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Absolute Continuity of Fuzzy Measures and Convergence of Sequence of Measurable Functions

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Abstract: In this note, the convergence of the sum of two convergent sequences of measurable functions is studied by means of two types of absolute continuity of fuzzy measures, i.e., strong absolute continuity of Type I, and Type VI. The discussions of convergence a.e. and convergence in measure are done in the general framework relating to a pair of monotone measures, and general results are shown. The previous related results are generalized.

Keywords: fuzzy measure; absolute continuity; convergence in measure; fuzzy integral

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1. Introduction

In fuzzy measure and fuzzy integral theory, many results in the classical measure theory no longer hold generally without additional conditions for fuzzy measures. For example, for Lebesgue measurable functions sequences $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$, we have

$$f_n \to f \; a.e.[m] \;\text{ and }\; g_n \to g \; a.e.[m] \implies f_n + g_n \to f + g \; a.e.[m] \tag{1}$$

and

$$f_n \xrightarrow{m} f \;\text{ and }\; g_n \xrightarrow{m} g \implies f_n + g_n \xrightarrow{m} f + g, \tag{2}$$

where m is the Lebesgue measure and the convergence concerns convergence almost everywhere and convergence in measure [1,2]. However, for a fuzzy measure μ, in general, the above Equations (1) and (2) may not be true.

The above Equations (1) and (2) were generalized to fuzzy measure spaces under the conditions of weak null-additivity and pseudometric generating property of set functions [3].

In this note, we consider a pair of fuzzy measures λ and ν defined on the same measurable space $(X, \mathcal{A})$. Under the condition of strong absolute continuity of Type I (resp. strong absolute continuity of Type VI), we obtain the following result:

$$f_n \to f \; a.e.[\nu] \;\text{ and }\; g_n \to g \; a.e.[\nu] \implies f_n + g_n \to f + g \; a.e.[\lambda] \tag{3}$$

$$\Big(\text{resp. } \; f_n \xrightarrow{\nu} f \;\text{ and }\; g_n \xrightarrow{\nu} g \implies f_n + g_n \xrightarrow{\lambda} f + g \Big). \tag{4}$$

Comparing Equation (3) with Equation (1), and Equation (4) with Equation (2), respectively, we see that the general results are obtained in the framework concerning a pair of monotone measures. The previous related results in [3,4] (see also [5]) are recovered.

2. Preliminaries

Let $(X, \mathcal{A})$ be a measurable space, i.e., X is a nonempty set and $\mathcal{A}$ is a σ-algebra of subsets of X. Let $\mathcal{F}$ denote the collection of all $\mathcal{A}$-measurable functions $f : X \to [0,1]$ (such a function f is also called a fuzzy set on X). The symbols " $\vee$ " and " $\wedge$ " denote logical addition and logical multiplication (i.e., for any $a, b \in [0,1]$, $a \vee b = \max\{a, b\}$, and $a \wedge b = \min\{a, b\}$), respectively.

2.1. Fuzzy Measures and Integrals

A fuzzy measure on $(X, \mathcal{A})$ is a real valued set function $\mu : \mathcal{A} \to [0,1]$ satisfying the conditions: (FM1) $\mu(\emptyset) = 0$ and $\mu(X) > 0$; (FM2) $\mu(P) \leq \mu(Q)$ whenever $P \subset Q$ and $P, Q \in \mathcal{A}$ (see [6,7]).

A fuzzy measure is also known as "non-additive measure", "capacity", "monotone measure", "non-additive probability", etc (see [8–11]).

Let $\mathfrak{FM}$ denote the set of all fuzzy measures defined on $(X, \mathcal{A})$. For $\lambda, \nu \in \mathfrak{FM}$, let (λ, ν) denote the order pair of λ and ν, i.e., $(\lambda, \nu) \in \mathfrak{FM} \times \mathfrak{FM}$.

We recall fuzzy integral [6] (it is also called Sugeno integral, see [12,13]).

