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Characterization of Clifford Torus in Three-Spheres

Dong-Soo Kim [1,†], Young Ho Kim [2,*,†] and Jinhua Qian [3,†]

1 Department of Mathematics, Chonnam National University, Gwangju 61186, Korea; dosokim@chonnam.ac.kr
2 Department of Mathematics, Kyungpook National University, Daegu 41566, Korea
3 Department of Mathematics, Northeastern University, Shenyang 110004, China; qianjinhua@mail.neu.edu.cn
* Correspondence: yhkim@knu.ac.kr; Tel.: +82-53-950-7325
† These authors contributed equally to this work.

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Abstract: We characterize spheres and the tori, the product of the two plane circles immersed in the three-dimensional unit sphere, which are associated with the Laplace operator and the Gauss map defined by the elliptic linear Weingarten metric defined on closed surfaces in the three-dimensional sphere.

Keywords: elliptic linear Weingarten metric; finite-type immersion; Gauss map; isoparametric surface; torus

1. Introduction

A three-dimensional sphere has been an interesting geometric model space since Poincaré's conjecture was proposed. Furthermore, the complete surfaces of the unit three-sphere $\mathbb{S}^3(1)$ in the four-dimensional Euclidean space $\mathbb{E}^4$ have unique and special geometric properties. For example, there are no complete surfaces immersed in $\mathbb{S}^3(1)$ with constant extrinsic Gaussian curvature K_N satisfying $K_N < -1$ and $-1 < K_N < 0$. Here, the extrinsic Gaussian curvature K_N is defined by the determinant of the shape operator of a surface in $\mathbb{S}^3(1)$ ([1], p. 138). However, there are infinitely many complete and flat surfaces in $\mathbb{S}^3(1)$ such as the tori $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$, the product of two plane circles, where $r_1^2 + r_2^2 = 1$. Among them, the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ is minimal and flat in $\mathbb{S}^3(1)$, and its closed geodesics are mapped onto closed curves of the finite-type in $\mathbb{S}^3(1)$. There are many papers devoted to characterizing the Clifford torus with different view points by dealing with minimal surfaces of the three-sphere [2–4]. By means of isometric immersion and the Gauss map of submanifolds, the tori in $\mathbb{S}^3(1)$ were studied in [5] in terms of the notion of finite-type immersion, and in [6], they were characterized with the so-called II-metric and II-Gauss map. The framework of finite-type immersion has been introduced and developed since the 1970s in generalizing the theory of minimal submanifolds in Euclidean space [7]. By definition, an isometric immersion $x : M \to \mathbb{E}^m$ of a Riemannian manifold M into a Euclidean space $\mathbb{E}^m$ is said to be of the finite-type if the immersion x can be represented as a sum of finitely many eigenvectors of the Laplace operator Δ of M in the following:

$$x = x_0 + x_1 + \cdots + x_k,$$

where x_0 is a constant vector and $x_1, ..., x_k$ are non-constant vectors satisfying $\Delta x_i = \lambda_i x_i$ for some $\lambda_i \in \mathbb{R}$, $i = 1, 2, ..., k$. If all of $\lambda_1, ..., \lambda_k$ are different, the immersion x is called k-type or the submanifold M is said to be of the k-type (cf. [7]). The simplest finite-type is of course the one-type. In this case, the immersion x satisfies:

$$\Delta x = kx + C \tag{1}$$

for some non-zero constant k and a constant vector C. It is well known that a submanifold M of the Euclidean space $\mathbb{E}^m$ is of the one-type if and only if M is a minimal submanifold of $\mathbb{E}^m$ or a minimal submanifold of a hypersphere of $\mathbb{E}^m$ [7]. From this point of view, spherical submanifolds, i.e., submanifolds lying in a sphere, draw our attention in studying finite-type submanifolds in Euclidean space.

Let $\mathbb{S}^{m-1}(1)$ be the unit hypersphere of $\mathbb{E}^m$ centered at the origin and $x : M \to \mathbb{S}^{m-1}$ an isometric immersion of a Riemannian manifold M into $\mathbb{S}^{m-1}(1)$. In this case, if the immersion x identified with the position vector in the ambient Euclidean space is of the finite-type, we call the spherical submanifold M finite-type. In particular, a spherical finite-type immersion $x : M \to \mathbb{S}^{m-1}$ of a Riemannian manifold M into $\mathbb{S}^{m-1}(1)$ is said to be mass-symmetric if x_0 is the center of the unit sphere $\mathbb{S}^{m-1}(1)$.

