

Article

New Oscillation Criteria for Advanced Differential Equations of Fourth Order

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Abstract: The main objective of this paper is to establish new oscillation results of solutions to a class of fourth-order advanced differential equations with delayed arguments. The key idea of our approach is to use the Riccati transformation and the theory of comparison with first and second-order delay equations. Four examples are provided to illustrate the main results.

Keywords: advanced differential equations; oscillations; Riccati transformations; fourth-order delay equations

1. Introduction

In the last decades, many researchers have devoted their attention to introducing more sophisticated analytical and numerical techniques to solve mathematical models arising in all fields of science, technology and engineering. Fourth-order advanced differential equations naturally appear in models concerning physical, biological and chemical phenomena, having applications in dynamical systems such as mathematics of networks and optimization, and applications in the mathematical modeling of engineering problems, such as electrical power systems, materials and energy, also, problems of elasticity, deformation of structures, or soil settlement, see [1].

The present paper deals with the investigation of the oscillatory behavior of the fourth order advanced differential equation of the following form

$$\left(a(v) (y''''(v))^\beta \right)' + \sum_{i=1}^j q_i(v) g(y(\eta_i(v))) = 0, \quad v \geq v_0, \quad (1)$$

where $j \geq 1$ and β is a quotient of odd positive integers. Throughout the paper, we suppose the following assumptions:

$a \in C^1([v_0, \infty), (0, \infty))$, $a'(v) \geq 0$, $q_i, \eta_i \in C([v_0, \infty), \mathbb{R})$, $q_i(v) \geq 0$, $\eta_i(v) \geq v$, $i = 1, 2, \dots, j$, $g \in C(\mathbb{R}, \mathbb{R})$ such that $g(x)/x^\beta \geq \ell > 0$, for $x \neq 0$ and under the condition

$$\int_{v_0}^{\infty} \frac{1}{a^{1/\beta}(s)} ds = \infty. \quad (2)$$

During this decade, several works have been accomplished in the development of the oscillation theory of higher order advanced equations by using the Riccati transformation and the theory of

comparison between first and second-order delay equations. Further, the oscillation theory of fourth and second order delay equations has been studied and developed by using an integral averaging technique and the Riccati transformation, see [2–23].

In this paper, we are aimed to complement the results reported in [24–26], therefore we discuss their findings and results below.

Moazz et al. [27] considered the fourth-order differential equation

$$\left(a(v) (y'''(v))^\gamma \right)' + q(v) y^\alpha(\eta(v)) = 0,$$

where γ, α are quotients of odd positive integers.

Grace et al. [28] considered the equation

$$\left(a(v) (y''(v))^\gamma \right)'' + q(v) g(y(\eta(v))) = 0, \tag{3}$$

where $\eta(v) \leq v$.

Zhang et al. in [29] studied qualitative behavior of the fourth-order differential equation

$$\left(a(v) (w'''(v))^\beta \right)' + q(v) w(\sigma(v)) = 0,$$

where $\sigma(v) \leq v, \beta$ is a quotient of odd positive integers and they used the Riccati transformation.

Agarwal and Grace [24] considered the equation

$$\left(\left(y^{(\kappa-1)}(v) \right)^\beta \right)' + q(v) y^\beta(\eta(v)) = 0, \tag{4}$$

where κ is even, and they established some new oscillation criteria by using the comparison technique. Among others, they proved it oscillatory if

$$\liminf_{v \rightarrow \infty} \int_v^{\eta(v)} (\eta(s) - s)^{\kappa-2} \left(\int_{\eta(v)}^\infty q(v) dv \right)^{1/\beta} ds > \frac{(\kappa - 2)!}{e}. \tag{5}$$

Agarwal et al. in [25] extended the Riccati transformation to obtain new oscillatory criteria for ODE (4) under the condition

$$\limsup_{v \rightarrow \infty} v^{\beta(\kappa-1)} \int_v^\infty q(s) ds > ((\kappa - 1)!)^\beta. \tag{6}$$

