relations to recent works.

Remark 4.

1. *Our result in Theorem 1 extends the results in [17,26,30,34,35] from Hilbert spaces to uniformly convex and q-uniformly smooth Banach spaces. Furthermore, when $\theta_n = 0$ in Algorithm 1, Theorem 1 reduces to the results in [33] and other related papers.*
2. *Our Theorem 2 extends the results in [16,19,21,22,24,36] from Hilbert spaces to uniformly convex and q-uniformly smooth Banach spaces.*
3. *Shehu in [37] obtained a nonasymptotic $O(1/n)$ convergence rate result for a Krasnoselskii–Mann iteration with inertial extrapolation step in real Hilbert spaces under the stringent condition of Boţ et al. [17] (Theorem 5). In this paper, we obtain the results for Krasnoselskii–Mann iteration with inertial extrapolation step under easy assumptions and give some complexity results in uniformly convex Banach spaces.*
4. *Themelis and Patrinos in [38] study a Newton-type generalization of the classical Krasnoselskii–Mann iteration in Hilbert spaces and obtained superlinear convergence when the direction satisfies Dennis-More condition in Hilbert spaces. However, Themelis and Patrinos in [38] do not consider Krasnoselskii–Mann iteration with inertial steps. Our results here involve inertial Krasnoselskii–Mann iteration obtained in a higher space viz, uniformly convex Banach space which extends Hilbert space.*
5. *In [39], Phon-on et al. established inertial S-iteration in Banach spaces and obtained convergence under boundedness of some generated sequence. In this paper, the boundedness assumption of any generated sequence is dispensed with in our results. Therefore, our results improve on the results of this paper.*

4. Numerical Illustration

In this section, we present two numerical examples in order to illustrate the behaviour of our proposed method. For the first example we are concern with the split convex feasibility problem (SCFP) (Censor and Elfving [40]) in an infinite-dimensional Hilbert space. Let H_1 and H_2 be two real Hilbert spaces and $T : H_1 \to H_2$ a bounded and linear operator and T^* its adjoint. Let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex sets. The split convex feasibility problem is formulated as follows:

$$\text{find a point } x \in C \text{ such that } Tx \in Q.$$

So, if we take $Ax := \nabla\left(\frac{1}{2}||Tx - P_Q Tx||^2\right) = T^*(I - P_Q)Tx$, where P_Q is the metric projection onto Q, ∇ is the gradient and $B = \partial i_C$ is the characteristic function of the set C ($i_C(x) = 0$ if $x \in C$ and $i_C(x) = \infty$ if $x \notin C$). So, the SCFP has a inclusion structure as in (1). It can be seen that A is Lipschitz continuous with constant $L = ||T||^2$ and B is maximal monotone, see e.g., [41].

Example 1. *Let $H_1 = L_2([0, 2\pi]) = H_2$ and norm $||x|| := \left(\int_0^{2\pi} |x(t)|^2 dt \right)^{\frac{1}{2}}$ and inner product $\langle x, y \rangle := \int_0^{2\pi} x(t)y(t)dt, \ \forall x, y \in H$. Consider the half-space*

$$C := \{x \in L_2([0, 2\pi]) \mid \langle \mathbf{1}, x \rangle \leq 1\} = \left\{ x \in L_2([0, 2\pi]) \mid \int_0^{2\pi} x(t)dt \leq 1 \right\} \tag{28}$$

where $\mathbf{1} \equiv 1 \in L_2([0, 2\pi])$. In addition, let the closed ball centered at $\sin \in L_2([0, 2\pi]$ with radius 4.

$$Q := \{x \in L_2([0, 2\pi]) \mid ||x - \sin||_{L_2}^2 \leq 16\} = \left\{ x \in L_2([0, 2\pi]) \mid \int_0^{2\pi} |x(t) - \sin(t)|^2 dt \leq 16 \right\}. \tag{29}$$

Consider the mapping $T : L^2([0, 2\pi]) \to L^2([0, 2\pi])$ such that $(Tx)(s) = x(s)$, $\forall x \in L^2([0, 2\pi])$. Then $(T^*x)(s) = x(s)$ and $\|T\| = 1$. So, we wish to solve the following problem:

$$\text{find } x^* \in C \text{ such that } Tx^* \in Q. \tag{30}$$

Observe since $(Tx)(s) = x(s)$, $\forall x \in L^2([0, 2\pi])$, (30) reduces to the well-known convex feasibility problem of the form.

$$\text{find } x^* \in C \cap Q. \tag{31}$$

Moreover, the solution set of (30) is nonempty since clearly $x(t) = 0$ is a solution. As explained before, we define $Ax := \nabla\left(\frac{1}{2}\|Tx - P_Q Tx\|^2\right) = T^*(I - P_Q)Tx$ and $B = \partial i_C$ and translate (30) to an inclusion formulation as in (1).