The notion of finite-type immersion can be extended to any smooth map $\phi : M \to \mathbb{E}^m$ of M into the Euclidean space $\mathbb{E}^m$. A smooth map ϕ is said to be of the finite-type if ϕ can be expressed as a sum of finitely many eigenvectors of Δ such as:

$$\phi = \phi_0 + \phi_1 + ... + \phi_k,$$

where ϕ_0 is a constant vector and $\phi_1, ..., \phi_k$ are non-constant vectors satisfying $\Delta\phi_i = \lambda_i\phi_i$ for some $\lambda_i \in \mathbb{R}$, $i = 1, 2, ..., k$. Among such maps, the Gauss map is one of the most typical and meaningful smooth maps with geometric meaning.

Let us consider how the Gauss map plays an important role in this regard. Let $Gr(n, m)$ be the Grassmann manifold consisting of all oriented n-planes in $\mathbb{E}^m$ passing through the origin. Let M be an n-dimensional submanifold of the Euclidean m-space $\mathbb{E}^m$. Now, we choose an adapted local orthonormal frame $\{e_1, e_2, \cdots, e_m\}$ in $\mathbb{E}^m$ such that $e_1, e_2, \cdots, e_n$ are tangent to M and $e_{n+1}, e_{n+2}, \cdots, e_m$ normal to M. An oriented n-plane passing through a point o can be identified with $(e_1 \wedge e_2 \wedge \cdots \wedge e_n)(o)$. Then, the Grassmann manifold $Gr(n, m)$ is regarded as a submanifold of the Euclidean space $\mathbb{E}^N$, where $N = \left(\begin{array}{c} m \\ n \end{array} \right)$. We define an inner product $\ll \cdot, \cdot \gg$ on $G(n, m) \subset \mathbb{E}^N$ by:

$$\ll e_{i_1} \wedge \cdots \wedge e_{i_n}, e_{j_1} \wedge \cdots \wedge e_{j_n} \gg = det(\langle e_{i_l}, e_{j_k} \rangle),$$

where l, k run over the range $\{1, 2, ..., n\}$. Then, $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} | 1 \leq i_1 < \cdots < i_n \leq m\}$ is an orthonormal basis of $\mathbb{E}^N$, and the Grassmann manifold $Gr(n, m)$ is a spherical submanifold contained in the unit hypersphere $\mathbb{S}^{N-1}(1)$. The smooth map carrying a point p in M to an oriented n-plane in $\mathbb{E}^m$ by the parallel translation of the tangent space of M at p to an n-plane passing through the origin in $\mathbb{E}^m$ is called the Gauss map, which is represented by $\eta : M \to Gr(n, m) \subset \mathbb{E}^N$ via $\eta(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$. In this regard, B.-Y.Chen et al. initiated the study of submanifolds of Euclidean space with the finite-type Gauss map [8].

On the other hand, it is also interesting to consider the case of the Gauss map η satisfying some differential equations such as $\Delta\eta = f\eta$ for some smooth function f, which looks similar to an eigenvalue problem, but is not exactly: for example, the helicoid and the right cone in $\mathbb{R}^3$ have the Gauss map η, which satisfies respectively, $\Delta\eta = f\eta$ and $\Delta\eta = f(\eta + C)$ for some non-vanishing function f and a non-zero constant vector C [9].

Inspired by this, in [9], one of the authors defined the notion of the pointwise one-type Gauss map. The Gauss map η of a submanifold M in the Euclidean space $\mathbb{E}^m$ is said to be of the pointwise one-type if it satisfies:

$$\Delta\eta = f(\eta + C)$$

for some non-zero smooth function f and a constant vector C. In particular, it is said to be of the pointwise one-type of the first kind if the constant vector C is zero. If $C \neq 0$, it is said to be of the pointwise one-type of the second kind.

A surface M in $\mathbb{S}^3(1)$ is called Weingarten if some relationship between its two principal curvatures $\kappa_1,\ \kappa_2$ is satisfied, namely if there is a smooth function (the Weingarten function) of two variables satisfying $W(\kappa_1, \kappa_2) = 0$. Especially, a surface in $\mathbb{S}^3(1)$ is called linear Weingarten if its mean curvature H and the external Gaussian curvature K_N satisfy:

$$2aH + bK_N = c \geq 0$$

for some constants a, b and c, which are not all zero at the same time. In particular, $a^2 + bc > 0$ gives the ellipticity for the differential equations of the coordinate functions of a parametrization $x = x(s,t)$ relative to the principal curvatures, and it enables for the symmetric tensor $\sigma = aI + bII$ defining a Riemannian metric on the surface, where I is the induced metric on M and II the second fundamental form. Briefly speaking, choose an orthonormal basis $\{e_1, e_2\}$ at a point $p \in M$ diagonalizing the shape operator S, i.e.,

$$Se_i = -\kappa_i e_i,$$

where $i = 1, 2$. Then,

$$\begin{aligned}
&\sigma(e_1, e_1)\sigma(e_2, e_2) - \sigma(e_1, e_2)^2 \\
&= (a + b\kappa_1)(a + b\kappa_2) \\
&= a^2 + b(2aH + bK_N) \\
&= a^2 + bc > 0.
\end{aligned}$$