Let $(\mu, f) \in \mathfrak{FM} \times \mathcal{F}$. The fuzzy integral of f on X with respect to μ, is defined by

$$(S) \int f \, d\mu = \bigvee_{0 \leq \alpha \leq 1} \left[\alpha \wedge \mu(\{x \in X \mid f(x) \geq \alpha\}) \right]. \tag{5}$$

For $A \in \mathcal{A}$, we define (S)$\int_A f d\mu = (S) \int f \chi_A d\mu$, where χ_A is the characteristic function of A.

Note that the Sugeno integral is a special kind of nonlinear integral.

We define the conjugate $\overline{\mu}$ of μ by

$$\overline{\mu}(A) = \mu(X) - \mu(X \setminus A), \quad A \in \mathcal{A}.$$

Then $\overline{\mu} \in \mathfrak{FM}$ and $\overline{\overline{\mu}} = \mu$.

2.2. Convergence of Sequence of Measurable Functions

Let $\mu \in \mathfrak{FM}$ and $f, f_n \in \mathcal{F}$ $(n = 1, 2, \ldots)$. We say that

(1) $\{f_n\}_{n \in \mathbb{N}}$ *converges almost everywhere to* f *on* X *with respect to* μ, and denote it by $f_n \xrightarrow{a.e.} f[\mu]$, if there is a subset $E \in \mathcal{A}$ such that $\mu(E) = 0$ and $f_n \to f$ on $X \setminus E$;

(2) $\{f_n\}_{n \in \mathbb{N}}$ *converges pseudo-almost everywhere to* f *on* X *with respect to* μ, and denote it by $f_n \xrightarrow{p.a.e.} f[\mu]$, if there is a subset $F \in \mathcal{A}$ such that $\mu(X \setminus F) = \mu(X)$ and $f_n \to f$ on $X \setminus F$;

(3) $\{f_n\}_{n \in \mathbb{N}}$ *converges to* f *in measure* μ (resp. *pseudo-in measure* μ) on X, denoted by $f_n \xrightarrow{\mu} f$ (resp. $f_n \xrightarrow{p.\mu} f$), if for any $\sigma > 0$, $\lim_{n \to \infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \sigma\}) = 0$ (resp. $\lim_{n \to +\infty} \mu(\{x \in X \mid |f_n(x) - f(x)| < \sigma\}) = \mu(X)$).

Proposition 1. *Let* $\mu \in \mathfrak{FM}$. *Then*

(1) $f_n \xrightarrow{a.e.} f[\mu]$ *if and only if* $f_n \xrightarrow{p.a.e.} f[\overline{\mu}]$;

(2) $f_n \xrightarrow{\mu.} f$ *if and only if* $f_n \xrightarrow{p.\overline{\mu}} f$.

2.3. Absolute Continuity of Fuzzy Measures

We recall several types of absolute continuity of fuzzy measures.

Definition 1. [14] *Let* $(\lambda, \nu) \in \mathfrak{FM} \times \mathfrak{FM}$. *We say that*

(1) λ *is absolutely continuous of Type I with respect to* ν, *denoted by* $\lambda \ll_I \nu$, *if for any* $A \in \mathcal{A}$, *we have*

$$\nu(A) = 0 \implies \lambda(A) = 0; \tag{6}$$

(2) λ is absolutely continuous of Type VI with respect to ν, denoted by $\lambda \ll_{VI} \nu$, if for any $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$, we have

$$\nu(A_n) \to 0 \ (n \to \infty) \implies \lambda(A_n) \to 0 \ (n \to \infty). \tag{7}$$

Inspired by the concepts of weak null-additivity and pseudometric generating property of set functions we generalized the above two types of absolute continuity, i.e., Type I and Type VI (see [3]). Let us first recall weak null-additivity and pseudometric generating property of fuzzy measures.