Authors in [26] studied oscillatory behavior of Equation (4) where $\beta = 1$ and if there exists a function $\tau \in C^1([v_0, \infty), (0, \infty))$, also, they proved oscillatory by using the Riccati transformation if

$$\int_v^\infty \left(\tau(s) q(s) - \frac{(\kappa - 2)! (\tau'(s))^2}{2^{3-2\kappa} s^{\kappa-2} \tau(s)} \right) ds = \infty. \tag{7}$$

To compare the conditions, we apply the previous results to the equation

$$y^{(4)}(v) + \frac{q_0}{v^4} y(3v) = 0, v \geq 1, \tag{8}$$

1. By applying Condition (5) in [24], we get

$$q_0 > 13.6$$

2. By applying Condition (6) in [25], we get

$$q_0 > 18.$$

3. By applying Condition (7) in [26], we get

$$q_0 > 576.$$

The main aim of this paper is to establish new oscillation results of solutions to a class of fourth-order differential equations with delayed arguments and they essentially complement the results reported in [24–26].

The rest of the paper is organized as follows. In Section 2, four lemmas are given to prove the main results. In Section 3, we establish new oscillation results for Equation (1), comparisons are carried out with oscillations of first and second-order delay differential equations and some examples are presented to illustrate the main results. Some conclusions are discussed in Section 4.

2. Some Auxiliary Lemmas

In this section, the following some auxiliary lemmas are provided

Lemma 1 ([23]). Suppose that $y \in C^\kappa([v_0, \infty), (0, \infty))$, $y^{(\kappa)}$ is of a fixed sign on $[v_0, \infty)$, $y^{(\kappa)}$ not identically zero and there exists a $v_1 \geq v_0$ such that

$$y^{(\kappa-1)}(v)y^{(\kappa)}(v) \leq 0,$$

for all $v \geq v_1$. If we have $\lim_{v \rightarrow \infty} y(v) \neq 0$, then there exists $v_\theta \geq v_1$ such that

$$y(v) \geq \frac{\theta}{(\kappa-1)!} v^{\kappa-1} |y^{(\kappa-1)}(v)|,$$

for every $\theta \in (0, 1)$ and $v \geq v_\theta$.

Lemma 2 ([30]). Let β be a ratio of two odd numbers, $V > 0$ and U are constants. Then

$$Ux - Vx^{(\beta+1)/\beta} \leq \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \frac{U^{\beta+1}}{V^\beta},$$

for all positive x .

Lemma 3 ([9]). If $y^{(i)}(v) > 0, i = 0, 1, \dots, \kappa$, and $y^{(\kappa+1)}(v) < 0$, then

$$\frac{y(v)}{v^\kappa/\kappa!} \geq \frac{y'(v)}{v^{\kappa-1}/(\kappa-1)!}.$$

Lemma 4 ([7]). Suppose that y is an eventually positive solution of Equation (1). Then, there exist two possible cases:

$$\begin{aligned} (S_1) \quad & y(v) > 0, y'(v) > 0, y''(v) > 0, y'''(v) > 0, y^{(4)}(v) < 0, \\ (S_2) \quad & y(v) > 0, y'(v) > 0, y''(v) < 0, y'''(v) > 0, y^{(4)}(v) < 0, \end{aligned}$$

for $v \geq v_1$, where $v_1 \geq v_0$ is sufficiently large.

3. Oscillation Criteria

In this section, we shall establish some oscillation criteria for fourth order advanced differential Equation (1).

Remark 1. It is well known (see [31]), the differential equation

$$\left[a(v) (y'(v))^\beta \right]' + q(v) y^\beta(g(v)) = 0, \quad v \geq v_0, \tag{9}$$

where $\beta > 0$ is the ratio of odd positive integers, $a, q \in C([v_0, \infty), \mathbb{R}^+)$ is nonoscillatory if and only if there exists a number $v \geq v_0$, and a function $\zeta \in C^1([v, \infty), \mathbb{R})$, satisfying the following inequality

$$\zeta'(v) + \gamma a^{-1/\beta}(v) (\zeta(v))^{(1+\beta)/\beta} + q(v) \leq 0, \text{ on } [v, \infty).$$

In what follows, we compare the oscillatory behavior of Equation (1) with the second-order half-linear equations of the type in Equation (9). There are numerous results concerning the oscillation of (9), which included Hille and Nehari types, Philos type, etc.