If necessary, the unit normal vector can be chosen by taking the opposite direction with a unit normal vector for σ to be positive definite. We call the surface (M, σ) with the Riemannian metric σ an elliptic linear Weingarten surface (ELW) and σ an elliptic linear Weingarten metric (ELW) [10,11].

In the present paper, two-spheres and the tori $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ $(r_1^2 + r_2^2 = 1)$ in $\mathbb{S}^3(1)$ are characterized with the notion of the ELW metric and its Laplace operator.

We assume that a surface of the sphere $\mathbb{S}^3(1)$ is complete and connected unless stated otherwise.

2. Preliminaries

Let $\mathbb{E}^4$ be the four-dimensional Euclidean space with the canonical metric tensor $\langle \cdot, \cdot \rangle$ and $\mathbb{S}^3(1)$ the unit hypersphere centered at the origin in $\mathbb{E}^4$.

Let M be a surface in $\mathbb{S}^3(1)$. We denote the Levi–Civita connection by $\tilde{\nabla}$ of $\mathbb{S}^3(1)$ and the induced connection ∇ of M in $\mathbb{S}^3(1)$. We use the same notation $\langle \cdot, \cdot \rangle$ as the canonical metric tensors of $\mathbb{E}^4$, $\mathbb{S}^3(1)$ and M.

The Gauss and Weingarten formulas of M in $\mathbb{S}^3(1)$ are respectively given by:

$$\tilde{\nabla}_X Y = \nabla_X Y + \langle SX, Y \rangle N, \quad \tilde{\nabla}_X N = -SX.$$

for vector fields X, Y, and Z tangent to M, where N is the unit normal vector field associated with the orientation of M in $\mathbb{S}^3(1)$ and $S : TM \to TM$ is the shape operator (or Weingarten map), where TM is the tangent bundle of M. Let H and K_N be the mean curvature and the extrinsic Gaussian curvature of M in $\mathbb{S}^3(1)$ defined by $H = \frac{1}{2}\text{tr}S$ and $K_N = \det S$ of M, respectively. M is said to be flat if its Gaussian curvature $K = 1 + K_N$ in $\mathbb{E}^4$ vanishes and M is said to be minimal in $\mathbb{S}^3(1)$ if the mean curvature H vanishes. In particular, the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ is minimal in $\mathbb{S}^3(1)$ and flat in $\mathbb{E}^4$, which is of the one-type in $\mathbb{E}^4$ [5,7].

Let M be a linear Weingarten surface of $\mathbb{S}^3(1)$. Then, a linear combination of its mean curvature H and its extrinsic Gaussian curvature K_N is constant on M, that is there exist three real numbers a, b, c with $(a, b, c) \neq (0, 0, 0)$ such that:

$$2aH + bK_N = c. \tag{2}$$

For convenience, we may assume that $c \geq 0$. It requires $a^2 + bc > 0$ for (2) to be elliptic for the differential equations of the coordinate functions of a parametrization $x = x(s, t)$ for M relative to the principal curvatures.

Let $x : M \to \mathbb{S}^3(1)$ be an isometric immersion induced from $\mathbb{E}^4$ in a natural manner, and we assume that $\{s, t\}$ is a local coordinate system of M. We may regard x as the position vector of the point of M in $\mathbb{E}^4$.

We put:

$$E_1 = \langle x_s, x_s \rangle, F_1 = \langle x_s, x_t \rangle, G_1 = \langle x_t, x_t \rangle, E_2 = \langle x_{ss}, N \rangle, F_2 = \langle x_{st}, N \rangle, G_2 = \langle x_{tt}, N \rangle.$$

Then, we have the first and second fundamental forms, respectively,

$$I = E_1 ds^2 + 2F_1 ds dt + G_1 dt^2,$$

$$II = E_2 ds^2 + 2F_2 ds dt + G_2 dt^2.$$

Together with the first fundamental form I and the second fundamental form II of M, the first and second fundamental forms I and II define a Riemannian metric $\sigma = aI + bII$ on M as shown briefly in the Introduction [10,11].

