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Effective Method for Solving Different Types of Nonlinear Fractional Burgers' Equations

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Abstract: In this study, a relatively new method to solve partial differential equations (PDEs) called the fractional reduced differential transform method (FRDTM) is used. The implementation of the method is based on an iterative scheme in series form. We test the proposed method to solve nonlinear fractional Burgers equations in one, two coupled, and three dimensions. To show the efficiency and accuracy of this method, we compare the results with the exact solutions, as well as some established methods. Approximate solutions for different values of fractional derivatives together with exact solutions and absolute errors are represented graphically in two and three dimensions. From all numerical results, we can conclude the efficiency of the proposed method for solving different types of nonlinear fractional partial differential equations over existing methods.

Keywords: fractional calculus; fractional reduced differential transform method; Caputo derivative; Burgers' equation

1. Introduction

Mathematical modeling of nonlinear systems is a major challenge for scientists currently. The study of the exact and approximate solutions helps us to understand the applications of these mathematical models. Finding an analytical solution is a very difficult task in most cases. Fractional order derivatives provide researchers with new ways of modeling numerous types of phenomena in science. Different kinds of fractional derivatives and their properties were considered in [1–5]. A survey of several diverse applications that have arisen from fractional calculus was given by Podlubny [4]. Abuasad et al. [6] applied a fractional multi-step differential transformed method to find approximate solutions to one of the most important to epidemiology and mathematical ecology, the fractional stochastic susceptible-infective-susceptible (SIS) epidemic model with imperfect vaccination. The most vital criteria that defined fractional derivatives were shown by Ross [7]. Several researchers are trying to find techniques to solve different types of fractional differential equations by modifying the traditional methods, while others have linked two or more methods to find numerical or analytical solutions of fractional equations. In [8,9], we modified the definition of the beta fractional derivative to find exact and approximate solutions of time fractional diffusion equations in different dimensions.

The differential transform method (DTM) was first applied to electrical circuit problems by Zhou [10]. The DTM was well addressed in [11–15]. To overcome the long computations of DTM, Keskin and Oturanc [16] presented an effective and powerful technique called the reduced differential transform method (RDTM). Srivastava et al. [17] used RDTM for solving the $(1+n)$-dimensional Burgers equation. Keskin and Oturance [18] proposed the so-called fractional reduced differential transform method (FRDTM).

The central benefits of FRDTM is that it can solve different classes of linear and nonlinear partial fractional differential equations (PFDE) in high dimensions. In most cases of linear PFDE, we can find the exact solutions; nevertheless, in the case of nonlinear PFDE, the approximate solutions are satisfying in comparison with exact solutions of non-fractional equations. Abuasad et al. [19] introduced a new modification of the fractional reduced differential transform method (m-FRDTM) to find exact and approximate solutions for multi-term time fractional diffusion equations (MT-TFDEs). Gupta [20] studied eight different cases to obtain the approximate analytical solutions of the Benney–Lin equation with the fractional time derivative by FRDTM and the homotopy perturbation method (HPM). Rawashdeh [21] used FRDTM to find approximate analytical solutions to the time fractional Sharma–Tasso–Olver equation and the time fractional damped Burger equation. Srivastava et al. [22] solved the generalized time fractional order biological population model (GTFBPM) by FRDTM. Rawashdeh [23] employed FRDTM to solve the nonlinear fractional Harry Dym equation. Singh and Srivastava [24] presented an approximate series solution of the multi-dimensional (heat-like) diffusion equation with the time fractional derivative using FRDTM. Singh [25] presented FRDTM to compute an alternative approximate solution of the initial valued autonomous system of linear and nonlinear fractional partial differential equations. Rawashdeh [26] proposed FRDTM to solve the one-dimensional space and time fractional Burgers equations and the time fractional Cahn–Allen equation, Arshad et al. [27] presented a general form of FRDTM to solve the wave-like problem, the Zakharov–Kuznetsov equation, and the coupled Burgers equation, and Abuasad et al. [28] proposed FRDTM for finding exact and approximate solutions of the fractional Helmholtz equation.

In the comparison between DTM, RDTM, FRDTM, and MsDTM (multi-step differential transform method), we find that DTM is an improved method of the Taylor series method, which needs additional computational work for large orders, and it decreases the size of the computational domain and is appropriate for numerous problems [14]. Meanwhile, RDTM is simpler than DTM, and the total number of calculations essential in RDTM is much fewer than that in traditional DTM [16]. FRDTM is a modified method of RDTM for fractional order derivatives. The multi-step differential transform method (MsDTM) is able to overcome the key disadvantages of the DTM and RDTM, which are that the achieved series solution frequently converges in a very insignificant space and the range of convergence is a very slow procedure or entirely divergent given a wider space [6].

Burgers' equation (BE) was proposed by Burgers [29] to describe the mathematical model of turbulence. It is the simplest nonlinear diffusion equation arising in fluid mechanics. BE can be transformed into the diffusion equation using the Hopf–Cole transformation [30]. BE has been applied to turbulence problems [31], traffic flow [32], and plane waves [33]. Several numerical methods have been used to solve BE, for instance the finite element method and generalized boundary element method. Esen and Tabozan [34] solved the time fractional order BE by the quadratic B-spline Galerkin method. Li et al. [35] proposed a linear implicit finite difference scheme for solving generalized time fractional BE, and Miškinis [36] presented some important properties of the fractional BE and the relation with BE of integer order. Non-perturbative analytical solutions for the generalized BE with time and space fractional derivatives were derived using the Adomian decomposition method by Momani [37]. Safari et al. [38] applied the variational iteration method (VIM) and the Adomian decomposition method (ADM) to the fractional KdV–Burgers–Kuramoto equation.

In this paper, we give approximate solutions of the time fractional Burgers' equation in one, two coupled, and three dimensions. The rest of this article is organized as follows: In Section 2, we present the basic definitions of the fractional derivatives in brief. Section 3 gives the idea of FRDTM. In Section 4, we illustrate an application of this method to the fractional Burgers equation in different dimensions. Section 5 is the conclusion.

2. Preliminaries and Fractional Derivative Order

This section gives some important definitions, such as the Gamma function and the basic definitions of the fractional derivatives.

2.1. The Gamma Function

The most basic interpretation of the (complete) Gamma function $\Gamma(x)$ is simply the generalization of the factorial to complex and real arguments. The Gamma function can be defined as [4]:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, \mathrm{d}t, \quad x > 0. \tag{1}$$

2.2. Fractional Derivative

In this paper, we will use the Caputo fractional derivative. The initial conditions for fractional order differential equations with the Caputo fractional derivative are in a form involving only the limit values of integer order derivatives at the lower terminal initial time $(t = a)$, such as $y'(a), y''(a), \ldots$ [4]. The fractional derivative of a constant function is zero.

Definition 1. *The Caputo fractional derivative is defined as [5]:*

$$ {}^{c}\mathbf{D}_a^\alpha f(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \dfrac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} \mathrm{d}\tau, & n-1 < \alpha < n, \\ \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} f(t), & \alpha = n, \end{cases} \tag{2}$$

Ross [7] indicated five criteria for the fractional derivative: If $\mathbf{D}_x^\alpha f(x)$ represents the fractional Caputo derivative, where $\alpha > 0,\ \beta > 0$, then all of the following need to be satisfied:

- The fractional derivative of an analytical function is analytic.
- Backward compatibility: when the order is positive, the integer fractional derivative gives the same result as the ordinary derivative.
- Identity: the zero order derivative of a function returns the function itself.
- Linearity: the operator must be linear.
- The index law of fractional integrals holds, that is,

$$\mathbf{D}_x^{-\alpha}\, \mathbf{D}_x^{-\beta} f(x) = \mathbf{D}_x^{-\alpha-\beta} f(x). \tag{3}$$

Other criteria for the fractional derivatives can be found in [39,40].