Let $\mu \in \mathfrak{FM}$. (1) μ is called *weakly null-additive* [11,15], if for any $P, Q \in \mathcal{A}$, we have

$$\mu(P) = \mu(Q) = 0 \Longrightarrow \mu(P \cup Q) = 0. \tag{8}$$

(2) μ is called to have *pseudometric generating property* (briefly, $(p.g.p.)$) [4,16], if for any sequences $\{P_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and $\{Q_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$, we have

$$\mu(P_n) \vee \mu(Q_n) \to 0 \ (n \to \infty) \Longrightarrow \mu(P_n \cup Q_n) \to 0 \ (n \to \infty). \tag{9}$$

Definition 2. (Li et al. [3]) *Let $(\lambda, \nu) \in \mathfrak{FM} \times \mathfrak{FM}$. We say that*

(1) λ is strongly absolute continuous of Type I with respect to ν, denoted by $\lambda \ll_{I}^{(s)} \nu$, if

$$\nu(A) = \nu(B) = 0 \implies \lambda(A \cup B) = 0; \tag{10}$$

(2) λ is strongly absolute continuous of Type VI with respect to ν, denoted by $\lambda \ll_{VI}^{(s)} \nu$, if for any sequences $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$, we have

$$\nu(A_n) \vee \nu(B_n) \to 0 \ (n \to \infty) \implies \lambda(A_n \cup B_n) \to 0 \ (n \to \infty). \tag{11}$$

Note: Comparing Equation (10) with Equation (8), and Equation (11) with Equation (9), respectively, we have (1) $\mu \ll_{I}^{(s)} \mu$ if and only ifμ is weakly null-additive, and (2) $\mu \ll_{VI}^{(s)} \mu$ if and only if μ has pseudometric generating property.

Proposition 2. *Let $(\lambda, \nu) \in \mathfrak{FM} \times \mathfrak{FM}$.*

(1) If $\lambda \ll_{VI} \nu$, then $\lambda \ll_{I} \nu$.

(2) If $\lambda \ll_{VI}^{(s)} \nu$, then $\lambda \ll_{I}^{(s)} \nu$.

(3) $\lambda \ll_{I}^{(s)} \nu$ implies $\lambda \ll_{I} \nu$, and $\lambda \ll_{VI}^{(s)} \nu$ implies $\lambda \ll_{VI} \nu$.

The inverse statements in (1), (2) and (3) may not hold.

3. Main Results

The following Theorems 1–3 are our main results.

Theorem 1. *Let $(\lambda, \nu) \in \mathfrak{FM} \times \mathfrak{FM}$. Then, we have*

(1) for all $f \in \mathcal{F}$ and all $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$,

$$f_n \xrightarrow{a.e.} f[\nu] \implies f_n \xrightarrow{a.e.} f[\lambda]$$

if and only if $\lambda \ll_{I} \nu$;

(2) for all $f \in \mathcal{F}$ and all $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$,

$$f_n \xrightarrow{\nu} f \implies f_n \xrightarrow{\lambda} f$$

if and only if $\lambda \ll_{VI} \nu$.

Proof. From Definition 1 it is easy to obtain the conclusions. □

Theorem 2. *Let $(\lambda, \nu) \in \mathfrak{FM} \times \mathfrak{FM}$. Then the following are equivalent:*
(1) $\lambda \ll_I^{(s)} \nu$;
(2) for all $f, g \in \mathcal{F}$ *and all* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, *we have*

$$f_n \xrightarrow{a.e.} f[\nu] \text{ and } g_n \xrightarrow{a.e.} g[\nu] \implies f_n \vee g_n \xrightarrow{a.e.} f \vee g[\lambda];$$

(3) for all $f, g \in \mathcal{F}$ *and all* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, *we have*

$$f_n \xrightarrow{a.e.} f[\nu] \text{ and } g_n \xrightarrow{a.e.} g[\nu] \implies f_n + g_n \xrightarrow{a.e.} f + g[\lambda].$$

Proof. $(1) \Rightarrow (2)$. If $f_n \xrightarrow{a.e} f[\nu]$ and $g_n \xrightarrow{a.e} g[\nu]$, then there exist $F, G \in \mathcal{A}$ with $\nu(F) = \nu(G) = 0$ such that $f_n \to f$ on $X \setminus F$ and $g_n \to g$ on $X \setminus G$. Since $\lambda \ll_I^{(s)} \nu$, we have $\lambda(F \cup G) = 0$. Noting that $f_n \vee g_n \to f \vee g$ on $X \setminus F \cup G$, therefore $f_n \vee g_n \xrightarrow{a.e.} f \vee g[\lambda]$.