3. The Gauss Map of the ELW Surface of $\mathbb{S}^3(1)$ in $\mathbb{E}^4$

Let $M = (M, \sigma)$ be an ELW surface of $\mathbb{S}^3(1)$ with the Riemannian metric σ. For a two-dimensional surface (M, σ), we can adopt an isothermal coordinate system. Let u, v be the isothermal coordinates for the metric σ. Then, we have:

$$\sigma = (aE_1 + bE_2)du^2 + 2(aF_1 + bF_2)dudv + (aG_1 + bG_2)dv^2 = \lambda(du^2 + dv^2) \tag{3}$$

for some positive valued function λ. Using the first and second fundamental forms I and II, we have the shape operator S of the form:

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \tag{4}$$

where:

$$S_{11} = \frac{1}{E_1G_1 - F_1^2}(G_1E_2 - F_1F_2), \quad S_{12} = \frac{1}{E_1G_1 - F_1^2}(G_1F_2 - F_1G_2),$$

$$S_{21} = \frac{1}{E_1G_1 - F_1^2}(-E_2F_1 + E_1F_2), \quad S_{22} = \frac{1}{E_1G_1 - F_1^2}(E_1G_2 - F_1F_2).$$

Equation (3) gives:

$$\lambda = aE_1 + bE_2 = aG_1 + bG_2 \quad \text{and} \quad aF_1 + bF_2 = 0$$

and the Laplacian Δ^σ with respect to the Riemannian metric σ by:

$$\begin{aligned}\Delta^\sigma &= -\frac{1}{\sqrt{\det \sigma}}(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}) \\ &= -\frac{1}{\lambda}(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}).\end{aligned} \tag{5}$$

In turn, we get:

$$\lambda^2 = (aE_1 + bE_2)(aG_1 + bG_2) - (aF_1 + bF_2)^2,$$

from which,

$$\lambda^2 = \{a^2 + b(2aH + bK_N)\}(E_1G_1 - F_1^2).$$

Since $2aH + bK_N = c \geq 0$, we get

$$\lambda^2 = (a^2 + bc)(E_1G_1 - F_1^2).$$

Without loss of generality, we may assume that $a^2 + bc = 1$. Then, we get:

$$\lambda = \sqrt{E_1G_1 - F_1^2}.$$

We then have the Gauss map $\eta : M \to \Lambda^2\mathbb{E}^4 = \mathbb{E}^6$ of M by:

$$\eta = \frac{x_u \wedge x_v}{||x_u \wedge x_v||} = \frac{x_u \wedge x_v}{\lambda}.$$

For later use, we compute the Laplacian Δ^σ of η associated with the Riemannian metric σ. After a long and straightforward computation by applying the Christoffel symbols and using the Gauss and Weingarten formulas on M as an immersed surface in $\mathbb{S}^3(1) \subset \mathbb{E}^4$ several times, we have:

Lemma 1. *Let M be an ELW surface of $\mathbb{S}^3(1)$ with the Riemannian metric σ. In terms of the isothermal coordinates u, v with respect to the metric σ, we have:*

$$\begin{aligned}-\Delta^\sigma \eta &= \frac{1}{\lambda}(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2})\eta \\ &= \lambda(f_1 + g_1)\eta + (f_2 + g_2)N \wedge x_u + (f_3 + g_3)N \wedge x_v \\ &+ (f_4 + g_4)N \wedge x + (f_5 + g_5)x \wedge x_u + (f_6 + g_6)x \wedge x_v,\end{aligned} \tag{6}$$

where:

$$f_1 = -\frac{G_1E_2^2}{\lambda^3} - \frac{E_1}{\lambda} - \frac{E_1F_2^2}{\lambda^3} + \frac{2E_2F_1F_2}{\lambda^3}, \tag{7}$$

$$f_2 = \frac{E_2}{2\lambda^3}(G_1(E_1)_v - F_1(G_1)_u) - (\frac{F_2}{\lambda})_u - \frac{F_2}{2\lambda^3}(G_1(E_1)_u - 2F_1(F_1)_u + F_1(E_1)_v), \tag{8}$$

$$f_3 = (\frac{E_2}{\lambda})_u + \frac{E_2}{2\lambda^3}(-F_1(E_1)_v + E_1(G_1)_u) - \frac{F_2}{2\lambda^3}(-F_1(E_1)_u - E_1(E_1)_v + 2E_1(F_1)_u),$$

