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Article

The Dirichlet Problem of Hessian Equation in Exterior Domains

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Abstract: In this paper, we will obtain the existence of viscosity solutions to the exterior Dirichlet problem for Hessian equations with prescribed asymptotic behavior at infinity by the Perron's method. This extends the Ju–Bao results on Monge–Ampère equations $\det D^2u = f(x)$.

Keywords: exterior Dirichlet problem; asymptotic behavior; Hessian equation; viscosity solution

1. Introduction

In this paper, we shall study the exterior Dirichlet problem of Hessian equation

$$S_k(D^2u) = f(x), \qquad x \in \mathbb{R}^n \setminus \overline{\Omega} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded, strictly convex set and $0 \in \Omega$, $f \in C^0(\mathbb{R}^n)$ is a positive function.

If $k = 1$, (1) is reduced to Poisson's equation $\Delta u = f(x)$. If $k \geq 2$, (1) is fully nonlinear elliptic. When $k = n$, we can derive the Monge–Ampère equation $\det D^2u = f(x)$. There are many results of interior Hessian equations, see [1–7] and the correlative literatures. For example, Caffarelli–Nirenberg–Spruck [2] obtained the existence result for the interior Dirichlet problem of Hessian equations.

Besides the interior Dirichlet problems, the exterior Dirichlet problem has also been extensively studied. The exterior Dirichlet problem is closely related to the classical theorem of Jörgens ($n = 2$ [8]), Calabi ($n \leq 5$ [9]), and Pogorelov ($n \geq 2$ [10]) which states that any classical convex solution of $\det D^2u = 1$ in $\mathbb{R}^n$ must be a quadratic polynomial. A more simplified and analytical proof of the theorem was obtained by Cheng–Yau [11]. Jost–Xin [12] proved the theorem in different ways. Later, Caffarelli [13] generalized the conclusion to the viscosity solution. However Trudinger–Wang [14] showed that if $u \in C^2(\Omega)$ is a convex function of $\det D^2u = 1$ and Ω is a convex set of $\mathbb{R}^n$with $\lim_{x\to\partial\Omega} u(x) = \infty$, then u is quadratic and $\Omega = \mathbb{R}^n$.

In 2003, Caffarelli–Li [15] proved the existence result to Monge–Ampère equation in $\mathbb{R}^n (n \geq 3)$. That is, let

$$\begin{cases} \det D^2u &= 1, & x \in \mathbb{R}^n \setminus \Omega, \\ u &= \tilde{\varphi}(x), & x \in \partial\Omega, \end{cases} \tag{2}$$

where $\tilde{\varphi} \in C^2(\partial\Omega)$. Let $\tilde{b} \in \mathbb{R}^n, \tilde{c} \in \mathbb{R}$ and

$$\mathcal{A} = \{A \text{ is a real } n \times n \text{ symmetric positive definite matrice satisfying } \det A = 1.\}$$

There is some constant c^* and $A \in \mathcal{A}$, c^* depends on $n, \Omega, \tilde{\varphi}, \tilde{b}$, for every $\tilde{c} > c^*$, then the problem (2) has a function $u \in C^\infty(\mathbb{R}^n \backslash \overline{\Omega}) \cap C^0(\mathbb{R}^n \backslash \Omega)$ satisfying the asymptotic behavior

$$\limsup_{|x| \to \infty} (|x|^{n-2} |u(x) - (\frac{1}{2} x' A x + \tilde{b} \cdot x + \tilde{c})|) < \infty. \tag{3}$$

If $n = 2$, by complex variable methods, Ferrer et al. [16,17] investigated the Dirichlet problem earlier. Then the exterior Dirichlet problem of Monge–Ampère equation was investigated by [18–23] and related literatures. For instance, in [20], Ju–Bao proved the existence result to $\det D^2 u = f(x)$ with $f = f_0(|x|) + O(|x|^{-\beta})$, $\beta > 0$ on exterior domains. Motivated by [15], the second author and Bao [24] first studied the Dirichlet problem of Hessian equation $S_k(D^2 u) = 1$ on exterior domains. The existence result with the asymptotic behavior (3) was obtained for $A = (C_n^k)^{-1/k} I$ and I is the identity matrix in [24]. More excellent achievements about the exterior problem of Hessian equations can be referred to [25–27]. Specially, Bao–Li–Li [25] extended the asymptotic behavior (3) to more general A and Cao–Bao [26] studied the exterior problem of Hessian equation $S_k(D^2 u) = f(x)$ in $\mathbb{R}^n$ where $f = 1 + O(|x|^{-\beta})$ with $\beta > 2$.