3. Fractional Reduced Differential Transform Method

In this section, we give the basic definitions and properties of FRDTM [22,24,27,28]. Consider a function of $(n+1)$ variables $w(t, x_1, x_2, \ldots, x_n)$, such that:

$$w(t, x_1, x_2, \ldots, x_n) = m_1(x_1) m_2(x_2) \cdots m_n(x_n) h(t),$$

then from the properties of the one-dimensional differential transform method (DTM) and motivated by the components of the form $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} t^{\alpha j}$, we write the general solution function $w(t, x_1, x_2, \ldots, x_n)$

as an infinite linear combination of such components to obtain the second half of Equation (4), then we have:

$$\begin{aligned} w(t, x_1, x_2, ..., x_n) &= \sum_{i_1=0}^{\infty} m_1(i_1)x_1^{i_1} \sum_{i_2=0}^{\infty} m_2(i_2)x_2^{i_2} \cdots \sum_{i_n=0}^{\infty} m_n(i_n)x_n^{i_n} \sum_{j=0}^{\infty} h(j)t^{\alpha j} \\ &= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \sum_{j=0}^{\infty} W(i_1, i_2, ..., i_n, j)x_1^{i_1}x_2^{i_2} \cdots x_n^{i_n} t^{\alpha j}, \end{aligned} \tag{4}$$

where $W(i_1, i_2, ..., i_n, j) = m_1(i_1)m_2(i_2) \cdots m_n(i_n)h(j)$ is referred to as the spectrum of $w(t, x_1, x_2, ..., x_n)$. Furthermore, the lowercase $w(t, x_1, x_2, ..., x_n)$ is used for the original function, while its fractional reduced transformed function is represented by the uppercase $W_k(x_1, x_2, ..., x_n)$, which is called the T-function.

Let $w(t, x_1, x_2, ..., x_n)$ be analytical and continuously differentiable with respect to $n + 1$ variables t, $x_1, x_2, ...$ to x_n in the domain of interest, then the FRDTM in n dimensions of $w(t, x_1, x_2, ..., x_n)$ is given by:

$$W_k(x_1, x_2, ..., x_n) = \frac{1}{\Gamma(k\alpha + 1)}[\mathbf{D}_t^{\alpha k}(w(t, x_1, x_2, ..., x_n))]_{t=t_0}, \tag{5}$$

where $k = 0, 1, 2, \ldots$, with time fractional derivative.

The inverse FRDTM of $W_k(x_1, x_2, ..., x_n)$ is defined by:

$$w(t, x_1, x_2, ..., x_n) := \sum_{k=0}^{\infty} W_k(x_1, x_2, ..., x_n)(t - t_0)^{k\alpha}. \tag{6}$$

From (5) and (6), we have:

$$w(t, x_1, x_2, ..., x_n) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha + 1)}[\mathbf{D}_t^{\alpha k}(w(t, x_1, x_2, ..., x_n))]_{t=t_0}(t - t_0)^{k\alpha}.$$

In particular, for $t_0 = 0$, the above equation becomes:

$$w(t, x_1, x_2, ..., x_n) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha + 1)}[\mathbf{D}_t^{\alpha k}(w(t, x_1, x_2, ..., x_n))]_{t=0}t^{k\alpha}. \tag{7}$$

From the above definition, it can be seen that the concept of FRDTM is derived from the power series expansion of a function. Then, the inverse transformation of the set of values $\{W_k(x_1, x_2, ..., x_n)\}_{k=0}^{m}$ gives an approximate solution as:

$$\tilde{w}_m(t, x_1, x_2, ..., x_n) = \sum_{k=0}^{m} W_k(x_1, x_2, ..., x_n)t^{\alpha k}, \tag{8}$$

where n is the order of the approximate solution. Therefore, the exact solution is given by:

$$w(t, x_1, x_2, ..., x_n) = \lim_{m \to \infty} \tilde{w}_m(t, x_1, x_2, ..., x_n). \tag{9}$$

In Table 1, we give some properties of FRDTM, where the transform is essentially a generating function and $\delta(k - r)$ is defined by:

$$\delta(k - r) = \begin{cases} 1, & k = r \\ 0, & k \neq r. \end{cases} \tag{10}$$

where $w = w(t, x_1, x_2..., x_n), u = u(t, x_1, x_2..., x_n), W_k = W_k(x_1, x_2..., x_n), U_k = U_k(x_1, x_2..., x_n)$.

Table 1. Fundamental operations of the fractional reduced differential transform method (FRDTM) [14,16,27].

Original Function	Transformed Function
$w = c_1 u \pm c_2 v$	$W_{\alpha k} = c_1 U_{\alpha k} \pm c_2 V_{\alpha k}$
$w = uv$	$W_{\alpha k} = \sum_{i=0}^{k} U_{\alpha i} V_{\alpha(k-i)}$
$w = \mathbf{D}_t^{m\alpha} u$	$W_{\alpha k} = \dfrac{\Gamma(\alpha(k+m)+1)}{\Gamma(k\alpha+1)} U_{\alpha(k+m)}$
$w = \dfrac{\partial^h u}{\partial x_i^h}$	$W_{\alpha k} = \dfrac{\partial^h U_{\alpha k}}{\partial x_i^h}, i = 1, 2, ..., n$
$w = x_i^m t^r$	$W_{k\alpha} = x_i^m \delta(\alpha k - r), i = 1, 2, ..., n$
$w = x_i^m t^r u$	$W_{\alpha k} = x_i^m \sum_{i=0}^{k} \delta(\alpha i - r) U_{\alpha(k-r)}, i = 1, 2, ..., n$

4. Numerical Experiments

To demonstrate the efficiency of FRDTM as an approximate tool for solving different types of nonlinear fractional diffusion equations, we apply the proposed algorithm to three nonlinear time fractional BEs of one-dimensional, (3 + 1)-dimensional, and (2 + 1)-dimensional coupled equations.

4.1. Example 1

Consider the following one-dimensional nonlinear time fractional BE, where $0 < \alpha \leq 1$ [41]:

$$\mathbf{D}_t^{\alpha} u + au\frac{\partial u}{\partial x} - c\frac{\partial^2 u}{\partial x^2} = 0, \tag{11}$$

with the initial condition:

$$u(x,0) = u_0(x,t) = \frac{2[b - bc\tanh(bx)]}{a}, \tag{12}$$

where a, b, and c are arbitrary constants and $a \neq 0$. Applying the appropriate properties given in Table 1 to Equation (11), we obtain the following recurrence relation:

$$U_{k+1} = \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)} \left(c\frac{\partial^2 U_k}{\partial x^2} - a\sum_{r=0}^{k} U_r \frac{\partial U_{k-r}}{\partial x} \right), \tag{13}$$

where $k = 0, 1, 2, \ldots$ and $U_k = U_k(x,t)$. From (13), we obtain the inverse transform coefficients of $t^{k\alpha}$ as follows:

$$U_0 = \frac{2[b - b\,c\,\tanh(bx)]}{a}, \quad U_1 = \frac{4b^3 c\,\mathrm{sech}^2(bx)}{a\,\Gamma(\alpha+1)},$$
$$U_2 = \frac{16\,b^5\,c\,\tanh(bx)\,\mathrm{sech}^2(bx)}{a\,\Gamma(2\alpha+1)}, \quad \ldots.$$

Continuing in the same manner and after some successive approximations, the differential inverse transform of $\{U_k\}_{k=0}^{\infty}$ will give the following series solution,