$(2) \Rightarrow (1)$. Let $P, Q \in \mathcal{A}$ and $\nu(P) = \nu(Q) = 0$. Denote $f_n = \chi_P$ and $g_n = \chi_Q$, where χ_P and χ_Q are the characteristic functions of P and Q, respectively. It follows from $f_n \xrightarrow{a.e} 0[\nu]$ and $g_n \xrightarrow{a.e} 0[\nu]$ that $f_n \vee g_n \xrightarrow{a.e.} 0[\lambda]$. So $\lambda(P \cup Q) = \lambda(\{x \in X | \, f_n(x) \vee g_n(x) \nrightarrow 0\}) = 0$.

$(1) \Leftrightarrow (3)$. The proof is similar. □

Theorem 3. *Let $(\lambda, \nu) \in \mathfrak{FM} \times \mathfrak{FM}$. Then the following are equivalent:*
(1) $\lambda \ll_{VI}^{(s)} \nu$;
(2) for all $f, g \in \mathcal{F}$ *and all* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, *we have*

$$f_n \xrightarrow{\nu} f \text{ and } g_n \xrightarrow{\nu} g \implies f_n + g_n \xrightarrow{\lambda} f + g;$$

(3) for all $f, g \in \mathcal{F}$ *and all* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, *we have*

$$f_n \xrightarrow{\nu} f \text{ and } g_n \xrightarrow{\nu} g \implies f_n \vee g_n \xrightarrow{\lambda} f \vee g.$$

Proof. $(1) \Rightarrow (2)$. Suppose that $f_n \xrightarrow{\nu} f$ and $g_n \xrightarrow{\nu} g$. For any given $\epsilon > 0$, we denote

$$F_n(\epsilon) = \Big\{x \in X \big| |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\Big\}$$

and

$$G_n(\epsilon) = \Big\{x \in X \big| |g_n(x) - g(x)| \geq \frac{\epsilon}{2}\Big\}.$$

Then

$$\Big\{x \in X \big| |(f_n(x) + g_n(x)) - (f(x) + g(x))| \geq \epsilon\Big\} \subset F_n(\epsilon) \cup G_n(\epsilon).$$

Since $f_n \xrightarrow{\nu} f$ and $f_n \xrightarrow{\nu} g$, we have

$$\nu\Big(F_n(\epsilon)\Big) \bigvee \nu\Big(G_n(\epsilon)\Big) \longrightarrow 0 \ (n \to \infty).$$

Therefore, from $\lambda \ll_{VI}^{(s)} \nu$, we have

$$\lambda\Big(F_n(\epsilon) \cup G_n(\epsilon)\Big) \longrightarrow 0 \ (n \to \infty).$$

Hence it is clear that

$$\lambda\Big(\Big\{x \in X | \ |(f_n(x) + g_n(x)) - (f(x) + g(x))| \geq \epsilon\Big\}\Big) \longrightarrow 0 \ (n \to \infty).$$

This shows that $f_n + g_n \xrightarrow{\lambda} f + g$.

(2) $\Rightarrow$ (3). If $f_n \xrightarrow{\nu} f$ and $g_n \xrightarrow{\nu} g$, then $f_n + g_n \xrightarrow{\lambda} f + g$.

Noting that $f_n(x) \vee g_n(x) \leq f_n(x) + g_n(x)$, for any $\epsilon > 0$ we have

$$\{x \in X|\ |f_n(x) \vee g_n(x)| \geq \epsilon\} \subset \{x \in X|\ |f_n(x) + g_n(x)| \geq \epsilon\}.$$

So

$$\lim_{n \to +\infty} \lambda(\{x \in X \mid |\ f_n(x) \vee g_n(x)| \geq \epsilon\}) \leq \lim_{n \to +\infty} \lambda(\{x \in X \mid |\ f_n(x) + g_n(x)| \geq \epsilon\}) = 0.$$

This shows $f_n \vee g_n \xrightarrow{\lambda} 0$.