$$f_4 = \frac{2E_1F_2}{\lambda} - \frac{2E_2F_1}{\lambda},$$

$$f_5 = (\frac{F_1}{\lambda})_u - \frac{E_1}{2\lambda^3}((E_1)_v G_1 - F_1(G_1)_u) + \frac{F_1}{2\lambda^3}(G_1(E_1)_u - 2F_1(F_1)_u + F_1(E_1)_v),$$

$$f_6 = -(\frac{E_1}{\lambda})_u - \frac{E_1}{2\lambda^3}(E_1(G_1)_u) - \frac{F_1}{2\lambda^3}(F_1(E_1)_u - 2E_1(F_1)_u),$$

$$g_1 = -\frac{G_1F_2^2}{\lambda^3} - \frac{G_1}{\lambda} - \frac{E_1G_2^2}{\lambda^3} + \frac{2F_1F_2G_2}{\lambda^3},$$

$$g_2 = \frac{F_2}{2\lambda^3}(2G_1(F_1)_v - G_1(G_1)_u - F_1(G_1)_v) - (\frac{G_2}{\lambda})_v - \frac{G_2}{2\lambda^3}(G_1(E_1)_v - F_1(G_1)_u),$$

$$g_3 = (\frac{F_2}{\lambda})_v + \frac{F_2}{2\lambda^3}(-2F_1(F_1)_v + F_1(G_1)_u + E_1(G_1)_v) - \frac{G_2}{2\lambda^3}(-F_1(E_1)_v + E_1(G_1)_u),$$

$$g_4 = \frac{2F_1G_2}{\lambda} - \frac{2F_2G_1}{\lambda},$$

$$g_5 = (\frac{G_1}{\lambda})_v - \frac{F_1}{2\lambda^3}(2G_1(F_1)_v - G_1(G_1)_u - F_1(G_1)_v) + \frac{G_1}{2\lambda^3}(G_1(E_1)_v - F_1(G_1)_u),$$

$$g_6 = -(\frac{F_1}{\lambda})_v + \frac{F_1}{2\lambda^3}(2F_1(F_1)_v - F_1(G_1)_u - E_1(G_1)_v) + \frac{G_1}{2\lambda^3}(-F_1(E_1)_v + E_1(G_1)_u).$$

4. Closed ELW Surfaces in $\mathbb{S}^3(1)$ with the Pointwise One-Type Gauss Map

Let $M = (M, \sigma)$ be a closed and ELW surface of $\mathbb{S}^3(1)$ with the ELW metric σ. Here, a closed surface means a compact surface without a boundary. In this section, we assume that $a^2 + bc = 1$ unless otherwise stated.

Let $x : M \to \mathbb{S}^3(1) \subset \mathbb{E}^4$ be an isometric immersion of M into $\mathbb{S}^3(1)$. M is said to be of the σ-finite-type if x admits a finite sum of eigenvectors of the Laplace operator Δ^σ defined by the metric σ satisfying:

$$x = x_0 + \sum_{i=1}^{k} x_i, \tag{9}$$

where x_0 is a constant vector and x_i are non-constant $\mathbb{E}^4$-valued maps satisfying $\Delta^\sigma x_i = \lambda_i x_i$ with $\lambda_1, \lambda_2, \cdots, \lambda_k$, and $\lambda_i \in \mathbb{R}$ $(i = 1, 2, ..., k)$. It is said to be of the σ-infinite-type otherwise. When such λ_i are different, i.e., $\lambda_1 < \lambda_2 < \cdots < \lambda_k$, we call it the σ-k-type. Just as is given by (1), if M is of the σ-one-type, we have:

$$\Delta^\sigma x = kx + C$$

for some non-zero k and a constant vector C.

We need the following lemma for later use.

Lemma 2. *Let M be a surface of $\mathbb{S}^3(1)$ with the ELW metric $\sigma = aI + bII$. If M has the pointwise one-type Gauss map of the first kind with respect to the metric σ, then $E_1 - G_1$ and F_1 satisfy:*

$$(E_1)_u + 2(F_1)_v - (G_1)_u = 0 \quad \textit{and} \quad (E_1)_v - 2(F_1)_u - (G_1)_v = 0. \tag{10}$$