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} U_k t^{k\alpha} \\ &= \frac{2[b - bc\tanh(bx)]}{a} + \frac{4b^3 c\,\mathrm{sech}^2(bx)}{a\,\Gamma(\alpha+1)} t^{\alpha} \\ &+ \frac{16\,b^5\,c\,\tanh(bx)\,\mathrm{sech}^2(bx)}{a\,\Gamma(2\alpha+1)} t^{2\alpha} + \cdots. \end{aligned} \tag{14}$$

If $\alpha = 1$ (non-fractional case) in Equation (14), then the third order approximate solution of FRDTM gives a reliable solution, which is compared with the exact solution in Table 2.

$$\begin{aligned} u_3(x,t) &= \\ \frac{1}{3a}\times & \Big[-48b^7ct^3\mathrm{sech}^4(bx) + 4b^3ct\mathrm{sech}^2(bx)\left(8b^4t^2 + 6b^2t\tanh(bx) + 3\right) \\ & -6bc\tanh(bx) + 6b\Big]. \end{aligned} \tag{15}$$

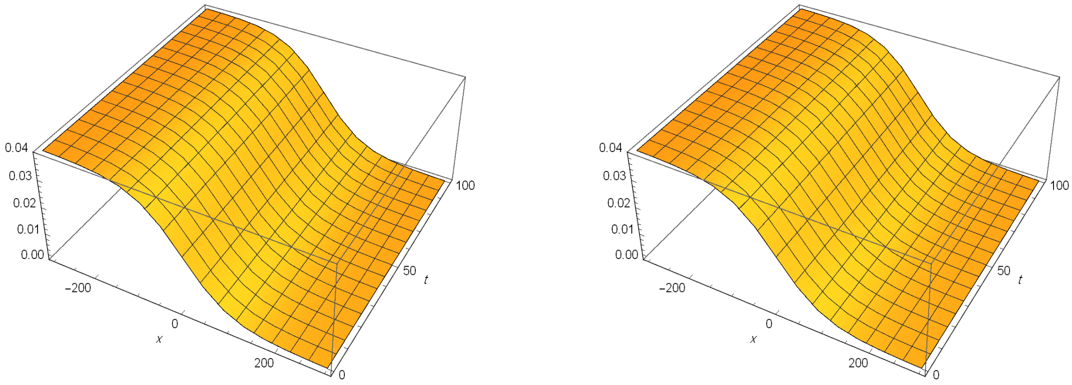
The exact solution of the non-fractional BE (11) is given [41] as:

$$u(x,t) = \frac{2b}{a} - \frac{a\,b\,c}{a}\tanh[b(x-2bt)]. \tag{16}$$

Figure 1 shows the exact solution of non–fractional solution and the three-dimensional plot of the third order approximate solution of FRDTM ($\alpha = 1$), while Figure 2 depicts the third order approximate solutions for ($\alpha = 0.9, 0.5$); Figure 3 depicts solutions in two-dimensional plots for different values of α; Figure 4 shows the absolute errors between the exact solution and third order solution for $\alpha = 1$. Table 2 presents the numerical solutions for $u_3(x,t)$ by FRDTM at different values of α and a comparison of absolute errors at $\alpha = 1$ for FRDTM with fractional variational iteration method (FVIM).

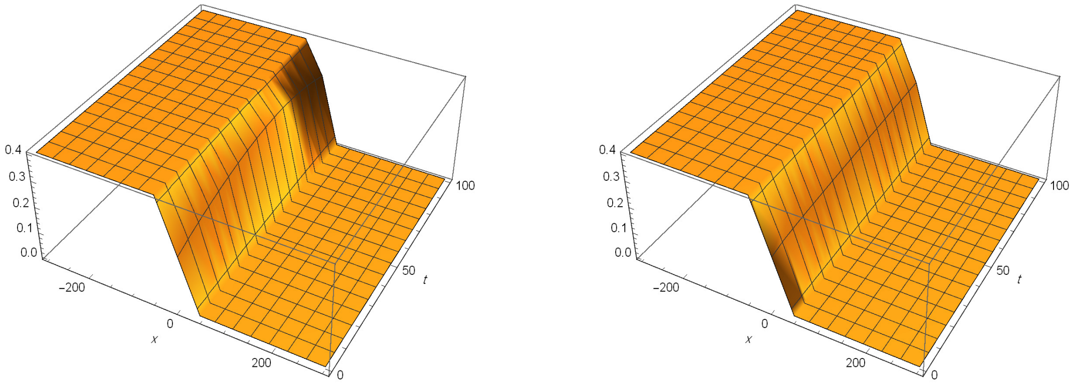
Table 2. Approximate numerical solution for $u_3(x,t)$ by FRDTM at different values of α and a comparison of absolute errors at $\alpha = 1$ for FRDTM with FVIM.

		Non-Fractional Order				**Fractional Order**	
x	t	$\alpha = 1$	u_{Exact}	$\lvert u_{FRDTM} - u_{Exact}\rvert$	$\lvert u_{FVIM} - u_{Exact}\rvert$	$\alpha = 0.9$	$\alpha = 0.8$
	0	0.0399011	0.0399011	0	0	0.0399011	0.0399011
	20	0.0399019	0.0399019	1.6237×10^{-14}	8.36083×10^{-12}	0.0399017	0.0399016
	40	0.0399027	0.0399027	2.59348×10^{-13}	6.67572×10^{-11}	0.0399022	0.0399019
−300	60	0.0399034	0.0399034	1.31092×10^{-12}	2.2487×10^{-10}	0.0399027	0.0399022
	80	0.0399042	0.0399042	4.13682×10^{-12}	5.31996×10^{-10}	0.0399032	0.0399025
	100	0.039905	0.039905	1.00842×10^{-11}	1.03705×10^{-9}	0.0399036	0.0399028
	0	0.0352319	0.0352319	0	0	0.0352319	0.0352319
	20	0.0352654	0.0352654	1.40942×10^{-13}	2.47381×10^{-10}	0.0352577	0.0352517
	40	0.0352987	0.0352987	2.23993×10^{-12}	1.98016×10^{-9}	0.03528	0.0352663
−100	60	0.0353318	0.0353318	1.12634×10^{-11}	6.68676×10^{-9}	0.035301	0.0352794
	80	0.0353646	0.0353646	3.5358×10^{-11}	1.58587×10^{-8}	0.0353212	0.0352916
	100	0.0353973	0.0353973	8.57394×10^{-11}	3.09908×10^{-8}	0.0353409	0.0353031
	0	0.00476812	0.00476812	0	0	0.00476812	0.00476812
	20	0.00480182	0.00480182	1.42837×10^{-13}	1.78436×10^{-11}	0.00479407	0.00478797
	40	0.00483572	0.00483572	2.30062×10^{-12}	1.41591×10^{-10}	0.00481666	0.00480275
100	60	0.00486984	0.00486984	1.17243×10^{-11}	4.73909×10^{-10}	0.00483819	0.00481609
	80	0.00490415	0.00490415	3.73003×10^{-11}	1.11383×10^{-9}	0.00485909	0.00482858
	100	0.00493868	0.00493868	9.16666×10^{-11}	2.15664×10^{-9}	0.00487955	0.00484049
	0	0.0000989049	0.0000989049	0	4.20128×10^{-19}	0.0000989049	0.0000989049
	20	0.0000996974	0.0000996974	1.62847×10^{-14}	8.22805×10^{-22}	0.0000995152	0.0000993717
	40	0.000100496	0.000100496	2.60942×10^{-13}	6.59551×10^{-11}	0.000100047	0.0000997194
300	60	0.000101301	0.000101301	1.32305×10^{-12}	2.2304×10^{-10}	0.000100555	0.000100034
	80	0.000102113	0.000102113	4.18795×10^{-12}	5.29741×10^{-10}	0.000101048	0.000100328
	100	0.000102931	0.000102931	1.02402×10^{-11}	1.03671×10^{-9}	0.000101532	0.000100609



(**a**) (**b**)

Figure 1. (**a**) The exact solution and (**b**) the third order approximation u_3 for $\alpha = 1, a = 1, b = 0.01$, and $c = 1$.