Theorem 1. Assume that Equation (2) holds. If the differential equations

$$\left(\frac{2a^{1/\beta}(v)}{(\theta v^2)^\beta} (y'(v))^\beta \right)' + \sum_{i=1}^j q_i(v) y^\beta(v) = 0 \tag{10}$$

and

$$y''(v) + y(v) \int_v^\infty \left(\frac{1}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta = 0 \tag{11}$$

are oscillatory for some constant $\theta \in (0, 1)$, then every solution of Equation (1) is oscillatory.

Proof. By contradiction, assume that y is a positive solution of Equation (1). Then, we can suppose that $y(v)$ and $y(\eta_i(v))$ are positive for all $v \geq v_1$ sufficiently large. From Lemma 4, we have two possible cases (S₁) and (S₂).

Let case (S₁) holds, then with the help of Lemma 1, we get

$$y'(v) \geq \frac{\theta}{2} v^2 y'''(v), \tag{12}$$

for every $\theta \in (0, 1)$ and for all large v .

Define

$$\varphi(v) := \tau(v) \left(\frac{a(v) (y'''(v))^\beta}{y^\beta(v)} \right), \tag{13}$$

we see that $\varphi(v) > 0$ for $v \geq v_1$, where there exists a positive function $\tau \in C^1([v_0, \infty), (0, \infty))$ and

$$\begin{aligned} \varphi'(v) &= \tau'(v) \frac{a(v) (y'''(v))^\beta}{y^\beta(v)} + \tau(v) \frac{(a(y''')^\beta)'(v)}{y^\beta(v)} \\ &\quad - \beta \tau(v) \frac{y^{\beta-1}(v) y'(v) a(v) (y'''(v))^\beta}{y^{2\beta}(v)}. \end{aligned}$$

Using Equations (12) and (13), we obtain

$$\begin{aligned} \varphi'(v) &\leq \frac{\tau'_+(v)}{\tau(v)} \varphi(v) + \tau(v) \frac{(a(v) (y'''(v))^\beta)'(v)}{y^\beta(v)} \\ &\quad - \beta \tau(v) \frac{\theta}{2} v^{\kappa-2} \frac{a(v) (y'''(v))^{\beta+1}}{y^{\beta+1}(v)} \\ &\leq \frac{\tau'(v)}{\tau(v)} \varphi(v) + \tau(v) \frac{(a(v) (y'''(v))^\beta)'(v)}{y^\beta(v)} \\ &\quad - \frac{\beta \theta v^2}{2(\tau(v) a(v))^{1/\beta}} \varphi(v)^{\frac{\beta+1}{\beta}}. \end{aligned} \tag{14}$$

From Equations (1) and (14), we obtain

$$\varphi'(v) \leq \frac{\tau'(v)}{\tau(v)} \varphi(v) - \ell \tau(v) \frac{\sum_{i=1}^j q_i(v) y^\beta(\eta_i(v))}{y^\beta(v)} - \frac{\beta \theta v^2}{2(\tau(v) a(v))^{\frac{1}{\beta}}} \varphi(v)^{\frac{\beta+1}{\beta}}.$$

Note that $y'(v) > 0$ and $\eta_i(v) \geq v$, thus, we get

$$\varphi'(v) \leq \frac{\tau'(v)}{\tau(v)} \varphi(v) - \ell \tau(v) \sum_{i=1}^j q_i(v) - \frac{\beta \theta v^2}{2(\tau(v) a(v))^{\frac{1}{\beta}}} \varphi(v)^{\frac{\beta+1}{\beta}}. \tag{15}$$