(3) $\Rightarrow$ (1). For any $\{P_n\}_{n \in \mathbb{N}} \subset \mathcal{A}, \{Q_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ with $\lim_{n \to \infty} \mu(P_n) \vee \mu(Q_n) = 0$, we define measurable function sequences $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$ and $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$ by

$$f_n(x) = \chi_{P_n}(x) = \begin{cases} 0 & \text{if } x \notin P_n \\ 1 & \text{if } x \in P_n \end{cases}$$

and

$$g_n(x) = \chi_{Q_n}(x) = \begin{cases} 0 & \text{if } x \notin Q_n \\ 1 & \text{if } x \in Q_n, \end{cases}$$

n = 1, 2,..., then $f_n \xrightarrow{\nu} 0$ and $g_n \xrightarrow{\nu} 0$. Thus, $f_n \vee g_n \xrightarrow{\lambda} 0$. Therefore for $\epsilon = \frac{1}{2}$, we have

$$\lim_{n \to +\infty} \lambda(\{x \in X|\ f_n(x) \vee g_n(x) \geq \frac{1}{2}\}) = 0.$$

Noting $f_n \vee g_n = \chi_{P_n} \vee \chi_{Q_n} = \chi_{P_n \cup Q_n}$, we have

$$\{x \in X|\ f_n(x) \vee g_n(x) \geq \frac{1}{2}\} = \{x \in X|\ \chi_{P_n \cup Q_n}(x) \geq \frac{1}{2}\} = P_n \cup Q_n.$$

So

$$\lim_{n \to +\infty} \lambda(P_n \cup Q_n) = 0.$$

Thus we have $\lambda \ll_{VI}^{(s)} \nu$. $\square$

When we take $(\lambda, \nu) = (\mu, \mu), (\overline{\mu}, \overline{\mu}), (\overline{\mu}, \mu)$, and $(\mu, \overline{\mu})$, respectively, and combine Proposition 1, then the previous results obtained in [3] (Li et al.) are recovered by Theorems 2 and 3, respectively.

Corollary 1. ([3], Theorem 1) *Let $\mu \in \mathfrak{FM}$. Then,*

(1) $\mu \ll_I^{(s)} \mu$ *(i.e., μ is weakly null-additive) if and only if for any $f, g, f_n, g_n \in \mathcal{F}$, $f_n \xrightarrow{a.e} f\ [\mu]$ and $g_n \xrightarrow{a.e} g\ [\mu] \implies f_n + g_n \xrightarrow{a.e} f + g\ [\mu]$.*

(2) $\overline{\mu} \ll_I^{(s)} \overline{\mu}$ *(i.e., $\overline{\mu}$ is weakly null-additive) if and only if for any $f, g, f_n, g_n \in \mathcal{F}$, $f_n \xrightarrow{p.a.e} f\ [\mu]$ and $g_n \xrightarrow{p.a.e} g\ [\mu] \implies f_n + g_n \xrightarrow{p.a.e} f + g\ [\mu]$.*

(3) $\overline{\mu} \ll_I^{(s)} \mu$ *if and only if for any $f, g, f_n, g_n \in \mathcal{F}$, $f_n \xrightarrow{a.e} f\ [\mu]$ and $g_n \xrightarrow{a.e} g\ [\mu] \implies f_n + g_n \xrightarrow{p.a.e} f + g\ [\mu]$.*

(4) $\mu \ll_I^{(s)} \overline{\mu}$ *if and only if for any $f, g, f_n, g_n \in \mathcal{F}$, $f_n \xrightarrow{p.a.e} f\ [\mu]$ and $g_n \xrightarrow{p.a.e} g\ [\mu] \implies f_n + g_n \xrightarrow{a.e} f + g\ [\mu]$.*

Corollary 2. ([3], Theorem 2) *Let $\mu \in \mathfrak{FM}$. Then,*

(1) $\mu \ll_{VI}^{(s)} \mu$ *(i.e., μ has (p.g.p.)) if and only if for any $f, g, f_n, g_n \in \mathcal{F}$, $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g \implies f_n + g_n \xrightarrow{\mu} f + g$.*