(**a**) (**b**)

Figure 2. The third order approximation u_3 for (**a**) $\alpha = 0.9$ and (**b**) $\alpha = 0.5$, $a = 1, b = 0.1$, and $c = 1$.

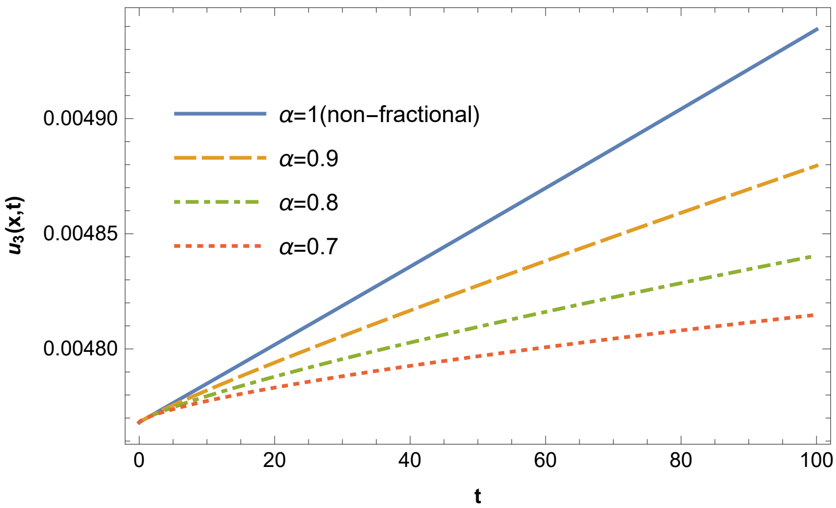


Figure 3. The FRDTM approximate solutions $u_3(x,t)$ for $\alpha = 1, 0.9, 0.8, 0.7$; $t \in [0,100]$, and $x = 100$.

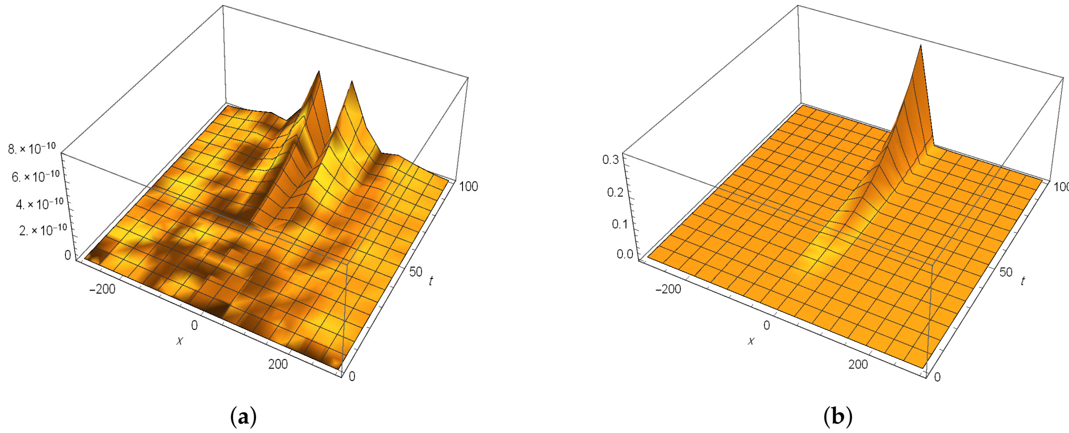


Figure 4. The absolute errors between the exact solution and third order solution for $\alpha = 1, a = 1, c = 1$ with (**a**) $b = 0.01$ and (**b**) $b = 0.1$.

4.2. Example 2

Consider the following (3 + 1)-dimensional time fractional BE [17]:

$$\mathbf{D}_t^\alpha u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - u\frac{\partial u}{\partial x}, \tag{17}$$

where $0 < \alpha \leq 1$, with the initial condition:

$$u(x,y,z,0) = u_0(x,y,z) = x + y + z. \tag{18}$$

Applying the appropriate properties given in Table 1 to Equation (17), we obtain the following recurrence relation:

$$U_{k+1} = \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)}\left(\frac{\partial^2 U_k}{\partial x^2} + \frac{\partial^2 U_k}{\partial y^2} + \frac{\partial^2 U_k}{\partial z^2} + \sum_{r=0}^{k} U_r \frac{\partial U_{k-r}}{\partial x}\right), \tag{19}$$

where $k = 0, 1, 2, \ldots$ and $U_k = U_k(x,y,z)$. From (19), we obtain the inverse transform coefficients of $t^{k\alpha}$ as follows:

$$U_0 = x + y + z, \;\; U_1 = \frac{x+y+z}{\Gamma(\alpha+1)},$$
$$U_2 = \frac{2(x+y+z)}{\Gamma(2\alpha+1)}, \;\; U_3 = \frac{\left[4\Gamma(\alpha+1)^2 + \Gamma(2\alpha+1)\right](x+y+z)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}, \ldots.$$

Continuing in the same manner and after a few iterations, the differential inverse transform of $\{U_k\}_{k=0}^{\infty}$ will give the following series solution:

$$\begin{aligned} u(x,y,z,t) &= \sum_{k=0}^{\infty} U_k t^{k\alpha} \\ &= x + y + z + \frac{x+y+z}{\Gamma(\alpha+1)}t^\alpha + \frac{2(x+y+z)}{\Gamma(2\alpha+1)}t^{2\alpha} \\ &\quad + \frac{\left[4\Gamma(\alpha+1)^2 + \Gamma(2\alpha+1)\right](x+y+z)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}t^{3\alpha} \ldots . \end{aligned} \tag{20}$$

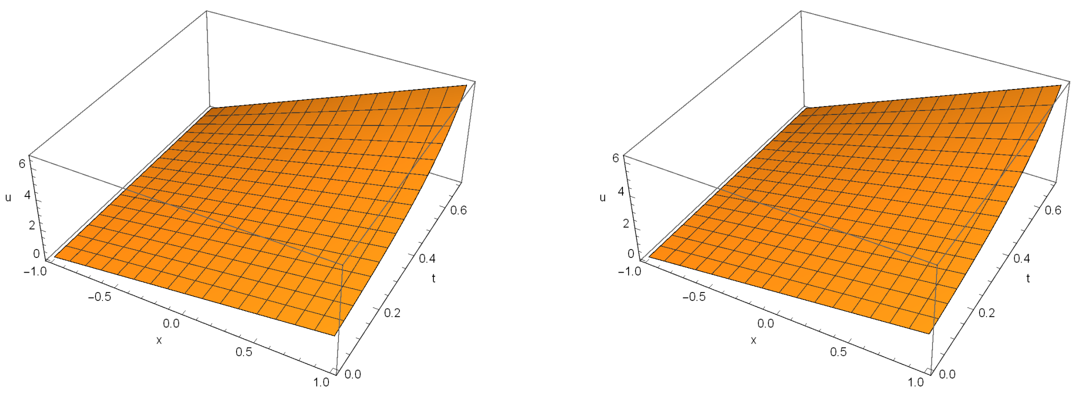
If $\alpha = 1$, then the FRDTM gives the same results as the RDTM. Equation (17) subject to (18) was solved in [17] for non-fractional BE, and the exact solution is:

$$u(x, y, z, t) = \frac{x + y + z}{1 - t}, \quad \text{provided that} \ \ 0 \leq t < 1. \tag{21}$$

The three-dimensional plots of the exact solution and seventh term of the FRDTM solution of (17) with the initial condition (18) are shown in Figures 5 and 6 for different values of $\alpha = 1, 0.8, 0.5$ for $t \in [0, 0.7]$, $x \in [-1, 1]$, and $y = 0.5, z = 0.5$. Figure 7 depicts solutions in two-dimensional plots for the exact solution and different values of $\alpha = 1$ non-fractional, $0.9, 0.8,$ and 0.7 for $t \in [0, 0.8]$ and $y = 0.2, z = 0.1, x = 0.3$. The plot of absolute errors between the exact solution and the seventh order approximate solution of FRDTM is given in Figure 8. Table 3 shows the approximate numerical solution by FRDTM at different values of $\alpha = 1, 0.9, 0.8,$ and 0.7 and comparison of absolute errors for different values of $t = 0, 0.2, 0.4,$ and 0.6 and different values of $x = -300, -100, 100,$ and 300.