We implement our algorithm with different starting point $x_0(t) = x_1(t)$, $t \in [0, 2\pi]$. We choose the stopping criterion $\|x_n - y_n\| < 10^{-5}$ and other parameters are chosen as $\varepsilon_n = 1/n^2, \lambda_n = 1/n, \theta = 0.5, r = 0.5$. To justify our algorithm's name we compare it with the standard Krasnoselskii–Mann, which is the update of x_{n+1} in Algorithm 1 with w_n replaced by x_n and $\lambda_n \in (0, 1)$. The results for different starting points are presented in Table 1.

Recall the definition of the operator $T_r^{A,B}$ (10) and following [41] (Example 23.4) and [42] we get the following results. For $z \in L_2([0, 2\pi]$ we have

$$(I + \lambda_n B)^{-1}(z) = (I + \lambda_n \partial i_C)^{-1}(z) = \arg\min_{u \in L^2([0,2\pi])} \left\{ i_C(u) + \frac{1}{2\lambda_n}\|u - z\|_{L_2}^2 \right\} = P_C(z). \tag{32}$$

Moreover, by [42]

$$P_C(z) = \begin{cases} \frac{1 - \int_0^{2\pi} z(t)dt}{4\pi^2} 1 + z, & \int_0^{2\pi} z(t)dt > 1 \\ \\ z, & \int_0^{2\pi} z(t)dt \le 1. \end{cases}$$

For $w \in L_2([0, 2\pi]$ we also have

$$P_Q(w) = \begin{cases} \sin + \frac{4}{\sqrt{\int_0^{2\pi} |w(t) - \sin(t)|^2 dt}}(w - \sin), & \int_0^{2\pi} |w(t) - \sin(t)|^2 dt > 16 \\ \\ w, & \int_0^{2\pi} |w(t) - \sin(t)|^2 dt \le 16. \end{cases}$$

Table 1. Comparison of Algorithm 1 and the classical Krasnoselskii–Mann method.

Starting Points	CPU Time		Iterations	
	Algorithm 1	Classical KM	Algorithm 1	Classical KM
$x_0 = x_1 = \frac{t^2}{10}$ 0.054	0.201	7	17	
$x_0 = x_1 = \frac{2t}{16}$ 0.056	0.254	11	28	
$x_0 = x_1 = 2\sin(5t)^4 - 3\cos(-2t)$ 0.0653	0.103	4	15	
$x_0 = x_1 = \frac{t^2 - \exp(-t)}{525}$ 0.0732	0.142	5	15	
$x_0 = x_1 = \frac{1}{2}\exp\left(\frac{t}{3}\right)^{\frac{3}{2}}$	0.103	0.243	9	14

Example 2. *Take* $E = L_p([0, 2\pi]), 2 \le p < \infty$. *Then, E is 2-uniformly smooth and uniformly convex and so* $q = 2$ *in Algorithm 1. Define* $(Tx)(s) := \max\{-x(s), 0\}$ $\forall x \in L_p([0, 2\pi])$. *Then, T is nonexpansive and* $F(T) = 0$. *In the below numerical illustration we choose* $p = 4, 10, 100$, *starting points* $x_0 = x_1 = 2\sin(5t)$ *and other parameters as in the previous example. Based on the example setting and Remark 4 we find Shehu's*

algorithm [37] (Equation (3)) most suitable for comparison with our Algorithm 1. The results are reported next in Table 2.

Table 2. Comparison of Algorithm 1 and Shehu's Algorithm.

	CPU Time		$\|x\|_{L_p}$	
p	**Algorithm 1**	**Shehu's Algorithm**	**Algorithm 1**	**Shehu's Algorithm**
4	1.1250	1.2813	11.7172	11.7172
10	0.9531	1.3906	3.8892	4.8892
1000	0.0625	0.2500	2.1135	2.8135

5. Conclusions

In this paper, we give weak and strong convergence results for relaxed inertial forward-backward splitting method in uniformly convex and q-uniformly smooth Banach spaces under some appropriate conditions. Our results are new in Banach spaces, and generalize some existing results in the literature. In our future project, we will generalize our results in this paper to finding zero of maximal monotone operators in a more general Banach space.