If we set $\tau(v) = \ell = 1$ in Equations (15), then we find

$$\varphi'(v) + \frac{\beta \theta v^2}{2a^{\frac{1}{\beta}}(v)} \varphi(v)^{\frac{\beta+1}{\beta}} + \sum_{i=1}^j q_i(v) \leq 0.$$

Thus, we can see that Equation (10) is a nonoscillatory, which is a contradiction. Let suppose the case (S₂) holds. Define

$$\psi(v) := \vartheta(v) \frac{y'(v)}{y(v)},$$

we see that $\psi(v) > 0$ for $v \geq v_1$, where there exist a positive function $\vartheta \in C^1([v_0, \infty), (0, \infty))$. By differentiating $\psi(v)$, we obtain

$$\psi'(v) = \frac{\vartheta'(v)}{\vartheta(v)} \psi(v) + \vartheta(v) \frac{y''(v)}{y(v)} - \frac{1}{\vartheta(v)} \psi(v)^2. \tag{16}$$

Now, integrating Equation (1) from v to m and using $y'(v) > 0$, we obtain

$$a(m) (y'''(m))^\beta - a(v) (y'''(v))^\beta = - \int_v^m \sum_{i=1}^j q_i(s) g(y(\eta_i(s))) ds.$$

By virtue of $y'(v) > 0$ and $\eta_i(v) \geq v$, we get

$$a(m) (y'''(m))^\beta - a(v) (y'''(v))^\beta \leq -\ell y^\beta(v) \int_v^m \sum_{i=1}^j q_i(s) ds.$$

Letting $m \rightarrow \infty$, we see that

$$a(v) (y'''(v))^\beta \geq \ell y^\beta(v) \int_v^\infty \sum_{i=1}^j q_i(s) ds$$

and hence

$$y'''(v) \geq y(v) \left(\frac{\ell}{a(v)} \int_v^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta}.$$

Integrating again from v to ∞ , we get

$$y''(v) + y(v) \int_v^\infty \left(\frac{\ell}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta \leq 0. \tag{17}$$

From Equations (16) and (17), we obtain

$$\psi'(v) \leq \frac{\vartheta'(v)}{\vartheta(v)} \psi(v) - \vartheta(v) \int_v^\infty \left(\frac{\ell}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta - \frac{1}{\vartheta(v)} \psi(v)^2. \tag{18}$$

If we now set $\vartheta(v) = \ell = 1$ in Equation (18), then we obtain

$$\psi'(v) + \psi^2(v) + \int_v^\infty \left(\frac{1}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta \leq 0.$$

Thus, it can be seen that Equation (11) is non oscillatory, which is a contradiction. Hence, Theorem 1 is proved. \square

Remark 2. It is well known (see [19]) that if

$$\int_{v_0}^\infty \frac{1}{a(v)} dv = \infty, \text{ and } \liminf_{v \rightarrow \infty} \left(\int_{v_0}^v \frac{1}{a(s)} ds \right) \int_v^\infty q(s) ds > \frac{1}{4},$$

then Equation (9) with $\beta = 1$ is oscillatory.

Based on the above results and Theorem 1, we can easily obtain the following Hille and Nehari type oscillation criteria for (1) with $\beta = 1$.

Theorem 2. Let $\beta = \ell = 1$, and assuming that Equation (2) holds, if

$$\int_{v_0}^\infty \frac{\theta v^2}{2a(v)} dv = \infty$$

and

$$\liminf_{v \rightarrow \infty} \left(\int_{v_0}^v \frac{\theta s^2}{2a(s)} ds \right) \int_v^\infty \sum_{i=1}^j q_i(s) ds > \frac{1}{4}, \tag{19}$$

also, if

$$\liminf_{v \rightarrow \infty} v \int_{v_0}^v \int_v^\infty \left(\frac{1}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right) d\zeta dv > \frac{1}{4}, \tag{20}$$

for some constant $\theta \in (0, 1)$, then all solutions of Equation (1) are oscillatory.