(2) $\overline{\mu} \ll_{VI}^{(s)} \overline{\mu}$ *(i.e.,* $\overline{\mu}$ *has (p.g.p.)) if and only if for any* $f, g, f_n, g_n \in \mathcal{F}$, $f_n \xrightarrow{p.\mu} f$ *and* $g_n \xrightarrow{p.\mu} g \Longrightarrow f_n + g_n \xrightarrow{p.\mu} f + g$.
(3) $\overline{\mu} \ll_{VI}^{(s)} \mu$ *if and only if for any* $f, g, f_n, g_n \in \mathcal{F}$, $f_n \xrightarrow{\mu} f$ *and* $g_n \xrightarrow{\mu} g \Longrightarrow f_n + g_n \xrightarrow{p.\mu} f + g$.
(4) $\mu \ll_{VI}^{(s)} \overline{\mu}$ *if and only if for any* $f, g, f_n, g_n \in \mathcal{F}$, $f_n \xrightarrow{p.\mu} f$ *and* $g_n \xrightarrow{p.\mu} g \Longrightarrow f_n + g_n \xrightarrow{\mu} f + g$.

In the following we discuss fuzzy measures defined by fuzzy integral.

Given $\nu \in \mathfrak{F}\mathfrak{M}$ and $h \in \mathcal{F}$. Then the Sugeno integral of h with respect to ν determines a new fuzzy measure $\lambda_h \in \mathfrak{F}\mathfrak{M}$, as follows:

$$\lambda_h(A) = (S)\int_A h d\nu, \quad \forall A \in \mathcal{A}.$$

Proposition 3. *(1)* $\lambda_h \ll_I \nu$*, and* $\lambda_h \ll_{VI} \nu$*; (2)* $\lambda_h \ll_I^{(s)} \nu$*; (3) If* ν *is continuous from below, then* $\lambda_h \ll_{VI}^{(s)} \nu$*.*

As a direct result of Proposition 3 and Theorems 2 and 3, we have the following corollary.

Corollary 3. *Let* $\nu \in \mathfrak{F}\mathfrak{M}$*. Then,*
(1) for all $f, g \in \mathcal{F}$ *and all* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$*, we have*

$$f_n \xrightarrow{a.e.} f[\nu] \text{ and } g_n \xrightarrow{a.e.} g[\nu] \Longrightarrow f_n \vee g_n \xrightarrow{a.e.} f \vee g[\lambda_h];$$

(2) for all $f, g \in \mathcal{F}$ *and all* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$*, we have*

$$f_n \xrightarrow{a.e.} f[\nu] \text{ and } g_n \xrightarrow{a.e.} g[\nu] \Longrightarrow f_n + g_n \xrightarrow{a.e.} f + g[\lambda_h];$$

Furthermore, if ν *is continuous from below, then*
(3) for all $f, g \in \mathcal{F}$ *and all* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$*, we have*

$$f_n \xrightarrow{\nu} f \text{ and } g_n \xrightarrow{\nu} g \Longrightarrow f_n \vee g_n \xrightarrow{\lambda_h} f \vee g;$$

(4) for all $f, g \in \mathcal{F}$ *and all* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$, $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$*, we have*

$$f_n \xrightarrow{\nu} f \text{ and } g_n \xrightarrow{\nu} g \Longrightarrow f_n + g_n \xrightarrow{\lambda_h} f + g.$$

4. Conclusions

We have shown the equivalences between the convergence (a.e. or in measure) of the sum of two convergent sequences of measurable functions and several types of absolute continuity of fuzzy measures. The main results are Theorems 1–3. The characteristics of strong absolute continuity of Type I and Type VI of fuzzy measures have been described by using convergence of sequence of measurable functions. As we have seen, such descriptions were done in a more general context concerning a pair of monotone measures, the previous related results [3,15] become to be some special cases of our new results.

In our discussions we only involved theoretical methods, not presenting a specific application instance. In further research, we will focus on the application of these methods. For instance, we will generalize convergence in measure theorems of nonlinear integrals [17,18] in the general framework concerning a pair of monotone measures, and discuss the linearity of pan-integrals and concave integrals [19,20], our results will be useful.

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