Author Contributions: Y.S. and A.G. contributed equally to the this paper with regards to all aspects such as: conceptualization, methodology, software, validation, formal analysis, investigation, resources, data curation, writing—original draft preparation, writing—review and editing, visualization, supervision, project administration, and funding acquisition. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

isa　　　inverse strongly accretive
SCFP　split convex feasibility problem

References

1. Beck, A.; Teboulle, M. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM J. Imaging Sci.* **2009**, *2*, 183–202. [CrossRef]
2. Bertsekas, D.P.; Tsitsiklis, J.N. *Parallel and Distributed Computation: Numerical Methods*; Athena Scientific: Belmont, MA, USA, 1997.
3. Combettes, P.L.; Wajs, V.R. Signal recovery by proximal forward-backward splitting. *Multiscale Model. Simul.* **2005**, *4*, 1168–1200. [CrossRef]
4. Duchi, J.; Shalev-Shwartz, S.; Singer, Y.; Chandra, T. Efficient projections onto the l_1-ball for learning in high dimensions. In Proceedings of the 25th International Conference on Machine Learning, Helsinki, Finland, 5–9 July 2008.
5. Tibshirami, R. Regression shrinkage and selection via lasso. *J. Roy. Statist. Soc. Ser. B* **1996**, *58*, 267–288.
6. Lions, P.L.; Mercier, B. Splitting algorithms for the sum of two nonlinear operators. *SIAM J. Numer. Anal.* **1979**, *16*, 964–979. [CrossRef]
7. Passty, G.B. Ergodic convergence to a zero of the sum of monotone operators in Hilbert space. *J. Math. Anal. Appl.* **1979**, *72*, 383–390. [CrossRef]
8. Brézis, H.; Lions, P.L. Produits infinis de resolvantes. *Isr. J. Math.* **1978**, *29*, 329–345. [CrossRef]
9. Chen, G.H.G.; Rockafellar, R.T. Convergence rates in forward-backward splitting. *SIAM J. Optim.* **1997**, *7*, 421–444. [CrossRef]

10. Güler, O. On the convergence of the proximal point algorithm for convex minimization. *SIAM J. Control Optim.* **1991**, *29*, 403–419. [CrossRef]

11. Martinet, B. Régularisation d'inéquations variationnelles par approximations successives. *Rev. Française Informat. Recherche. Opérationnelle.* **1970**, *4*, 154–158.

12. Rockafellar, R.T. Monotone operators and the proximal point algorithm. *SIAM J. Control Optim.* **1976**, *14*, 877–898. [CrossRef]

13. Dunn, J.C. Convexity, monotonicity, and gradient processes in Hilbert space. *J. Math. Anal. Appl.* **1976**, *53*, 145–158. [CrossRef]

14. Polyak, B.T. Some methods of speeding up the convergence of iteration methods. *USSR Comput. Math. Math. Phys.* **1964**, *4*, 1–17. [CrossRef]

15. Tseng, P. A modified forward-backward splitting method for maximal monotone mappings. *SIAM J. Control Optim.* **2000**, *38*, 431–446. [CrossRef]

16. Alvarez, F.; Attouch, H. An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. *Set Valued Anal.* **2001**, *9*, 3–11. [CrossRef]

17. Bot, R.I.; Csetnek, E.R.; Hendrich, C. Inertial Douglas–Rachford splitting for monotone inclusion problems. *Appl. Math. Comput.* **2015**, *256*, 472–487.

18. Chen, C.; Chan, R.H.; Ma, S.; Yang, J. Inertial proximal ADMM for linearly constrained separable convex optimization. *SIAM J. Imaging Sci.* **2015**, *8*, 2239–2267. [CrossRef]

19. Lorenz, D.A.; Pock, T. An inertial forward-backward algorithm for monotone inclusions. *J. Math. Imaging Vis.* **2015**, *51*, 311–325. [CrossRef]

20. Moudafi, A.; Oliny, M. Convergence of a splitting inertial proximal method for monotone operators. *J. Comput. Appl. Math.* **2003**, *155*, 447–454. [CrossRef]

21. Pesquet, J.-C.; Pustelnik, N. A parallel inertial proximal optimization methods. *Pac. J. Optim.* **2012**, *8*, 273–305.