In this paper, we'll generalize the outcome of Monge–Ampère equation in [20] to the Hessian equation (1). We will study the exterior Dirichlet problem

$$\begin{cases} S_k(D^2 u) = f(x), & x \in \mathbb{R}^n \setminus \overline{\Omega}, \\ u = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{4}$$

In order to make the Hessian equation to be elliptic, we have to limit a class of functions. Set

$$\Gamma_k = \{\lambda \in \mathbb{R}^n | \sigma_j(\lambda) > 0, j = 1, 2, \cdots, k\}.$$

Let $u \in C^2(\mathbb{R}^n \backslash \Omega)$ and $\lambda(D^2 u)$ represent the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the Hessian matrix $D^2 u$. If $\lambda \in \Gamma_k$, then we call u k-convex.

Definition 1. *[24] A function $u \in C^0(\mathbb{R}^n \setminus \overline{\Omega})$ is known as a viscosity supersolution (respectively, subsolution) to $S_k(D^2 u) = f(x)$, if for any $t \in \mathbb{R}^n \setminus \overline{\Omega}$, $\varepsilon \in C^2(\mathbb{R}^n \setminus \overline{\Omega})$ satisfying*

$$\varepsilon(t) = u(t) \quad and \quad \varepsilon(x) \leq (respectively, \ \geq) u(x) \ on \ \mathbb{R}^n \setminus \overline{\Omega},$$

we get

$$S_k(D^2 \varepsilon(t)) \leq (respectively, \ \geq) f(t).$$

In viscosity supersolution, we need $\varepsilon(x)$ to be $k-$convex.

If $u \in C^0(\mathbb{R}^n \setminus \overline{\Omega})$ is a viscosity supersolution, meanwhile, a viscosity subsolution, we call that $u \in C^0(\mathbb{R}^n \setminus \overline{\Omega})$ is a viscosity solution to $S_k(D^2 u) = f(x)$.

If $u \in C^0(\mathbb{R}^n \setminus \overline{\Omega})$ is a viscosity supersolution (resp. solution, subsolution) to (1) and $u \geq$ (resp. $=$, $\leq$) $\varphi(x)$ on $\partial\Omega$, we call that $u \in C^0(\mathbb{R}^n \setminus \overline{\Omega})$ is a viscosity supersolution (resp. solution, subsolution) to (4).

Let

$$f(x) = f_0(|x|) + O(|x|^{-\beta}),$$

with $f_0 \in C^0([0, +\infty)), \beta > 0$, $|x|$ large enough and f_0 satisfies

$$b_1 r^\alpha \leq f_0(r) \leq b_2 r^\alpha, r \geq r_0,$$

for some positive constants b_1, b_2, r_0.

Theorem 1. *Let Ω be a smooth, bounded and strictly convex set of $\mathbb{R}^n$ for $n \geq 3, \varphi \in C^2(\partial\Omega), \partial\Omega \in C^2$. Suppose that f satisfies the above assumptions. In case*

$$\frac{k(2-min\{n,\beta\})}{k-1} < \alpha < \infty, \quad n+\alpha > 0, \quad and \quad \alpha+\beta > 0, \tag{5}$$

then for any given $b_0 \in \mathbb{R}^n$, there is some constant m^*, m^* depends on n, b_0, b_1, b_2, α, β, φ, Ω. For every $m > m^*$, there is a locally $k-$convex viscosity solution $u \in C^0(\mathbb{R}^n \setminus \Omega)$ to the exterior Dirichlet problem (4). In addition, u satisfies

$$\lim_{|x|\to\infty} sup|x|^{\alpha-\frac{\alpha}{k}-2+min\{n,\beta\}}|u(x)-g_0(|x|)-b_0x-m| < \infty, \tag{6}$$

where $g_0(|x|)$ is the radially symmetric solution of $S_k(D^2u) = f_0(|x|)$ in $\mathbb{R}^n$ with $g_0(0) = g_0^{'}(0) = 0$, given by (11).

Theorem 2. *Let f, f_0, g_0 be as in Theorem 1. For any given $b_0 \in \mathbb{R}^n$, there is some constant m^*, m^* depends on b_0, b_1, b_2, n, α, β, then for any $m > m^*$, the equation*

$$S_k(D^2u) = f(x), \qquad x \in \mathbb{R}^n \tag{7}$$

has an entire $k-$convex solution $u \in C^0(\mathbb{R}^n)$ in the viscosity sense. In addition, u satisfies (6).

This paper can be divided into the following sections. In the second part, we give the radially symmetric solution to $S_k(D^2u) = f_0$. The third and fourth parts will prove Theorems 1 and 2, respectively. In the last section, we show the importance of condition (5) by counterexample.