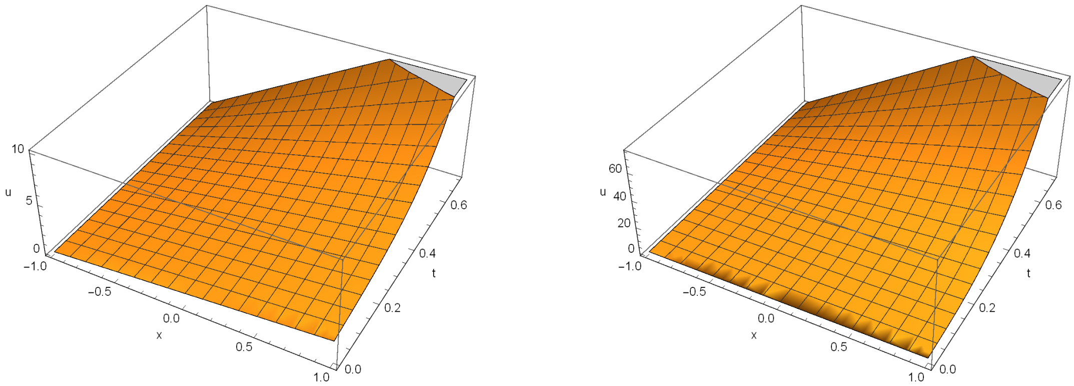
Table 3. Seventh order approximate numerical solutions by FRDTM at different values of α and comparison of absolute errors at $\alpha = 1$.

		Non-Fractional Order				**Fractional Order**	
x	t	$\alpha = 1$	u_{Exact}	Absolute error	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.7$
	0	0.2	0.2	0	0.2	0.2	0.2
-0.8	0.2	0.249999	0.25	6.4×10^{-7}	0.267012	0.339529	0.705107
	0.4	0.333115	0.333333	0.000218453	0.383913	0.677598	2.51926
	0.6	0.491602	0.5	0.00839808	0.639169	1.6304	7.11893
	0	0.5	0.5	0	0.5	0.5	0.5
-0.5	0.2	0.624998	0.625	1.6×10^{-6}	0.66753	0.848823	1.76277
	0.4	0.832787	0.833333	0.000546133	0.959782	1.694	6.29816
	0.6	1.229	1.25	0.0209952	1.59792	4.07599	17.7973
	0	1	1	0	1	1	1
0	0.2	1.25	1.25	3.2×10^{-6}	1.33506	1.69765	3.52553
	0.4	1.66557	1.66667	0.00109227	1.91956	3.38799	12.5963
	0.6	2.45801	2.5	0.0419904	3.19585	8.15198	35.5946
	0	1.5	1.5	0	1.5	1.5	1.5
0.5	0.2	1.875	1.875	4.8×10^{-6}	2.00259	2.54647	5.2883
	0.4	2.49836	2.5	0.0016384	2.87935	5.08199	18.8945
	0.6	3.68701	3.75	0.0629856	4.79377	12.228	53.3919
	0	2	2	0	2	2	2
1	0.2	2.49999	2.5	6.4×10^{-6}	2.67012	3.39529	7.05107
	0.4	3.33115	3.33333	0.00218453	3.83913	6.77598	25.1926
	0.6	4.91602	5.	0.0839808	6.39169	16.304	71.1893



(a) **(b)**

Figure 5. (**a**) $\alpha = 1$ (exact) and (**b**) $\alpha = 1$ (seven term FRDTM).



(**a**) (**b**)

Figure 6. The seventh order approximation u_7 for (a) $\alpha = 0.8$ and (b) $\alpha = 0.5$; $y = 0.5, z = 0.5$.

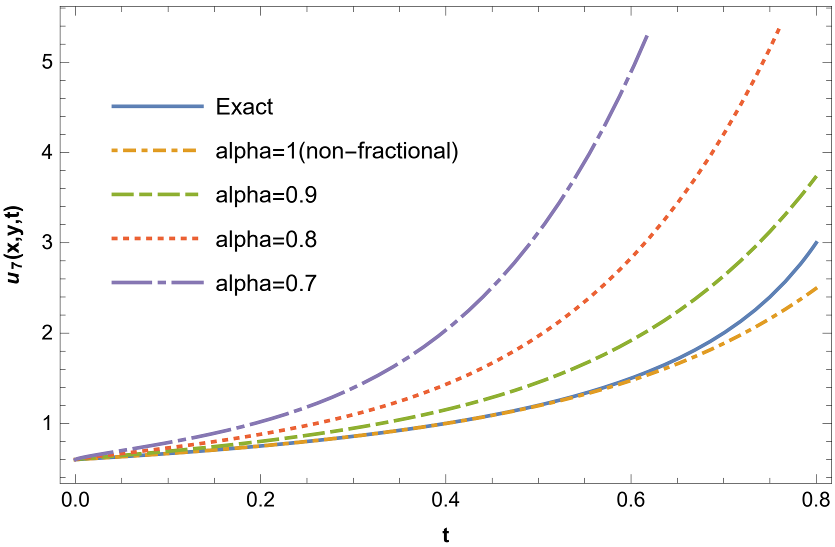
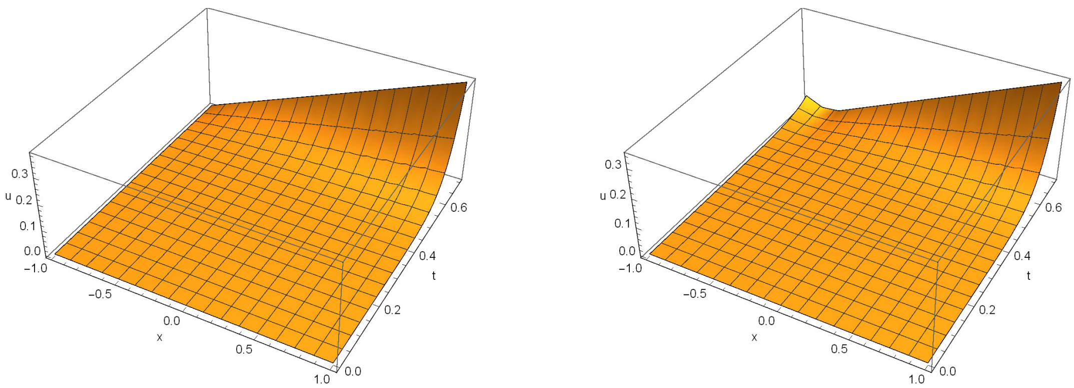


Figure 7. The seven term FRDTM solutions u_7 for $\alpha = 1$ (exact), $0.9, 0.8, 0.7$; $t \in [0, 0.8]$, and $y = 0.2, z = 0.1, x = 0.3$.



(**a**) (**b**)

Figure 8. The absolute errors between the exact solution and the seventh order solution for $\alpha = 1, z = 0.5$ with (**a**) $y = 0.5$ and (**b**) $y = 0.3$.