Proof. Suppose that the Gauss map is of the pointwise one-type of the first kind with respect to the metric σ, that is,

$$\Delta^\sigma \eta \wedge \eta = 0.$$

Since the vectors $x_s \wedge x_t$, $N \wedge x_t$, $x \wedge x_t$, $N \wedge x_s$ and $x \wedge x_s$ are linearly independent, we have from (6):

$$f_i + g_i = 0$$

for every $2 \leq i \leq 6$. In particular, $f_4 + g_4 = 0$ implies:

$$F_2(G_1 - E_1) + (E_2 - G_2)F_1 = 0. \tag{11}$$

Furthermore, $f_6 + g_6 = 0$ gives:

$$(E_1)_u + 2(F_1)_v - (G_1)_u = 0$$

since $\lambda^2 = E_1G_1 - F_1^2 > 0$. Similarly, $f_5 + g_5 = 0$ implies:

$$(E_1)_v - 2(F_1)_u - (G_1)_v = 0.$$

□

Remark 1. *Suppose that $b = 0$. Then, the ELW metric σ is nothing but the induced metric inherited from that of $\mathbb{S}^3(1)$. Therefore, we focus on the problem with the ELW metric σ with $b \neq 0$.*

Definition 1. *We call an ELW surface M with $b \neq 0$ the proper ELW surface and the ELW metric the proper ELW metric.*

Theorem 1. *Let $M = (M, \sigma)$ be a closed and proper ELW surface in $\mathbb{S}^3(1)$ with the pointwise one-type Gauss map of the first kind relative to the proper ELW metric σ. Then, M is of the σ-one-type if and only if M is one of the following:*

(1) a sphere $\mathbb{S}^2(r)$ with $0 < r \leq 1$.

(2) a torus $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ with $r_1^2 + r_2^2 = 1$ ($r_1 \neq r_2$).

Proof. By making use of the Gauss and Weingarten formulas, we get:

$$\Delta^\sigma x = A_1 x_u + A_2 x_v - \frac{1}{\lambda}(E_2 + G_2)N + \frac{1}{\lambda}(E_1 + G_1)x,$$

where:

$$A_1 = \frac{1}{2\lambda^2}\{G_1((E_1)_u - (G_1)_u + 2(F_1)_v) - F_1(-2(F_1)_u + (E_1)_v - (G_1)_v)\},$$

$$A_2 = \frac{1}{2\lambda^2}\{-F_1((E_1)_u - (G_1)_u + 2(F_1)_v) + E_1(-2(F_1)_u + (E_1)_v - (G_1)_v)\}.$$

By Lemma 2, we have:

$$\Delta^\sigma x = -\frac{1}{\lambda}(E_2 + G_2)N + \frac{1}{\lambda}(E_1 + G_1)x. \tag{12}$$

($\Rightarrow$) Suppose that M is of the σ-one-type, i.e.,

$$\Delta^\sigma x = kx + C \tag{13}$$

for some non-zero constant k and a constant vector C. Since M has the pointwise one-type Gauss map of the first kind relative to the proper ELW metric σ, we have from Lemma 1 that $f_i + g_i = 0$ ($i \geq 2$). Together with (13) and Lemma 2, we get:

$$\langle C, x_u \rangle = 0, \quad \langle C, x_v \rangle = 0, \tag{14}$$

$$\langle C, N \rangle = -\frac{1}{\lambda}(E_2 + G_2), \tag{15}$$

$$k + \langle C, x \rangle = \frac{1}{\lambda}(E_1 + G_1). \tag{16}$$

From Equation (14), we see that the right sides of (15) and (16) are constant. Thus, we may put:

$$\Delta^{\sigma} x = c_1 N + c_2 x$$

for constants $c_1 = -\frac{1}{(E_2+G_2)}$ and $c_2 = \frac{1}{(E_1+G_1)}$. Together with the above equation and (13), we get:

$$C = c_1 N + (c_2 - k)x.$$

Differentiating C with respect to u and v respectively and using (4), we obtain:

$$S_{11}c_1 = k - c_2, S_{12}c_1 = S_{21}c_1 = 0, S_{22}c_1 = k - c_2.$$

Case 1. If $c_1 \neq 0$, the surface M is totally umbilic in $\mathbb{S}^3(1)$ and M is a sphere $\mathbb{S}^2(r)$ with radius $0 < r \leq 1$.
Case 2. Suppose that $c_1 = 0$. Then, we get:

$$k = c_2 \quad \text{and} \quad E_2 + G_2 = 0$$

and consequently, (12) reduces to:

$$\Delta^{\sigma} x = c_2 x.$$

Since $aF_1 + bF_2 = 0$ and $E_2 + G_2 = 0$, Equation (11) gives:

$$(-\frac{a}{b}(G_1 - E_1) + 2E_2)F_1 = 0.$$

Subcase 2.1. Suppose $F_1 \neq 0$. Then, $E_2 = \frac{a}{2b}(G_1 - E_1)$ and $G_2 = \frac{a}{2b}(E_1 - G_1)$.