2. Radially Symmetric Solutions of $S_K(D^2U) = F_0$

Define $u(x) = u(r) = u(|x|)$ to be radially symmetric, where $r = |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$. By direct calculation, we can get

$$D_{ij}u(r) = (ru''(r) - u'(r))\frac{x_i x_j}{r^3} + u'(r)\frac{\gamma_{ij}}{r}, \qquad i,j = 1,\ldots,n,$$

where

$$\gamma_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

We can choose $x = (r, 0, \cdots 0)^T$, then

$$S_k(D^2u) = C_{n-1}^{k-1}u''(r)u'(r)^{k-1}r^{1-k} + C_{n-1}^{k}u'(r)^k r^{-k} = f_0(r). \tag{8}$$

From (8), we can get

$$(r^{n-k}u'(r)^k)' = \frac{nr^{n-1}f_0(r)}{C_n^k}, \tag{9}$$

and

$$u(r) = d + \int_{2R_1}^{r} a\tau^{1-\frac{n}{k}}[\int_1^{\tau} ns^{n-1}f_0(s)ds + \overline{b}]^{\frac{1}{k}}d\tau, \tag{10}$$

where R_1 is a positive number and

$$d = u(2R_1), \;\; a = (\frac{1}{C_n^k})^{\frac{1}{k}}, \;\; \overline{b} = C_n^k(u^{'}(1))^k.$$

Then the radially symmetric solution $g_0(r)$ of $S_k(D^2u) = f_0$ with $g_0(0) = 0, g_0'(0) = 0$ is

$$g_0(r) = \int_0^r a\tau^{1-\frac{n}{k}}[\int_0^\tau ns^{n-1}f_0(s)ds]^{\frac{1}{k}}d\tau. \tag{11}$$

3. Proof of Theorem 1

Known from [20], by subtracting a linear function from u, let us suppose that $b_0 = 0$ in (6). Set $\underline{f}$ and $\overline{f}$ be two positive functions and satisfy

$$0 < \underline{f}(|x|) = f_0(|x|) - c_1|x|^{-\beta} \leq f(x) \leq \overline{f}(|x|) = f_0(|x|) + c_2|x|^{-\beta}, \qquad x \in \mathbb{R}^n$$

for some positive numbers c_1, c_2 and $|x|$ large enough.

Lemma 1. *Let D be a smooth, bounded subset of $\mathbb{R}^n$, f be a positive function on $\overline{D}$ and a function $\underline{u} \in C^0(\overline{D})$ be a $k-$convex viscosity subsolution to $S_k(D^2u) = f(x)$, then there is a $k-$convex viscosity solution $u \in C^0(\overline{D})$ which satisfies*

$$\begin{cases} S_k(D^2u) = f(x), & in \;\; D, \\ u = \underline{u}, & on \;\; \partial D. \end{cases} \tag{12}$$

Lemma 2. *Let $D_1 \subset D_2$ be two smooth, bounded sets in $\mathbb{R}^n$ and $f \in C^0(\mathbb{R}^n)$ be positive. In the viscosity sense, assume that $k-$convex functions $w \in C^0(\overline{D}_2)$, $u \in C^0(\mathbb{R}^n \setminus D_1)$ satisfy*

$$\begin{aligned} S_k(D^2w) \geq f(x), \quad & x \in D_2, \\ S_k(D^2u) \geq f(x), \quad & x \in \mathbb{R}^n \setminus \overline{D}_1, \end{aligned}$$

respectively, besides

$$\begin{aligned} u < w, \qquad & on \; \partial D_1, \\ u > w, \qquad & on \; \partial D_2. \end{aligned}$$

Define

$$v(x) := \begin{cases} w(x), & in \; D_1, \\ max\{w(x), u(x)\}, & in \; D_2 \setminus \overline{D}_1, \\ u(x), & in \; \mathbb{R}^n \setminus \overline{D}_2. \end{cases}$$

Then in the viscosity sense, $v \in C^0(\mathbb{R}^n)$ is a $k-$convex function and satisfies

$$S_k(D^2v) \geq f(x), \quad x \in \mathbb{R}^n.$$

The proofs of Lemma 1 and Lemma 2 can be referred to [20,24], here we omit the proofs.

For some constant $\bar{m}$, let $S_{\bar{m}}$ be a set satisfying that a function $w \in S_{\bar{m}}$ if and only if

(1) $w \in C^0(\mathbb{R}^n \setminus \Omega)$ is a locally $k-$convex viscosity subsolution of $S_k(D^2u) = f(x)$ in $\mathbb{R}^n \setminus \overline{\Omega}$, $w \leq \varphi$ on $\partial\Omega$;

(2) $w(x) \leq \bar{m} + \overline{v}(x)$, for any $x \in \mathbb{R}^n \setminus \Omega$, with

$$\overline{v}(x) := \int_1^{|x|} a\tau^{1-\frac{n}{k}}[\int_1^\tau ns^{n-1}\underline{f}(s)ds]^{\frac{1}{k}}d\tau. \tag{13}$$

Lemma 3. *There is some constant $\tilde{m}$, $\tilde{m}$ depends on $\Omega, n, \varphi, \alpha, \beta$, then for any $\bar{m} > \tilde{m}$, there is a viscosity subsolution $\underline{u} \in S_{\bar{m}}$.*

Proof. Set $R_2 > R_1 > 1$ such that $R_2 > 3R_1$, $\Omega \subset\subset B_{R_1}(0)$. Define

$$C := \max_{x \in \overline{B_{R_2}}} f(x) > 0.$$

According to [15], for any $\varepsilon \in \partial\Omega$, there exists a $k-$convex solution $w_\varepsilon(x)$ to the following equation

$$S_k(D^2 u) = C, \quad in\ \mathbb{R}^n$$

with

$$w_\varepsilon(\varepsilon) = \varphi(\varepsilon), \quad w_\varepsilon < \varphi\ on\ \partial\Omega \setminus \{\varepsilon\}.$$