4.3. Example 3

Consider the following (2 + 1)-dimensional time–fractional coupled BE [42]:

$$\mathbf{D}_t^{\alpha_1} u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{22}$$

$$\mathbf{D}_t^{\alpha_2} v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{23}$$

where $0 < \alpha_1, \alpha_2 \leq 1,\ \ 0 \leq x, y \leq 1,\ \ t > 0$, and $R \gg 1$ is the Reynolds number, subject to the initial conditions:

$$u(x,y,0) = u_0(x,y,t) = \frac{1}{4}\left(\frac{1}{e^{\frac{1}{8}R(x-y)}+1} + 2\right), \tag{24}$$

$$v(x,y,0) = v_0(x,y,t) = \frac{1}{4}\left(\frac{1}{e^{\frac{1}{8}R(y-x)}+1} + 3\right). \tag{25}$$

Applying the appropriate properties given in Table 1 to Equation (22) and Equation (23), we obtain the following recurrence relation:

$$\begin{aligned} U_{k+1} &= \frac{\Gamma(k\alpha_1+1)}{\Gamma(\alpha_1(k+1)+1)} \\ &\times\left[\frac{1}{R}\left(\frac{\partial^2 U_k}{\partial x^2} + \frac{\partial^2 U_k}{\partial y^2}\right) - \sum_{r=0}^{k} U_r \frac{\partial U_{k-r}}{\partial x} - \sum_{r=0}^{k} V_r \frac{\partial U_{k-r}}{\partial y}\right], \end{aligned} \tag{26}$$

$$\begin{aligned} V_{k+1} &= \frac{\Gamma(k\alpha_2+1)}{\Gamma(\alpha_2(k+1)+1)} \\ &\times\left[\frac{1}{R}\left(\frac{\partial^2 V_k}{\partial x^2} + \frac{\partial^2 V_k}{\partial y^2}\right) - \sum_{r=0}^{k} U_r \frac{\partial V_{k-r}}{\partial x} - \sum_{r=0}^{k} V_r \frac{\partial V_{k-r}}{\partial y}\right], \end{aligned} \tag{27}$$

where $k = 0, 1, 2, \ldots$, $U_k = U_k(x,y)$, and $V_k = V_k(x,y)$. From (26) and (27), we obtain the inverse transform coefficients of $t^{k\alpha_i}, i = 1, 2$ as follows:

$$\begin{aligned}
U_0 &= \frac{1}{4}\left(\frac{1}{e^{\frac{1}{8}R(x-y)}+1} + 2\right), \\
V_0 &= \frac{1}{4}\left(\frac{1}{e^{\frac{1}{8}R(y-x)}+1} + 3\right), \\
U_1 &= -\frac{R\,\mathrm{sech}^2\left(\frac{1}{16}R(x-y)\right)}{512\,\Gamma(\alpha_1+1)}, \\
V_1 &= \frac{R\,\mathrm{sech}^2\left(\frac{1}{16}R(x-y)\right)}{512\,\Gamma(\alpha_2+1)}, \\
U_2 &= \frac{R^2\,\mathrm{sech}^4\left(\frac{1}{16}R(x-y)\right)}{65536\,\Gamma(2\alpha_1+1)\Gamma(\alpha_2+1)}\left[\Gamma(\alpha_2+1)\left(2\sinh\left(\frac{1}{8}R(x-y)\right)+1\right) - \Gamma(\alpha_1+1)\right], \\
V_2 &= \frac{R^2\,\mathrm{sech}^4\left(\frac{1}{16}R(x-y)\right)}{65536\,\Gamma(\alpha_1+1)\Gamma(2\alpha_2+1)}\left[\Gamma(\alpha_1+1)\left(-2\sinh\left(\frac{1}{8}R(x-y)\right)-1\right) + \Gamma(\alpha_2+1)\right].
\end{aligned}$$

Continuing in the same manner and after a few iterations, the differential inverse transform of $\{U_k\}_{k=0}^{\infty}$ and $\{V_k\}_{k=0}^{\infty}$ will give the following series solutions.

$$\begin{aligned} u(x,y,t) &= \sum_{k=0}^{\infty} U_k t^{k\alpha_1} \\ &= \frac{1}{4}\left(\frac{1}{e^{\frac{1}{8}R(x-y)}+1}+2\right) - \frac{R\,\mathrm{sech}^2\left(\frac{1}{16}R(x-y)\right)}{512\,\Gamma(\alpha_1+1)} t^{\alpha_1} \\ &+ \frac{R^2\,\mathrm{sech}^4\left(\frac{1}{16}R(x-y)\right)}{65536\,\Gamma(2\alpha_1+1)\Gamma(\alpha_2+1)}\left[\Gamma(\alpha_2+1)\left(2\sinh\left(\frac{1}{8}R(x-y)\right)+1\right)\right. \\ &\left. -\Gamma(\alpha_1+1)\right] t^{2\alpha_1} + \cdots, \end{aligned} \tag{28}$$

$$\begin{aligned} v(x,y,t) &= \sum_{k=0}^{\infty} V_k t^{k\alpha_2} \\ &= \frac{1}{4}\left(\frac{1}{e^{\frac{1}{8}R(y-x)}+1}+3\right) + \frac{R\,\mathrm{sech}^2\left(\frac{1}{16}R(x-y)\right)}{512\,\Gamma(\alpha_2+1)} t^{\alpha_2} \\ &+ \frac{R^2\,\mathrm{sech}^4\left(\frac{1}{16}R(x-y)\right)}{65536\,\Gamma(\alpha_1+1)\Gamma(2\alpha_2+1)}\left[\Gamma(\alpha_1+1)\left(-2\sinh\left(\frac{1}{8}R(x-y)\right)-1\right)\right. \\ &\left. +\Gamma(\alpha_2+1)\right] t^{2\alpha_2} + \cdots. \end{aligned} \tag{29}$$

The fourth term of the numerical solutions of Equations (22) and (23) was solved using coupled fractional reduced differential transform (CFRDTM) and compared with FVIM in [42]. CFRDTM depends directly on the generalized Taylor formula and therefore needs more computational steps than FRDTM.

The exact solutions for the non-fractional case where $\alpha_1 = \alpha_2 = 1$ of Equations (22) and (23) are:

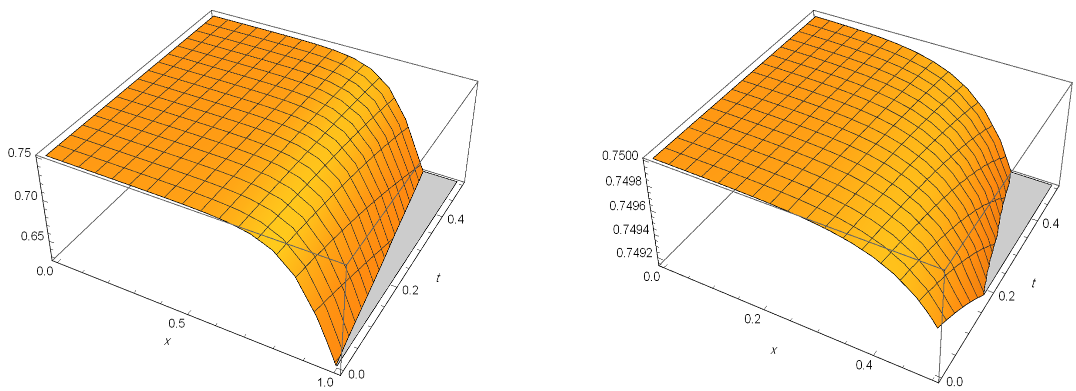
$$u(x,y,t) = \frac{3}{4} - \frac{1}{4[1+\exp((R/32)(-4x+4y-t))]}, \tag{30}$$