22. Attouch, H.; Alexandre, C. Convergence of a relaxed inertial forward–backward algorithm for structured monotone inclusions. *Appl. Math. Optim.* **2019**, *80*, 547–598. [CrossRef]

23. Cholamjiak W.; Cholamjiak P.; Suantai S. An inertial forward-backward splitting method for solving inclusion problems in Hilbert spaces. *J. Fixed Point Theory Appl.* **2018**, *20*, 42. [CrossRef]

24. Dong, Q.L.; Jiang, D.; Cholamjiak, P.; Shehu, Y. A strong convergence result involving an inertial forward-backward algorithm for monotone inclusions. *J. Fixed Point Theory Appl.* **2017**, *19*, 3097–3118. [CrossRef]

25. López, G.; Martín-Márquez, V.; Wang, F.; Xu H.K. Forward-Backward splitting methods for accretive operators in Banach spaces. *Abstr. Appl. Anal.* **2012**, *2012*, 109236. [CrossRef]

26. Maingé, P.E. Approximation method for common fixed points of nonexpansive mappings in Hilbert spaces. *J. Math. Anal. Appl.* **2007**, *325*, 469–479. [CrossRef]

27. Cioranescu, I. *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*; Kluwer Academic Publishers: Berlin, Germany, 1990.

28. Xu, H.K. Inequalities in Banach spaces with applications. *Nonlinear Anal.* **1991**, *16*, 1127–1138. [CrossRef]

29. Chidume, C. *Geometric Properties of Banach Spaces and Nonlinear Iterations*; Springer: Berlin, Germany, 2009.

30. Maingé, P.E. Convergence theorems for inertial KM-type algorithms. *J. Comput. Appl. Math.* **2008**, *219*, 223–236. [CrossRef]

31. Goebel, K.; Kirk, W.A. *Topics in Metric Fixed Point Theory*; Cambridge University Press: Cambridge, UK, 1990.

32. Xu, H.K. Iterative algorithms for nonlinear operators. *J. London. Math. Soc.* **2002**, *66*, 240–256. [CrossRef]

33. Reich, S. Weak convergence theorems for nonexpansive mappings in Banach spaces. *J. Math. Anal. Appl.* **1979**, *67*, 274–276. [CrossRef]

34. Dong, Q.L.; Yuan, H.B.; Cho, Y.J.; Rassias, T.M. Modified inertial Mann algorithm and inertial CQ-algorithm for nonexpansive mappings. *Optim. Lett.* **2018**, *12*, 87–102. [CrossRef]

35. Maingé, P.E. Inertial iterative process for fixed points of certain quasi-nonexpansive mappings. *Set Valued Anal.* **2007**, *15*, 67–79. [CrossRef]

36. Bot, R.I.; Csetnek, E.R. An inertial forward-backward-forward primal-dual splitting algorithm for solving monotone inclusion problems. *Numer. Algorithm* **2016**, *71*, 519–540. [CrossRef]

37. Shehu, Y. Convergence rate Analysis of inertial Krasnoselskii-Mann-type iteration with applications. *Numer. Funct. Anal. Optim.* **2018**, *39*, 1077–1091. [CrossRef]

38. Themelis, A.; Patrinos, P: SuperMann: A superlinearly convergent algorithm for finding fixed points of nonexpansive operators. *IEEE Trans. Automat. Control* **2019**, *64*, 4875–4890. [CrossRef]

39. Phon-on, A.; Makaje, N.; Sama-Ae, A.; Khongraphan, K. An inertial S-iteration process. *Fixed Point Theory Appl.* **2019**, *4*. [CrossRef]

40. Censor, Y.; Elfving, T. A multiprojection algorithm using Bregman projections in a product space. *Numer. Algorithms* **1994**, *8*, 221–239. [CrossRef]

41. Bauschke, H.H.; Combettes, P.L. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*; CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC; Springer: New York, NY, USA, 2011; ISBN 978-1-4419-9466-0.

42. Cegielski, A. *Iterative Methods for Fixed Point Problems in Hilbert Spaces*; Lecture Notes in Mathematics 2057; Springer: Berlin, Germany, 2012.