Let

$$W(x) := \sup_{\varepsilon \in \partial\Omega} w_\varepsilon(x), \quad x \in \overline{B_{R_2}(0)}.$$

Accordingly, $W(x)$ is a $k-$convex viscosity subsolution of $S_k(D^2u) = f(x)$ in $B_{R_2}(0)$, and satisfies

$$W(\varepsilon) \le \varphi(\varepsilon), \quad \varepsilon \in \partial\Omega.$$

Through the definition of W, for any $\varepsilon \in \partial\Omega$,

$$W(\varepsilon) \ge w_\varepsilon(\varepsilon) = \varphi(\varepsilon),$$

so we have

$$W = \varphi, \quad on\ \partial\Omega.$$

For $b \ge 0$, let

$$\underline{v}_b(x) = \inf_{x \in B_{R_1}} W(x) + \int_{2R_1}^{|x|} a\tau^{1-\frac{n}{k}} [\int_1^\tau ns^{n-1}\overline{f}(s)ds + b]^{\frac{1}{k}} d\tau.$$

Obviously, $\underline{v}_b$ is a locally $k-$convex viscosity subsolution of $S_k(D^2u) = f(x)$ in $\mathbb{R}^n \setminus B_1(0)$ and satisfies

$$\underline{v}_b(x) \le W(x), \quad for\ 1 \le |x| \le R_1.$$

Fix $b_1 > 0$ large enough such that for $b \ge b_1$, with $R_2 > 3R_1$, we can get

$$\underline{v}_b(x) \ge \inf_{x \in B_{R_1}} W(x) + \int_{2R_1}^{3R_1} a\tau^{1-\frac{n}{k}} [\int_1^\tau ns^{n-1}\overline{f}(s)ds + b]^{\frac{1}{k}} d\tau \ge 1 + W(x),\ |x| = R_2.$$

Moreover, $\underline{v}_b(x)$ can be rewritten as

$$\begin{aligned}
\underline{v}_b(x) &= d + \int_{2R_1}^{\infty} a\tau^{1-\frac{n}{k}} [\int_1^\tau ns^{n-1}\overline{f}(s)ds + b]^{\frac{1}{k}} d\tau - \int_{|x|}^{\infty} a\tau^{1-\frac{n}{k}} [\int_1^\tau ns^{n-1}\overline{f}(s)ds + b]^{\frac{1}{k}} d\tau \\
&= d + \int_{2R_1}^{\infty} a\tau^{1-\frac{n}{k}} [\int_1^\tau ns^{n-1}\overline{f}(s)ds + b]^{\frac{1}{k}} d\tau - \int_{2R_1}^{\infty} a\tau^{1-\frac{n}{k}} [\int_0^\tau ns^{n-1}f_0(s)ds]^{\frac{1}{k}} d\tau \\
&\quad + \int_{2R_1}^{\infty} a\tau^{1-\frac{n}{k}} [\int_0^\tau ns^{n-1}f_0(s)ds]^{\frac{1}{k}} d\tau - \int_{|x|}^{\infty} a\tau^{1-\frac{n}{k}} [\int_1^\tau ns^{n-1}\overline{f}(s)ds + b]^{\frac{1}{k}} d\tau \\
&= d + \int_{2R_1}^{\infty} a\tau^{1-\frac{n}{k}} [(\int_1^\tau ns^{n-1}\overline{f}(s)ds + b)^{\frac{1}{k}} - (\int_0^\tau ns^{n-1}f_0(s)ds)^{\frac{1}{k}}] d\tau \\
&\quad - \int_0^{2R_1} a\tau^{1-\frac{n}{k}} [\int_0^\tau ns^{n-1}f_0(s)ds]^{\frac{1}{k}} d\tau + \int_0^{|x|} a\tau^{1-\frac{n}{k}} [\int_0^\tau ns^{n-1}f_0(s)ds]^{\frac{1}{k}} d\tau \\
&\quad - \int_{|x|}^{\infty} a\tau^{1-\frac{n}{k}} [(\int_1^\tau ns^{n-1}\overline{f}(s)ds + b)^{\frac{1}{k}} - (\int_0^\tau ns^{n-1}f_0(s)ds)^{\frac{1}{k}}] d\tau.
\end{aligned}$$

So

$$\underline{v}_b(x) = \underline{\mu}(b) + g_0(|x|) - \int_{|x|}^{\infty} a\tau^{1-\frac{n}{k}}[(\int_1^{\tau} ns^{n-1}\overline{f}(s)ds + b)^{\frac{1}{k}} - (\int_0^{\tau} ns^{n-1}f_0(s)ds)^{\frac{1}{k}}]d\tau, \tag{14}$$

where

$$\underline{\mu}(b) = d + \int_{2R_1}^{\infty} a\tau^{1-\frac{n}{k}}[(\int_1^{\tau} ns^{n-1}\overline{f}(s)ds + b)^{\frac{1}{k}} - (\int_0^{\tau} ns^{n-1}f_0(s)ds)^{\frac{1}{k}}]d\tau - g_0(2R_1).$$