$$v(x,y,t) = \frac{3}{4} + \frac{1}{4[1+\exp((R/32)(-4x+4y-t))]}. \tag{31}$$

Figures 9 and 10 plot the exact solutions and the approximate solutions of $u_4(x,y,t)$ and $v_4(x,y,t)$, respectively, using FRDTM ($\alpha_1 = \alpha_2 = 1$) in three dimensions. Figure 11 shows the absolute errors between the exact solutions and the approximate solutions for $\alpha_1 = \alpha_2 = 1, y = 1$ and $R = 100$ with (a) $u_4(x,y,t)$ and (b) $v_4(x,y,t)$. The approximate solutions of $u_4(x,y,t)$ for (a) $\alpha_1 = 0.7$ and (b) $\alpha_1 = 0.5$ are plotted in Figure 12, while the approximate solutions of $v_4(x,y,t)$ for (a) $\alpha_1 = 0.7$ and (b) $\alpha_1 = 0.5$ are plotted in Figure 13. Table 4 shows the numerical approximate solutions of $u_4(x,y,t)$ at different values of $\alpha_1 = 1, 0.9, 0.8,$ and 0.7 and a comparison of the absolute errors for different values t and x, where Table 5 shows the numerical approximate solutions of $v_4(x,y,t)$ at different values of $\alpha_2 = 1, 0.9, 0.8,$ and 0.7 and the comparison of absolute errors for different values t and x.

Table 4. Fourth order approximate numerical solutions of $u_4(x, y, t)$ by FRDTM at different values of α_1 and comparison of absolute errors at $\alpha_1 = 1, y = 1$ and $R = 100$.

		Non-Fractional Order			Fractional Order		
x	t	$\alpha_1 = 1$	u_{Exact}	Absolute error	$\alpha_1 = 0.9$	$\alpha_1 = 0.8$	$\alpha_1 = 0.7$
0.1	0	0.749997	0.749997	0	0.749997	0.749997	0.749997
	0.1	0.749951	0.749996	0.0000441493	0.749951	0.749951	0.749951
	0.2	0.749911	0.749994	0.0000827147	0.749911	0.749911	0.749911
	0.3	0.74987	0.749992	0.00012135	0.74987	0.74987	0.74987
	0.4	0.749829	0.749989	0.000159937	0.749829	0.749829	0.749829
	0.5	0.749786	0.749984	0.000198136	0.749786	0.749786	0.749786
0.2	0	0.749989	0.749989	0	0.749989	0.749989	0.749989
	0.1	0.74983	0.749984	0.000154016	0.74983	0.74983	0.74983
	0.2	0.74969	0.749979	0.000288529	0.74969	0.74969	0.74969
	0.3	0.749548	0.749971	0.00042328	0.749548	0.749548	0.749548
	0.4	0.749403	0.74996	0.000557856	0.749403	0.749403	0.749403
	0.5	0.749255	0.749946	0.000691078	0.749255	0.749255	0.749255
0.3	0	0.74996	0.74996	0	0.74996	0.74996	0.74996
	0.1	0.749409	0.749946	0.000536592	0.749409	0.749409	0.749409
	0.2	0.748921	0.749926	0.00100496	0.748921	0.748921	0.748921
	0.3	0.748425	0.749899	0.00147406	0.748425	0.748425	0.748425
	0.4	0.747919	0.749862	0.0019425	0.747919	0.747919	0.747919
	0.5	0.747405	0.749811	0.00240619	0.747405	0.747405	0.747405
0.4	0	0.749862	0.749862	0	0.749862	0.749862	0.749862
	0.1	0.74795	0.749811	0.00186104	0.74795	0.74795	0.74795
	0.2	0.74626	0.749742	0.00348202	0.74626	0.74626	0.74626
	0.3	0.744543	0.749647	0.00510455	0.744543	0.744543	0.744543
	0.4	0.742794	0.749518	0.00672411	0.742794	0.742794	0.742794
	0.5	0.741015	0.749342	0.00832666	0.741015	0.741015	0.741015
0.4	0	0.749518	0.749518	0	0.749518	0.749518	0.749518
	0.1	0.742989	0.749342	0.0063529	0.742989	0.742989	0.742989
	0.2	0.737257	0.749102	0.011845	0.737257	0.737257	0.737257
	0.3	0.731444	0.748774	0.0173301	0.731444	0.731444	0.731444
	0.4	0.72553	0.748327	0.0227968	0.72553	0.72553	0.72553
	0.5	0.719519	0.747719	0.0281996	0.719519	0.719519	0.719519

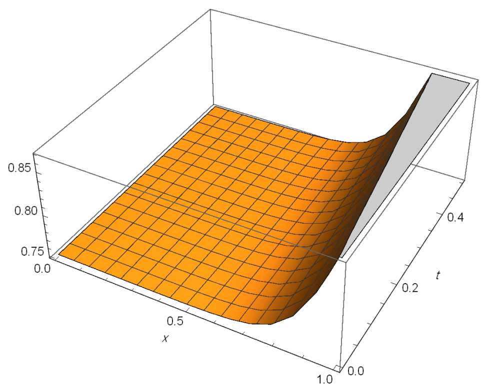


(**a**) (**b**)

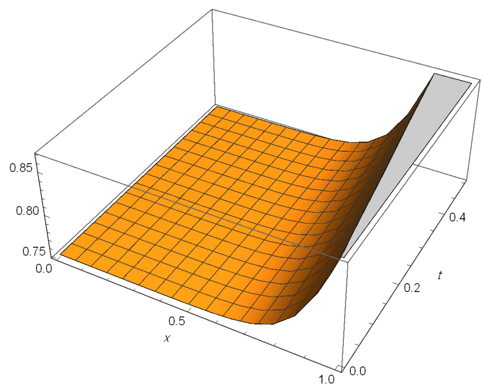
Figure 9. The approximate solution $u_4(x, y, t)$ for $\alpha_1 = 1$, $y = 1$, and $R = 100$: (**a**) FRDTM and (**b**) exact solution.

Table 5. Fourth order numerical approximate solutions of $v_4(x, y, t)$ by FRDTM at different values of α_2 and comparison of absolute errors at $\alpha_2 = 1$, $y = 1$ and $R = 100$.

		Non-Fractional Order			Fractional Order		
x	t	$\alpha_2 = 1$	v_{Exact}	Absolute error	$\alpha_2 = 0.9$	$\alpha_2 = 0.8$	$\alpha_2 = 0.7$
0.1	0	0.750003	0.749997	6.50356×10^{-6}	0.750003	0.750003	0.750003
	0.1	0.750049	0.749996	0.0000530386	0.750049	0.750049	0.750049
	0.2	0.750089	0.749994	0.0000948648	0.750089	0.750089	0.750089
	0.3	0.75013	0.749992	0.000137957	0.75013	0.75013	0.75013
	0.4	0.750171	0.749989	0.000182636	0.750171	0.750171	0.750171
	0.5	0.750214	0.749984	0.000229162	0.750214	0.750214	0.750214
0.2	0	0.750011	0.749989	0.0000226989	0.750011	0.750011	0.750011
	0.1	0.75017	0.749984	0.000185041	0.75017	0.75017	0.75017
	0.2	0.75031	0.749979	0.000330935	0.75031	0.75031	0.75031
	0.3	0.750452	0.749971	0.000481239	0.750452	0.750452	0.750452
	0.4	0.750597	0.74996	0.000637074	0.750597	0.750597	0.750597
	0.5	0.750745	0.749946	0.00079935	0.750745	0.750745	0.750745
0.3	0	0.75004	0.74996	0.0000792181	0.75004	0.75004	0.75004
	0.1	0.750591	0.749946	0.000644864	0.750591	0.750591	0.750591
	0.2	0.751079	0.749926	0.00115293	0.751079	0.751079	0.751079
	0.3	0.751575	0.749899	0.0016763	0.751575	0.751575	0.751575
	0.4	0.752081	0.749862	0.00221889	0.752081	0.752081	0.752081
	0.5	0.752595	0.749811	0.00278389	0.752595	0.752595	0.752595
0.4	0	0.750138	0.749862	0.000276389	0.750138	0.750138	0.750138
	0.1	0.75205	0.749811	0.00223874	0.75205	0.75205	0.75205
	0.2	0.75374	0.749742	0.00399813	0.75374	0.75374	0.75374
	0.3	0.755457	0.749647	0.00580973	0.755457	0.755457	0.755457
	0.4	0.757206	0.749518	0.00768747	0.757206	0.757206	0.757206
	0.5	0.758985	0.749342	0.0096425	0.758985	0.758985	0.758985
0.5	0	0.750482	0.749518	0.000963367	0.750482	0.750482	0.750482
	0.1	0.757011	0.749342	0.00766873	0.757011	0.757011	0.757011
	0.2	0.762743	0.749102	0.0136418	0.762743	0.762743	0.762743
	0.3	0.768556	0.748774	0.0197828	0.768556	0.768556	0.768556
	0.4	0.77447	0.748327	0.0261433	0.77447	0.77447	0.77447
	0.5	0.780481	0.747719	0.0327624	0.780481	0.780481	0.780481