In the following theorem, we compare the oscillatory behavior of Equation (1) with the first-order differential equations:

Theorem 3. Assume that Equation (2) holds, if the differential equations

$$x'(v) + \ell \sum_{i=1}^j q_i(v) \left(\frac{\theta v^2}{2a^{1/\beta}(v)} \right)^\beta x(\eta(v)) = 0 \tag{21}$$

and

$$z'(v) + v z(v) \int_v^\infty \left(\frac{\ell}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta = 0 \tag{22}$$

are oscillatory for some constant $\theta \in (0, 1)$, then every solutions of Equation (1) is oscillatory.

Proof. We prove this theorem by contradiction again, assume that y is a positive solution of Equation (1). Then, we can suppose that $y(v)$ and $y(\eta_i(v))$ are positive for all $v \geq v_1$ sufficiently large. From Lemma 4, we have two possible cases (S_1) and (S_2).

In the case where (S_1) holds, from Lemma 1, we see

$$y(v) \geq \frac{\theta v^2}{2a^{1/\beta}(v)} \left(a^{1/\beta}(v) y'''(v) \right),$$

for every $\theta \in (0, 1)$ and for all large v . Thus, if we set

$$x(v) = a(v) (y'''(v))^\beta > 0,$$

then we see that ψ is a positive solution of the inequality

$$x'(v) + \ell \sum_{i=1}^j q_i(v) \left(\frac{\theta v^2}{2a^{1/\beta}(v)} \right)^\beta x(\eta(v)) \leq 0. \tag{23}$$

From [20] [Theorem 1], we conclude that the corresponding Equation (21) has a positive solution, which is a contradiction. In the case where (S_2) holds. From Lemma 3, we get

$$y(v) \geq v y'(v), \tag{24}$$

From Equations (17) and (24), we get

$$y''(v) + v y'(v) \int_v^\infty \left(\frac{\ell}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta \leq 0.$$

Now, we set

$$z(v) = y'(v).$$

Thus, we find ψ is a positive solution of the inequality

$$z'(v) + v z(v) \int_v^\infty \left(\frac{\ell}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta \leq 0. \tag{25}$$

From ([20], Theorem 1), we conclude that the corresponding Equation (22) has a positive solution, which is a contradiction again. Thus the proof is completed. \square

Corollary 1. Let Equation (2) hold, if

$$\liminf_{v \rightarrow \infty} \int_v^{\eta_i(v)} \ell \sum_{i=1}^j q_i(s) \left(\frac{\theta s^2}{2a^{1/\beta}(s)} \right)^\beta ds > \frac{6^\beta}{e} \tag{26}$$

and

$$\liminf_{v \rightarrow \infty} \int_v^{\eta_i(v)} s \int_v^\infty \left(\frac{\ell}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta ds > \frac{1}{e} \tag{27}$$

for some constant $\theta \in (0, 1)$, then every solutions of Equation (1) is oscillatory.

Example 1. Consider a differential equation

$$\left(v^3 (w'''(v))^3 \right)' + \frac{q_0}{v^6} w^3(2v) = 0, \quad v \geq 1, \tag{28}$$

where q_0 is a constant. Let $\beta = 3$, $a(v) = v^3$, $q(v) = q_0/v^6$ and $\eta(v) = 2v$. If we set $\ell = 1$, then Condition (26) becomes

$$\begin{aligned} \liminf_{v \rightarrow \infty} \int_v^{\eta_i(v)} \ell \sum_{i=1}^j q_i(s) \left(\frac{\theta s^2}{2a^{1/\beta}(s)} \right)^\beta ds &= \liminf_{v \rightarrow \infty} \int_v^{2v} \frac{q_0}{s^6} \left(\frac{\theta s^2}{2s^{1/3}} \right)^3 ds \\ &= \liminf_{v \rightarrow \infty} \left(\frac{q_0 \theta^3}{8} \right)^3 \int_v^{2v} \frac{q_0}{s} ds \\ &= \frac{q_0 \theta^3 \ln 2}{8} > \frac{6^3}{e} \end{aligned}$$

and Condition (27) holds. Therefore, from Corollary 1, all solutions of Equation (28) are oscillatory if $q_0 > 1728 / (\theta^3 e \ln 2)$ for some constant $\theta \in (0, 1)$.