When τ is large enough, we can get

$$\begin{aligned}
& a\tau^{1-\frac{n}{k}}[(\int_1^{\tau} ns^{n-1}\overline{f}(s)ds + b)^{\frac{1}{k}} - (\int_0^{\tau} ns^{n-1}f_0(s)ds)^{\frac{1}{k}}] \\
=& a\tau^{1-\frac{n}{k}}[(\int_{\tau_0}^{\tau} ns^{n-1}(O(s^{\alpha}) + C_3 s^{-\beta})ds + d_1)^{\frac{1}{k}} - (\int_{\tau_0}^{\tau} ns^{n-1}(O(s^{\alpha}))ds + d_2)^{\frac{1}{k}}] \\
=& a(d_3)^{\frac{1}{k}}\tau^{1+\frac{\alpha}{k}}[(1 + \frac{d_4}{d_3}\tau^{-\alpha-\beta} + \frac{d_5}{d_3}\tau^{-n-\alpha})^{\frac{1}{k}} - (1 + \frac{d_6}{d_3}\tau^{-n-\alpha})^{\frac{1}{k}}] \\
\approx& a(d_3)^{\frac{1}{k}}\tau^{1+\frac{\alpha}{k}}[\frac{d_5 - d_6}{kd_3}\tau^{-n-\alpha} + \frac{d_4}{kd_3}\tau^{-\alpha-\beta}] \\
=& O(\tau^{1-n-\alpha+\frac{\alpha}{k}}) + O(\tau^{1-\beta-\alpha+\frac{\alpha}{k}}) \\
=& O(\tau^{\frac{\alpha}{k}+1-\alpha-min\{n,\beta\}}),
\end{aligned}$$

where C_3 is a constant, $d_i (i = 1, 2, \ldots, 6)$ depends on $n,\ \tau_0,\ \alpha,\ \beta,\ k,\ b,\ C_3$. In view of $-\frac{k(min\{n,\beta\}-2)}{k-1} < \alpha < \infty$, then $2 + \frac{\alpha}{k} - \alpha - min\{n, \beta\} < 0$ and we have

$$\underline{v}_b(x) = \underline{\mu}(b) + g_0(|x|) + O(|x|^{\frac{\alpha}{k}-\alpha+2-min\{n,\beta\}}),\ \ |x| \to \infty. \tag{15}$$

Moreover, from (13) and the above, we can see that

$$\overline{v}(x) = \underline{\mu}_0 + g_0(|x|) - \int_{|x|}^{\infty} a\tau^{1-\frac{n}{k}}[(\int_1^{\tau} ns^{n-1}\underline{f}(s)ds)^{\frac{1}{k}} - (\int_0^{\tau} ns^{n-1}f_0(s)ds)^{\frac{1}{k}}]d\tau, \tag{16}$$

where

$$\underline{\mu}_0 = \int_1^{\infty} a\tau^{1-\frac{n}{k}}[(\int_1^{\tau} ns^{n-1}\underline{f}(s)ds)^{\frac{1}{k}} - (\int_0^{\tau} ns^{n-1}f_0(s)ds)^{\frac{1}{k}}]d\tau - g_0(1).$$

Similarly, we can get

$$\overline{v}(x) = \underline{\mu}_0 + g_0(|x|) + O(|x|^{\frac{\alpha}{k}+2-min\{n,\beta\}-\alpha}), \qquad |x| \to \infty. \tag{17}$$

Since

$$a\tau^{1-\frac{n}{k}}[(\int_1^{\tau} ns^{n-1}\overline{f}(s)ds + b)^{\frac{1}{k}} - (\int_0^{\tau} ns^{n-1}f_0(s)ds)^{\frac{1}{k}}] \tag{18}$$

$$\geq a\tau^{1-\frac{n}{k}}[(\int_1^{\tau} ns^{n-1}\underline{f}(s)ds)^{\frac{1}{k}} - (\int_0^{\tau} ns^{n-1}f_0(s)ds)^{\frac{1}{k}}], \tag{19}$$

therefore

$$\underline{v}_b(x) \leq \overline{v}(x) + \underline{\mu}(b) - \underline{\mu}_0,\ \ \forall x \in \mathbb{R}^n \setminus B_1(0). \tag{20}$$

Obviously, $\underline{\mu}(b)$ for b is continuous, monotonic increasing, and $\underline{\mu}(b) \to \infty$ as $b \to \infty$. Fix $b_2 > 0$ large enough such that for $b > b_2$,

$$W(x) \leq \overline{v}(x) + \underline{\mu}(b) - \underline{\mu}_0,\ \ |x| \leq R_2. \tag{21}$$