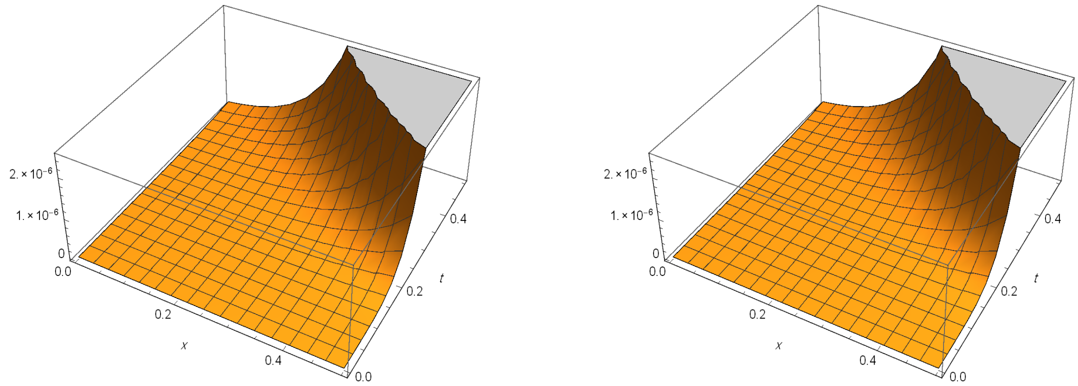


(**a**)



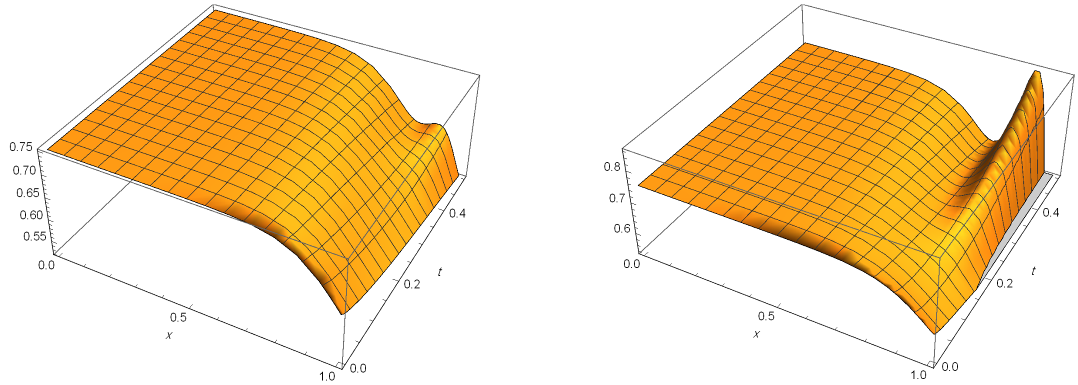
(**b**)

Figure 10. The approximate solution $v_4(x, y, t)$ for $\alpha_2 = 1$, $y = 1$, and $R = 100$: (**a**) FRDTM and (**b**) exact solution.



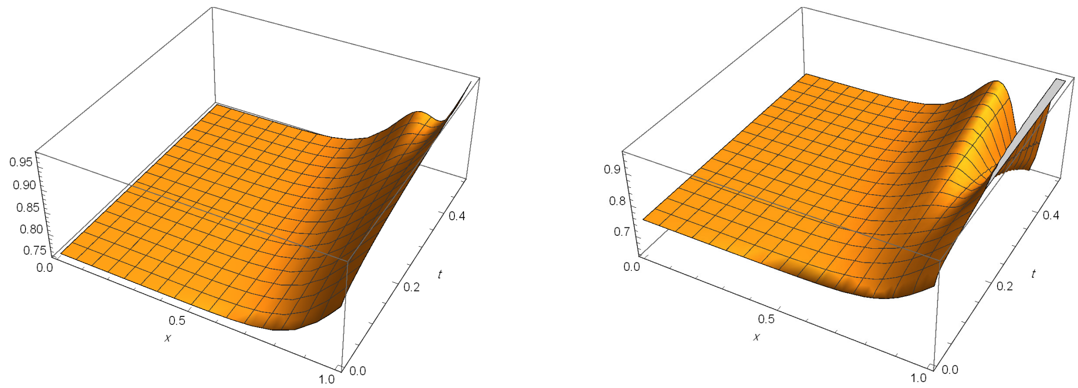
(**a**) (**b**)

Figure 11. The absolute errors between the exact solutions and the fourth order solutions for $\alpha = 1, y = 1$ and $R = 100$ with (**a**) $u_4(x, y, t)$ and (**b**) $v_4(x, y, t)$.



(**a**) (**b**)

Figure 12. The $u_4(x, y, t)$ for (**a**) $\alpha_1 = 0.7$ and (**b**) $= \alpha_1 = 0.5, y = 1$, and $R = 100$.



(**a**) (**b**)

Figure 13. The $v_4(x, y, t)$ for (**a**) $\alpha_2 = 0.7$ and (**b**) $= \alpha_1 = 0.5, y = 1$, and $R = 100$.

5. Results and Conclusions

In this work, we applied FRDTM for the nonlinear fractional Burgers equations in three different dimensions. Example 1 indicated that the third order approximate solution was accurate in comparison with the exact solution; see Figure 1. The maximum absolute errors were notably small with order 10^{-10}; see Figure 4. While the results in Table 2 indicated the efficiency of FRDTM over FVIM such that the maximum difference between the third order FRDTM solutions and the exact solution was of order 10^{-11}, the maximum difference between the second order FVIM solutions and the exact solution was of order 10^{-8}. Example 2 showed that only seven terms of the FRDTM were enough to give a good approximate solution; see Figures 5 and 7 for the exact solutions and different values of α.

The numerical results from Table 3 indicated that the approximate solutions were close to the exact solution especially if $0 < t < 0.3$. Example 3 for solving $(2+1)$-dimensional time fractional coupled BE showed remarkable accuracy between the fourth order approximate solutions using FRDTM and the exact solution; see Figures 9 and 10. Tables 4 and 5 with Figure 11 showed the absolute errors between the approximate solutions and the exact solutions, which indicated the importance of FRDTM for solving different types of nonlinear fractional coupled equations.

In conclusion, the study of the figures and tables in the three previous examples illustrated the importance of using FRDTM; only small amounts of computations gave rapid convergence to the exact solutions, and only a few iterations were enough to yield good accuracy with exact solutions. These results showed certainly that FRDTM was a reliable and powerful method for solving different types of nonlinear fractional partial differential equations over existing methods.

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