Example 2. Let the equation

$$y^{(4)}(v) + \frac{q_0}{v^4} y(2v) = 0, \quad v \geq 1, \tag{29}$$

where $q_0 > 0$ is a constant. Let $\beta = 1$, $a(v) = 1$, $q(v) = q_0/v^4$ and $\eta(v) = 2v$. If we set $\ell = 1$, then Condition (19) becomes

$$\begin{aligned} \liminf_{v \rightarrow \infty} \left(\int_{v_0}^v \frac{\theta s^2}{2a(s)} ds \right) \int_v^\infty \sum_{i=1}^j q_i(s) ds &= \liminf_{v \rightarrow \infty} \left(\frac{v^3}{3} \right) \int_v^\infty \frac{q_0}{s^4} ds \\ &= \frac{q_0}{9} > \frac{1}{4} \end{aligned}$$

and Condition (20) becomes

$$\begin{aligned} \liminf_{v \rightarrow \infty} \int_{v_0}^v \int_v^\infty \left(\frac{1}{a(\zeta)} \int_\zeta^\infty \sum_{i=1}^j q_i(s) ds \right)^{1/\beta} d\zeta dv &= \liminf_{v \rightarrow \infty} \left(\frac{q_0}{6v} \right) \\ &= \frac{q_0}{6} > \frac{1}{4}. \end{aligned}$$

Therefore, from Theorem 2, all solutions of Equation (29) are oscillatory if $q_0 > 2.25$.

Remark 3. We compare our result with the known related criteria

The condition	(5)	(6)	(7)	our condition
The criterion	$q_0 > 25.5$	$q_0 > 18$	$q_0 > 1728$	$q_0 > 2.25$

Example 3. Consider a differential Equation (8) where $q_0 > 0$ is a constant. Note that $\beta = 1$, $\kappa = 4$, $a(v) = 1$, $q(v) = q_0/v^4$ and $\eta(v) = 3v$. If we set $\ell = 1$, then Condition (19) becomes

$$\frac{q_0}{9} > \frac{1}{4}.$$

Therefore, from Theorem 2, all the solutions of Equation (8) are oscillatory if $q_0 > 2.25$.

Remark 4. We compare our result with the known related criteria

The condition	(5)	(6)	(7)	our condition
The criterion	$q_0 > 13.6$	$q_0 > 18$	$q_0 > 576$	$q_0 > 2.25$

Example 4. Let the equation

$$y^{(4)}(v) + \frac{q_0}{v^2} y(cv) = 0, \quad v > 1, \tag{30}$$

where $q_0 > 0$, $c > 1$ are constants. Note that $\beta = 1$, $a(v) = 1$, $q(v) = q_0/v^2$ and $\eta(v) = cv$.

From ([14], Corollary 2.4), we have that the equation

$$y''(v) + \frac{q_0}{v^2}y(cv) = 0, \quad c > 1, \quad q_0 > 0,$$

is oscillatory if

$$q_0(1 + q_0 \ln c) > \frac{1}{4}.$$

Therefore, from Theorem 1, all the solutions of Equation (30) are oscillatory if $q_0(1 + q_0 \ln c) > 1/4$.

4. Conclusions

In this paper, the main aim to provide a study of asymptotic behavior of the fourth order advanced differential equation has been achieved. We used the theory of comparison with first and second-order delay equations and the Riccati substitution to ensure that every solution of this equation is oscillatory. The presented results complement a number of results reported in the literature. Furthermore, the findings of this paper can be extended to study a class of systems of higher order advanced differential equations.

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