Let $b^* = max\{b_1, b_2\}$, for any $b > b^*$, we define

$$\underline{u}_b(x) := \begin{cases} W(x), & |x| < R_1, \\ max\{W(x), \underline{v}_b(x)\}, & R_1 \le |x| < R_2, \\ \underline{v}_b(x). & |x| \ge R_2. \end{cases} \tag{22}$$

We know that

$$\underline{u}_b(x) = W(x) = \varphi(x), \quad on \ \partial\Omega.$$

Obviously, $\underline{u}_b(x)$ is a $k-$convex viscosity subsolution of $S_k(D^2u) = f(x)$. For any $m > m^* := \underline{\mu}(b^*)$, there is a constant $b > b^*$, and $m = \underline{\mu}(b)$. According to (20) and (21), we can know, for $m > m^*$,

$$\underline{u}_b(x) \le \overline{v}(x) + m - \underline{\mu}_0, \quad \forall x \in \mathbb{R}^n.$$

Therefore, $\underline{u}_b \in S_{m-\underline{\mu}_0}$. In addition, according to (15) , we have

$$\underline{u}_b(x) = \underline{\mu}(b) + g_0(|x|) + O(|x|^{\frac{\alpha}{k} - \alpha + 2 - min\{n,\beta\}}), \quad |x| \to \infty. \tag{23}$$

Then the lemma can be proved with $\tilde{m} = m^* - \underline{\mu}_0$. $\square$

Set $m > m^*$, define

$$u_m(x) := sup\{w(x) : w \in S_{m-\underline{\mu}_0}\}, \ \ x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

Lemma 4. *The function $u_m \in C^0(\mathbb{R}^n \setminus \Omega)$ is a locally $k-$convex solution to (4) and $u_m \le \overline{v} + m - \underline{\mu}_0$, $x \in \mathbb{R}^n \setminus \Omega$, in the viscosity sense.*

Proof. According to the definition of u_m and $S_{m-\underline{\mu}_0}$, it is clear that u_m is a locally $k-$convex viscosity subsolution to (1) and $u_m \le \overline{v} + m - \underline{\mu}_0$ *in* $\mathbb{R}^n \setminus \overline{\Omega}$.

Firstly, we just have to show that $u_m = \varphi$ on $\partial\Omega$. We can get by the proof of Lemma 3,

$$u_m(x) \ge \underline{u}_b(x), \quad in \ \mathbb{R}^n \setminus \overline{\Omega},$$

with $m = \underline{\mu}(b)$. Since $\underline{u}_b$ is continuous on $\partial\Omega$, then, for any $\varepsilon_0 \in \partial\Omega$, we have

$$\liminf_{x \to \varepsilon_0} u_m(x) \ge \underline{u}_b(\varepsilon_0) = \varphi(\varepsilon_0). \tag{24}$$

Now we prove

$$\limsup_{x \to \varepsilon_0} u_m(x) \le \varphi(\varepsilon_0).$$

For any $w \in S_{m-\underline{\mu}_0}$, w is a viscosity subsolution in $\mathbb{R}^n \setminus \overline{\Omega}$, that is, for every $t \in \mathbb{R}^n \setminus \overline{\Omega}$ and $\sigma \in C^2(\mathbb{R}^n \setminus \overline{\Omega})$ satisfying

$$\sigma(t) = w(t), \quad \sigma \ge w \ on \ \mathbb{R}^n \setminus \overline{\Omega},$$

then, $S_k(D^2\sigma(t)) \ge f(t) > 0$. So,

$$\Delta\sigma(t) \ge n[\frac{1}{C_n^k} S_k(D^2\sigma(t))]^{\frac{1}{k}} > 0.$$

From the above equation, we can see that w is a viscosity subsolution of $\Delta w = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$ and $w \le \varphi$ on $\partial\Omega$.

Fix a ball $B_R(0) \supset \Omega$. For the Dirichlet problem

$$\begin{cases} \Delta w^+ = 0, & x \in B_R(0) \setminus \overline{\Omega}, \\ w^+ = \varphi\,, & x \in \partial\Omega, \\ w^+ = u_m, & x \in \partial B_R(0), \end{cases}$$

there exists a classical solution $w^+ \in C^2(B_R(0) \setminus \overline{\Omega}) \bigcap C^0(\overline{B_R(0) \setminus \Omega})$. Using the comparison principle ([28,29]) for w^+ and $w \in S_{m-\underline{\mu}_0}$, we have

$$w \leq w^+ \; in \; \overline{B_R(0) \setminus \Omega},$$

then, $u_m \leq w^+ \; in \; B_R(0) \setminus \overline{\Omega}$ and

$$\limsup_{x \to \varepsilon_0} u_m(x) \leq w^+(\varepsilon_0) = \varphi(\varepsilon_0).$$

Secondly, we want to verify that $u_m(x)$ is a $k-$convex viscosity solution of $S_k(D^2u) = f(x)$. Fix a ball $B_\lambda(x_0) \subset \mathbb{R}^n \setminus \overline{\Omega}$, for any $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$. Then the Dirichlet problem

$$\begin{cases} S_k(D^2\hat{u}) = f(x), & x \in B_\lambda(x_0), \\ \hat{u} = u_m, & x \in \partial B_\lambda(x_0). \end{cases}$$

contains a $k-$convex viscosity solution $\hat{u} \in C^0(\overline{B_\lambda(x_0)})$.