We now compute the mean curvature H and the extrinsic Gaussian curvature K_N. Straightforward computation yields:

$$H = \frac{a}{4b\lambda^2}\left((E_1 - G_1)^2 + 4F_1^2\right), K_N = -\frac{a^2}{4\lambda^2 b^2}\left((E_1 - G_1)^2 + 4F_1^2\right).$$

Since $2aH + bK_N = c$, we see that $\frac{1}{\lambda^2}\left((E_1 - G_1)^2 + 4F_1^2\right)$ is constant, and hence, the mean curvature H and the extrinsic Gaussian curvature K_N are constant. Therefore, M is an isoparametric surface in $\mathbb{E}^4$. The classification theorem of isoparametric surfaces of $\mathbb{E}^4$ gives us that M is either a sphere $\mathbb{S}^2(r)$ $(0 < r \leq 1)$ or a torus $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$, where $r_1^2 + r_2^2 = 1$. Suppose that M is a torus $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ with $r_1^2 + r_2^2 = 1$. If we choose the parametrization of M by:

$$x(s,t) = (r_1 \cos s, r_1 \sin s, r_2 \cos t, r_2 \sin t),$$

we may take a unit normal vector N as $N = (r_2 \cos s, r_2 \sin s, -r_1 \cos t, -r_1 \sin t)$. Thus, we have the ELW metric:

$$\sigma = aI + bII = \begin{pmatrix} ar_1^2 - br_1 r_2 & 0 \\ 0 & ar_2^2 + br_1 r_2 \end{pmatrix}.$$

Since the Laplace operator Δ^σ is independent of the choice of the coordinates, we get:

$$\Delta^\sigma x = \frac{r_1}{(ar_1^2 - br_1r_2)}(\cos s, \sin s, 0, 0) + \frac{r_2}{(ar_2^2 + br_1r_2)}(0, 0, \cos t, \sin t). \tag{17}$$

It must satisfy $\Delta^\sigma x = kx + C$ for some $k \in \mathbb{R}$ and a constant vector C. Then, we get from (17) that the constant vector C vanishes and:

$$k = \frac{1}{ar_1^2 - br_1r_2} = \frac{1}{ar_2^2 + br_1r_2}. \tag{18}$$

If $r_1 = r_2 = 1/\sqrt{2}$, (18) implies $b = 0$, which is a contradiction. Hence, we have $r_1 \neq r_2$.

Subcase 2.2. Suppose that $F_1 = 0$. By Lemma 2, the function $E_1 - G_1$ is constant. If we differentiate $c_2 = \frac{E_1+G_1}{\lambda}$ with respect to u and v, we get:

$$(E_1 - G_1)^2(E_1)_u = (E_1 - G_1)^2(E_1)_v = 0$$

with the help of (10). Suppose that the open set $M_o = \{p \in M|(E_1)_u \neq 0\} \cup \{p \in M|(E_1)_v \neq 0\}$ is not empty. Let **O** be a connected component of M_o. Then, on **O**, we get $\lambda = E_1 = G_1$. This implies that the mean curvature H and the extrinsic Gaussian curvature K_N are constant on **O**. Thus, **O** is isoparametric, and it is contained in either a sphere $\mathbb{S}^2(r) \subset \mathbb{S}^3(1)$ $(0 < r \leq 1)$ or $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ with $r_1^2 + r_2^2 = 1$. By the connectedness of M, M, M is a sphere $\mathbb{S}^2(r) \subset \mathbb{S}^3(1)$ $(0 < r \leq 1)$ or $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ with $r_1^2 + r_2^2 = 1$ $(r_1 \neq r_2)$. Suppose that the interior U of $M - M_o$ is not empty. Then, E_1 and G_1 are constant on each component of U. Thus, U is flat and $K_N = -1$ on U. The mean curvature H is also constant on U. Using the continuity and connectedness of M, the surface M is flat, and thus, M is a torus $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ with $r_1^2 + r_2^2 = 1$ $(r_1 \neq r_2)$ in $\mathbb{S}^3(1)$.

Summing up the argument, M is either a sphere $\mathbb{S}^2(r) \subset \mathbb{S}^3(1)$ or a torus $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ with $r_1^2 + r_2^2 = 1$ $(r_1 \neq r_2)$.

($\Leftarrow$) Suppose that M is a sphere $\mathbb{S}^2(r)$ with $0 < r \leq 1$. If we choose a vector $C = -2HN + \frac{1}{r^2}x$, then C is a constant vector, and we easily see that M satisfies:

$$\Delta^\sigma x = kx + C,$$

where $k = 2 - 1/r^2$.