From the definition of u_m and $\overline{v}(x)$, then

$$\begin{cases} S_k(D^2(\overline{v} + m - \underline{\mu}_0)) \leq f(x), & in \; B_\lambda(x_0), \\ \overline{v} + m - \underline{\mu}_0 \geq u_m, & on \; \partial B_\lambda(x_0). \end{cases}$$

Applying the comparison principle to viscosity solutions, we have $\hat{u} \geq u_m$ and $\hat{u} \leq \overline{v} + m - \underline{\mu}_0$ on $\overline{B_\lambda(x_0)}$.

Set

$$\hat{w}_m(x) = \begin{cases} \hat{u}(x), & in \; B_\lambda(x_0), \\ u_m(x), & in \; \mathbb{R}^n \setminus (\Omega \cup B_\lambda(x_0)). \end{cases}$$

Then, $\hat{w}_m(x)$ is a locally $k-$convex viscosity subsolution, $u_m \leq \overline{v} + m - \underline{\mu}_0$ in $\mathbb{R}^n \setminus \Omega$ and $\hat{w}_m = \varphi$ on $\partial\Omega$. So $\hat{w}_m \in S_{m-\underline{\mu}_0}$. By the definition of u_m, we have $u_m \geq \hat{w}_m$ on $B_\lambda(x_0)$. We can know that $u_m \equiv \hat{u}$ on $B_\lambda(x_0)$ and $u_m \in C^0(\mathbb{R}^n \setminus \Omega)$ is a k-convex viscosity solution of (4). □

Proof of Theorem 1. According to the above, we just have to show that u_m satisfies (6). According to Lemma 4 and the definition of u_m, we have

$$\underline{u}_b \leq u_m \leq \overline{v} + m - \underline{\mu}_0, \quad in \; \mathbb{R}^n \setminus \Omega,$$

with $m = \underline{\mu}(b)$, by (17) and (23), we can know

$$\lim_{|x| \to \infty} sup \, |x|^{\alpha - \frac{\alpha}{k} - 2 + min\{n, \beta\}} |u_m(x) - g_0(|x|) - m| < \infty.$$

The theorem can be proved. □

4. Proof of Theorem 2

For some constant $\bar{m}$, let $\hat{S}_{\bar{m}}$ be a set satisfying that a function $w \in \hat{S}_{\bar{m}}$ if and only if

(1) $w \in C^0(\mathbb{R}^n)$ is a $k-$convex viscosity subsolution of $S_k(D^2u) = f(x)$ in $\mathbb{R}^n$.
(2) $w(x) \leq \overline{v}(x) + \bar{m}$, for any $x \in \mathbb{R}^n$.

According to Lemma 3, $\underline{u}_b$ is a $k-$convex viscosity subsolution of (1), $\underline{u}_b(x) \leq \overline{v}(x) + m - \underline{\mu}_0$, where $m = \underline{\mu}(b)$ and $b > b^*$. Then $\underline{u}_b \in \hat{S}_{m-\underline{\mu}_0}$ for $m > m^*$. Therefore,

$$\underline{u}_b(x) = g_0(|x|) + \underline{\mu}(b) + O(|x|^{\frac{\alpha}{k}+2-\alpha-min\{n,\beta\}}), \quad |x| \to \infty. \tag{25}$$

Lemma 5. *Define for $m > m^*$,*

$$\hat{u}_m(x) := sup\{w(x) : w \in \hat{S}_{m-\underline{\mu}_0}\}, \qquad x \in \mathbb{R}^n.$$

It is clear that $\hat{u}_m$ is a $k-$convex viscosity solution of $S_k(D^2u) = f(x)$ with $\hat{u}_m \leq \overline{v}(x) + m - \underline{\mu}_0$ in $\mathbb{R}^n$.

Proof. A similar method to prove this Lemma can be acquired in Lemma 4. □

Proof of Theorem 2. Known from Lemma 5, for any $m > m^*$, $\hat{u}_m \in C^0(\mathbb{R}^n)$ is a $k-$convex solution to (1). We need to prove (6). According to Lemma 5 and $\hat{u}_m$, then,

$$\underline{u}_b \leq \hat{u}_m \leq \overline{v} + m - \underline{\mu}_0, \ in\ \mathbb{R}^n,$$

with $m = \underline{\mu}(b)$, by (17) and (25), we can know

$$\lim_{|x|\to\infty} sup\ |x|^{\alpha-\frac{\alpha}{k}-2+min\{n,\beta\}}|\hat{u}_m(x) - g_0(|x|) - m| < \infty.$$