If M is a product of two plane circles $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ with $r_1^2 + r_2^2 = 1$, straightforward computation gives that $\Delta^\sigma x = kx$ for some non-zero constant k. This completes the proof. □

We now define the so-called orthogonal σ-k-type immersion of a Riemannian manifold into Euclidean space similarly given in [12].

Definition 2. *Let M be a closed and proper ELW surface of $\mathbb{S}^3(1)$ with the proper ELW metric σ. M is said to be of the orthogonal σ-k-type if the eigenvectors x_i are orthogonal with $||x_i|| = ||x_j||$ for $i \neq j$ in the spectral decomposition given in (9). M is also called σ-mass-symmetric if x_0 is the center of the sphere.*

Now, we consider the following characterization of the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ with the proper ELW metric σ.

Theorem 2. *Let M be a closed surface in $\mathbb{S}^3(1)$. Then, the following are equivalent:*
(1) *M is the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.*

(2) *M is a σ-mass symmetric and orthogonal σ-two-type proper ELW surface in $\mathbb{E}^4$, whose Gauss map is of the pointwise one-type Gauss map of the first kind with respect to the metric σ.*

Proof. (1) $\Rightarrow$ (2) Suppose that M is the Clifford torus parametrized by:

$$x(s,t) = 1/\sqrt{2}(\cos s, \sin s, \cos t, \sin t).$$

It is straightforward to show that:

$$x_1 = 1/\sqrt{2}(\cos s, \sin s, 0, 0), x_2 = 1/\sqrt{2}(0, 0, \cos t, \sin t)$$

are two eigenvectors corresponding to two different eigenvalues relative to Δ^σ. They are orthogonal and $x(s,t) = x_1 + x_2$ with $||x_1|| = ||x_2|| = 1/\sqrt{2}$. It is easy to show that its Gauss map has the pointwise one-type of the first kind. Therefore, M is σ-mass-symmetric and of the orthogonal σ-two-type.

(2) $\Rightarrow$ (1) Suppose that M is σ-mass symmetric and of the orthogonal σ-two-type in $\mathbb{E}^4$ with the pointwise one-type Gauss map of the first kind relative to the ELW metric σ. Then, due to Lemma 2, $E_1 - G_1$, $E_2 - G_2$, F_1, and F_2 are constant, and we have:

$$x = x_1 + x_2 \tag{19}$$

with $\Delta^\sigma x_1 = \lambda_1 x_1$ and $\Delta^\sigma x_2 = \lambda_2 x_2$ with $\langle x_1, x_2\rangle = 0$ and $||x_1|| = ||x_2||$ for two different real numbers λ_1 and λ_2. Applying Δ^σ to (19) and using (12), we get:

$$\{\lambda_1 - \frac{(E_1+G_1)}{\lambda}\}x_1 + \{\lambda_2 - \frac{(E_1+G_1)}{\lambda})\}x_2 + \frac{(E_2+G_2)}{\lambda}N = 0. \tag{20}$$

Suppose that there exists a point p in M such that $(E_2+G_2)(p) = 0$. It follows from (20) $\lambda_1 = \lambda_2$, which is a contradiction. Thus, the unit normal vector field N of M in $\mathbb{S}^3(1)$ is a linear combination of x_1 and x_2 such that:

$$N = \rho_1 x_1 + \rho_2 x_2$$

for some functions ρ_1 and ρ_2 with $\rho_1 \neq \rho_2$. Since $\langle N, x\rangle = 0$, we see that $\rho_1 = -\rho_2$. $\langle N, N\rangle = 1$ and $\langle x, x\rangle = 1$ imply $\rho_1 = \pm 1$ and $||x_1|| = ||x_2|| = 1/\sqrt{2}$. Thus, we have:

$$\lambda_1 + \lambda_2 = 2(E_2 + G_2)/\lambda$$

which is a constant, and in turn, so is $(E_1 + G_1)/\lambda$. Together with $\lambda = aE_1 + bG_1 = aG_1 + bG_2$ and $aF_1 + bF_2 = 0$, we see that the mean curvature H and the extrinsic Gaussian curvature K_N are constant, and hence, M is isoparametric in $\mathbb{E}^4$. Since M is of the σ-two-type, M must be the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$. $\square$

5. Conclusions

In this paper, spheres and the products of two plane circles immersed in the three-dimensional unit sphere are studied and characterized by means of the Laplacian and the Gauss map. Especially we consider the elliptic linear Weingarten metric on closed surfaces in the three-dimensional sphere and the Clifford torus is characterized with the elliptic linear Weingarten metric.

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