Theorem 2 can be proved. □

5. Example

In the last part, we demonstrate the importance of $\alpha > \dfrac{k(2-min\{n,\beta\})}{1-k}$ by a counterexample. Choose a ball $B_1(0) \subseteq B_H(0) \subset \mathbb{R}^n (n \geq 2)$ and a constant c, let $\alpha = -\dfrac{k(\beta-2)}{k-1}$ *for* $0 < \beta < n$. We shall obtain a locally $k-$convex radially symmetric solution which satisfies

$$\begin{cases} S_k(D^2u) = f(|x|), & in\ \mathbb{R}^n \setminus \overline{B_H(0)}, \\ u = c, & on\ \partial B_H(0), \end{cases} \tag{26}$$

and

$$\lim_{|x|\to\infty} sup|x|^{n-\beta}|u(|x|) - g_0(|x|) - b_1 - b_2 ln|x|| < \infty, \tag{27}$$

with $b_2 = a(1+\frac{\alpha}{n})^{-\frac{1}{k}}(\dfrac{n+\alpha}{kn-k\beta})$, b_1 depends on $n,\ \alpha,\ H,\ \beta,\ a,\ b,\ c$.

Let

$$f(|x|) = f_0(|x|) + |x|^{-\beta}, \qquad |x| \geq H, \tag{28}$$

where $f_0(|x|) = |x|^{\alpha}$.

Theorem 3. *Suppose $\alpha = -\dfrac{k(\beta-2)}{k-1}$ for $0 < \beta < n$. Define $u(x) = u(r) = g(|x|)$, where $r = |x|$. Let the locally $k-$convex function $u(x) \in C^0(\mathbb{R}^n \setminus B_H(0)) \bigcap C^2(\mathbb{R}^n \setminus \overline{B_H(0)})$ be a solution to (26). Then u satisfies (27).*

Proof. Suppose that $u(r) = u(x) = g(|x|), r = |x|$, *and* $u(x) \in C^0(\mathbb{R}^n \setminus B_H(0)) \bigcap C^2(\mathbb{R}^n \setminus \overline{B_H(0)})$ is a locally $k-$convex radial solution. For $H < r$, $g''(r)$ *and* $g'(r)$ are positive, respectively, and

$$g'(r) = ar^{1-\frac{n}{k}}(n\int_H^r v^{n-1}f(v)dv + b)^{\frac{1}{k}}$$

with $a = (\frac{1}{C_n^k})^{\frac{1}{k}}$, $b = C_n^k H^{n-k} g'(H)^k \geq 0$.

Set

$$h(r) = n\int_H^r v^{n-1}f(v)dv, \;\; h_0(r) = n\int_0^r v^{n-1}f_0(v)dv.$$

According to (28) and $H \leq r$, we obtain

$$h(r) = h_0(r) - b_0 + \frac{n}{n-\beta}r^{n-\beta},$$

where $b_0 = h_0(H) + \dfrac{n}{n-\beta}H^{n-\beta}$. We can give $g(|x|)$ as follows

$$\begin{aligned} g(|x|) &= \int_H^{|x|} a\tau^{1-\frac{n}{k}}[b + h(\tau)]^{\frac{1}{k}}d\tau + g(H) \\ &= g_0(|x|) - g_0(H) + c + \int_H^{|x|} a\tau^{1-\frac{n}{k}}[h_0(\tau)]^{\frac{1}{k}}[(1 + \frac{h(\tau) - h_0(\tau) + b}{h_0(\tau)})^{\frac{1}{k}} - 1]d\tau. \end{aligned}$$

In view of $\alpha = -\dfrac{k(\beta-2)}{k-1}$ for $0 < \beta < n$, with $h(\tau) = h_0(\tau) - b_0 + \dfrac{n}{n-\beta}r^{n-\beta}$, we can obtain

$$\begin{aligned} & a\tau^{1-\frac{n}{k}}[(1 + \frac{\frac{n}{n-\beta}\tau^{n-\beta} - b_0 + b}{h_0(\tau)})^{\frac{1}{k}} - 1][h_0(\tau)]^{\frac{1}{k}} \\ =& a(1+\frac{\alpha}{n})^{-\frac{1}{k}}(\frac{n+\alpha}{kn-k\beta})\tau^{-1} + a(1+\frac{\alpha}{n})^{1-\frac{1}{k}}(\frac{b-b_0}{k})\tau^{\beta-1-n} + O(\tau^{\beta-1-n}), \;\; as\ \tau \to \infty. \end{aligned}$$

Then,

$$g(|x|) = O(|x|^{\beta-n}) + g_0(|x|) + b_1 + b_2 ln|x|, \qquad as\ |x| \to \infty,$$

with $b_2 = a(1+\frac{\alpha}{n})^{-\frac{1}{k}}(\dfrac{n+\alpha}{kn-k\beta})$ and b_1 depends on $n,\ \beta,\ H,\ a,\ \alpha,\ c,\ b,\ k$. □

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